Loop Quantum Cosmology

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Symmetry

As usually, *symmetry assumptions* simplify complexity of full theory. Can be implemented in such a manner that *crucial properties* of background independent quantum geometry are *preserved.*

Based on classification of *invariant connections* on principal fiber bundles equipped with a given action of a symmetry group. Introduction of symmetries at *quantum level* possible (symmetric distributional states, induced representation of symmetric holonomy-flux algebra) but not completely worked out yet in general.

Purpose here: *Illustration* of new concepts in much simpler context of isotropic cosmology, outlook on appearance of *singularities.*

Isotropy

Symmetry group $S = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$ in flat model, endows space with background metric ${}^o q$ unique up to scale *(conformal structure as background)*. Also choose an orthonormal triad ${}^o e$ and co-triad ${}^o \omega$ compatible with ${}^o q$.

Allows to specify *invariant connections and densitized triads* such that $(s^* A, s_*^{-1} E) = (g^{-1} A g + g^{-1} dg,\ g^{-1} E g)$ for $s \in S$:

$$A = \tilde{c}\ {}^o\omega^i \tau_i, \quad E = \tilde{p} \sqrt{{}^o q}\ {}^o e_i \tau^i$$

with spatially constant $\tilde{c}$ and $\tilde{p}$ (τ^i generators of su(2)).

Symplectic structure:

$$\Omega_{\mathrm{grav}}(\delta_1, \delta_2) = \frac{1}{8\pi\gamma G} \int_{\mathcal{V}} \mathrm{d}^3 x \left(\delta_1 A_a^i(x) \delta_2 E_i^a(x) - \delta_2 A_a^i(x) \delta_1 E_i^a(x) \right)$$

integrated over fiducial cell $\mathcal{V}$ of coordinate size V_0 yields

$$\Omega_{\mathrm{grav}}^S = \frac{3 V_o}{8\pi\gamma G} \, \mathrm{d}\tilde{c} \wedge \mathrm{d}\tilde{p} \, .$$

Kinematics

Define $c = V_o^{1/3}\tilde{c}$ and $p = V_o^{2/3}\tilde{p}$ such that

$$\Omega^S_{\rm grav} \;=\; \frac{3}{8\pi\gamma G}\;{\rm d}c \wedge {\rm d}p$$

is independent of V_o. Classical phase space with canonical coordinates (c,p) *background independent.*

Elementary variables for quantization:
SU(2) *holonomy* along a straight edge e ($\mu \in \mathbb{R}$)

$$h_e(A) := \mathcal{P}\,\exp\int_e A = \cos\frac{\mu c}{2} + 2\,\sin\frac{\mu c}{2}\;(\dot{e}^{a\,o}\omega^i_a)\,\tau^i$$

and *flux* (constant test function f^i)

$$E_S(f) = \int_S E^c_i f^i \epsilon_{abc} {\rm d}x^a {\rm d}x^b = p\;V_o^{-2/3} A_f\;.$$

Representation

Non-trivial information captured by momentum p and *almost periodic functions* $g(c) = \sum_j \xi_j \; e^{i\mu_j c/2}$ where j runs over a finite number of integers, $\mu_j \in \mathbb{R}$ and $\xi_j \in \mathbb{C}$: *Cylindrical functions.*

Full theory: represent configuration variables (A_a^i) by *holonomies*, forming C^*-algebra Cyl as functions on space $\mathcal{A}$ of connections, then combine with momentum (flux) operators.

Here: C^*-algebra Cyl_S of *almost periodic functions on space of invariant connections,* topologically $\mathbb{R}$.

Gel'fand theorem: there is a *compact* Hausdorff space $\bar{\mathcal{A}}$ *(generalized connections)* such that the algebra of *all continuous functions* on $\bar{\mathcal{A}}$ is isomorphic to Cyl. Moreover, $\mathcal{A} \subset \bar{\mathcal{A}}$ densely.

Isotropy: $\bar{\mathcal{A}}_S \cong \bar{\mathbb{R}}_{\mathrm{Bohr}}$: *Bohr compactification of the real line,* $\mathbb{R} \subset \bar{\mathbb{R}}_{\mathrm{Bohr}}$ densely.

Compactness

Background independence leads to *holonomies,* which select special class of *almost periodic* functions among all continuous functions on $\mathbb{R}$.

Enlarging space by *compactification* introduces new continuity requirements such that now *all* continuous functions on the larger space are almost periodic.

Compact quantum configuration space $\bar{\mathbb{R}}_{\mathrm{Bohr}}$ because functions of holonomies form C^*-algebra, compact gauge group SU(2) required.

Basic algebra given by holonomies $e^{i\mu c/2}$ together with single flux $\hat{p}$ and relation $[e^{i\mu c/2}, \hat{p}] = -\frac{4}{3}\pi\gamma\ell_{\mathrm{P}}^2\mu e^{i\mu c/2}$.

Full theory: *unique representation,* not possible in homogeneous models. But: *relation between full theory and models* results in *distinguished representation* for models.

Hilbert space

Hilbert space: $L^2(\bar{\mathbb{R}}_{\rm Bohr}, {\rm d}\mu)$ with Haar measure on compact Abelian group $\bar{\mathbb{R}}_{\rm Bohr}$, explicitly:

$$\int_{\bar{\mathbb{R}}_{\rm Bohr}} f(c){\rm d}\mu(c) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(c){\rm d}c$$

Orthonormal basis: $|\mu\rangle$ for all $\mu \in \mathbb{R}$, connection representation $\langle c|\mu\rangle = e^{i\mu c/2}$, inner product $\langle\mu_1|\mu_2\rangle = \delta_{\mu_1,\mu_2}$.

Flux operator: p conjugate to c, $\hat{p} = -\frac{1}{3}i\gamma\ell_{\rm P}^2{\rm d}/{\rm d}c$.

Basic operators

$$\begin{aligned} \hat{p}|\mu\rangle &= \tfrac{1}{6}\gamma{\ell_{\rm P}}^2\mu|\mu\rangle \\ \widehat{e^{i\mu' c/2}}|\mu\rangle &= |\mu+\mu'\rangle \end{aligned}$$

$\longrightarrow$ $\hat{p}$ has *discrete spectrum* (normalizable eigenstates);

$\longrightarrow$ only $e^{i\mu c/2}$ quantized, not c itself:

$\langle\mu|e^{itc/2}|\mu\rangle = \langle\mu|\mu+t\rangle = \delta_{0,t}$ *not continuous in t:* operator for c directly does not exist.

Space-time geometry

Meaning of basic classical quantities ($\textit{triad } p, \textit{connection } c$)

$|\tilde{p}| = a^2$ $\mathrm{sgn}(\tilde{p})$: orientation

$\tilde{c} = \frac{1}{2}(k + \gamma\dot{a})$ $k = 0$: flat, $k = 1$: closed

Hamiltonian constraint (with matter Hamiltonian H_{matter}, flat):

$$H = -\frac{3}{8\pi\gamma^2 G} c^2 \sqrt{|p|} + H_{\mathrm{matter}}(p, \phi, p_\phi) = 0$$

Results in *Friedmann equation*

$$3\dot{a}^2 a = 8\pi G V_0^{-1} H_{\mathrm{matter}}(p, \phi, p_\phi)$$

Matter Hamiltonian, e.g.,

$$H_{\mathrm{matter}}(p, \phi, p_\phi) = V_0 \left(\tfrac{1}{2}|p|^{-3/2} p_\phi^2 + |p|^{3/2} V(\phi) \right)$$

for scalar. Problem: Need *inverse of* p*,* to be quantized.

Inverse scale factor

$\hat{p}$ has discrete spectrum containing zero: *no densely defined inverse.*

Observation: $\{c, \sqrt{|p|}\} = \frac{\mathrm{sgn}(p)}{2\sqrt{|p|}}\{c, p\} = \frac{4}{3}\pi\gamma G \frac{\mathrm{sgn}(p)}{\sqrt{|p|}}$.

Can quantize positive power of p, turn Poisson bracket into commutator. But: no operator for c.

Rewrite $\{c, \sqrt{|p|}\} = 2ie^{ic/2}\{e^{-ic/2}, \sqrt{|p|}\}$, allows to quantize all ingredients.

Closer to full theory: use $\hat{V} = |\hat{p}|^{3/2}$ and holonomies $h_i = \cos c/2 + 2\tau_i \sin c/2$:

$$\frac{\mathrm{sgn}p}{\sqrt{|p|}} = \frac{1}{2\pi\gamma G}\mathrm{tr}\left(\sum_{i=1}^{3}\tau^i h_i\{h_i^{-1}, V^{1/3}\}\right)$$

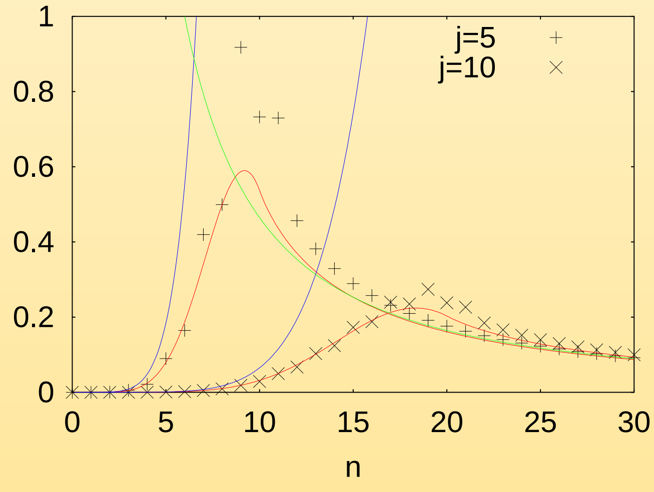
Quantized inverse scale factor

$$\frac{\widehat{\mathrm{sgn}(p)}}{\sqrt{|p|}} = -\frac{4i}{\gamma \ell_{\mathrm{P}}{}^2} \mathrm{tr}\left(\sum_i \tau^i h_i [h_i^{-1}, \hat{V}^{1/3}]\right)$$

$$\frac{\widehat{\mathrm{sgn}(p)}}{\sqrt{|p|}} |\mu\rangle = \frac{6}{\gamma \ell_{\mathrm{P}}{}^2} \left(V_{\mu+1}^{1/3} - V_{\mu-1}^{1/3}\right) |\mu\rangle$$

Properties: *finite* for all μ,

$\frac{\widehat{\mathrm{sgn}(p)}}{\sqrt{|p|}}|0\rangle = 0$.



Dynamics

Turn Hamiltonian constraint into operator, but c^2 *not almost periodic.*
Full theory: use *holonomies* as in

$$H_{\rm grav}[N] \propto \sum_v N(v)\epsilon^{IJK}{\rm tr}\left(h_{\alpha_{IJ}}h_{s_K}\{h_{s_K}^{-1},V\}\right)+O(\Delta)$$

with loops α_{IJ} of size Δ starting in any vertex of a given state.

Isotropy: Use $h_Ih_Jh_I^{-1}h_J^{-1}$ of “length” δ for $h_{\alpha_{IJ}}$ such that

$$\begin{aligned}\hat{H}_{\rm grav} &\approx \sum_{IJK}\epsilon^{IJK}{\rm tr}\left(h_Ih_Jh_I^{-1}h_J^{-1}h_K[h_K^{-1},\hat{V}]\right)\\ &\propto \sin^2(\delta c)\left(\sin(\tfrac{1}{2}\delta c)\hat{V}\cos(\tfrac{1}{2}\delta c)-\cos(\tfrac{1}{2}\delta c)\hat{V}\sin(\tfrac{1}{2}\delta c)\right)\end{aligned}$$

Operator limit $\delta\to 0$ *does not exist,* not even on general states.
Discreteness essential, nevertheless correct semiclassical behavior (c small).

Evolution

Constraint (non-symmetric ordering) acts as

$$\hat{H}_{\rm grav}|\mu\rangle = \frac{3}{16\pi G\gamma^3\delta^3\ell_{\rm P}^2}(V_{\mu+\delta}-V_{\mu-\delta})(|\mu+4\delta\rangle - 2|\mu\rangle + |\mu-4\delta\rangle)\,.$$

Triad representation: expand $|\psi\rangle = \sum_\mu \psi_\mu(\phi)|\mu\rangle$.
Leads to constraint equation:

$$\begin{aligned}(V_{\mu+5\delta}-V_{\mu+3\delta})\psi_{\mu+4\delta}(\phi) - 2(V_{\mu+\delta}-V_{\mu-\delta})\psi_\mu(\phi)\\ +(V_{\mu-3\delta}-V_{\mu-5\delta})\psi_{\mu-4\delta}(\phi) = -\frac{16\pi}{3}G\gamma^3\delta^3\ell_{\rm P}^2\hat{H}_{\rm matter}(\mu)\psi_\mu(\phi)\end{aligned}$$

as a *difference equation* for ψ_μ.

Extends solution through classical singularity $\mu=0$ to other orientation: *non-singular.*

Singularities

General scheme: two sides to classical singularity provided by *orientation factor of triad;* difference equation *does not break down,* unlike differential equations in Wheeler–DeWitt formulations.

So far realized in homogeneous models, spherical symmetry and cylindrical symmetry.

Classical knowledge about singularities needed to see position in superspace; *densitized triads* important:

For instance Bianchi I: Kasner solutions $a_I(t) = t^{\alpha_I}$ with $\sum_{I=1}^{3} \alpha_I = 1 = \sum_{I=1}^{3} \alpha_I^2$. One metric component always *diverges at singularity* $t = 0$.

Densitized triad variables: $|p^I| = a_J a_K$ ($\epsilon_{IJK} = 1$), $|p^I(t)| = t^{1-\alpha_I}$ with $1 - \alpha_I > 0$ for all I. All p^I become *zero at singularity.*