Quantum Geometry and Space-time Singularities

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General Relativity: A beautiful encoding of gravity in geometry. But, as a consequence, space-time itself ends at singularities. Big Bang thought of as the Beginning and Big Crunch the end. But general expectation: theory is pushed beyond its domain of applicability. Must incorporate Quantum Physics.

Idea (Loop Quantum Gravity): Retain the gravity $\leftrightarrow$ geometry duality by encoding new physics in Quantum Riemannian Geometry.

Goal of the talk: Because we already had an Introduction to Quantum Cosmology, I will focus on the Big Bang rather than Black hole singularities. Will argue that Physics does not stop at the Big Bang. Quantum geometry extends its life beyond singularities. Startling perspectives on the nature of space-time.

Phenomenology: informal discussions.

Organization:
1. Historical & Conceptual Setting
2. Loop Quantum Cosmology
3. Detailed Model.

Joint work with Tomasz Pawlowski and Param Singh

1. Conceptual and Historical Setting

- Some Long-Standing Questions expected to be answered by Quantum Gravity from first principles:

⋆ How close to the big-bang does a smooth space-time of GR make sense? (Onset of inflation?)
⋆ Is the Big-Bang singularity naturally resolved by quantum gravity? Or, is a new principle/ boundary condition at the Big Bang essential?
⋆ Is the quantum evolution across the 'singularity' deterministic?
(answer 'No' e.g. in the Pre-Big-Bang scenario)
⋆ What is on the other side? A quantum foam? Another large, classical universe? ...

- Emerging Scenario from LQC: vast classical regions bridged deterministically by quantum geometry. No new principle needed.

- In the classical theory, don't need full Einstein equations in all their complexity. Almost all work in physical cosmology based on homogeneous isotropic models and perturbations thereon. At least in a first step, can use the same strategy in the quantum theory: mini and midi-superspaces.

Older Quantum Cosmology (DeWitt, Misner, . . .)

- Since only finite number of DOF $a(t), \phi(t)$, field theoretical difficulties bypassed; analysis reduced to standard quantum mechanics.

- Quantum States: $\Psi(a,\phi)$; $\hat{a}\Psi(a,\phi) = a\Psi(a,\phi)$ etc.

Hamiltonian constraint: $R - K^2 + (\mathrm{tr}K)^2 = \rho$ becomes $a^2p^2 = H_\phi$.
Quantum evolution governed by the quantum constraint operator
i.e., the Wheeler-DeWitt differential equation

$$\ell_{\mathrm{Pl}}^4 \,\frac{\partial^2}{\partial a^2}(f(a)\Psi(a,\phi)) = \mathrm{const}\, G\ \hat{H}_\phi\,\Psi(a,\phi)$$

Evolution stops at $a = 0$ unless one makes unnatural assumptions
$\Rightarrow$ Singularity does *not* go away.

- New Development: Connection-dynamics and resulting quantum geometry. At large scales, effects negligible. However, dramatic differences near the big-bang. (Bojowald et al).

2. Loop Quantum Cosmology

- Spatial homogeneity and isotropy implies

$\star$ $$A_a = c\,\underbrace{\mathring{\omega}_a^i \sigma_i}_{\text{fixed}}, \qquad E^a = p\,\underbrace{\mathring{e}_i^a \sigma^i}_{\text{fixed}}$$

– holonomy: $h_e(c) = \cos\mu c\ \mathbf{1} + \sin\mu c\ \dot{e}^a \mathring{\omega}_a^i \sigma_i$

– $|p| = a^2$

$\star$ canonically conjugate variables:
c, p for gravity $\qquad$ ϕ, p_ϕ for matter

- Quantum theory:

Finite number of degrees of freedom $\Rightarrow$ Quantum Mechanics.

If x, p is the canonically conjugate pair, $[\hat{x}, \hat{p}] = i\,\hat{I}$.

If $\hat{U}(\lambda) = e^{i\lambda\hat{x}}$ and $\hat{V}(\mu) = e^{i\mu\hat{p}}$, then $\hat{U}(\lambda)\,\hat{V}(\mu) = e^{i\lambda\mu}\,\hat{V}(\mu)\,\hat{U}(\lambda)$.

von Neumann's uniqueness theorem: There is a unique IRR of $\hat{U}(\lambda), \hat{V}(\mu)$ by 1-parameter unitary groups on a Hilbert space satisfying: i) commutation relations; and ii) Weak continuity in λ, μ.

This is the standard Schrödinger representation.

New Quantum Mechanics

- Loop quantum cosmology (focus on gravity):

Key strategy: Follow full theory (Lewandowski, Sahlmann Lectures)

$\star$ States: depend on c only through holonomies
$\Rightarrow$ Almost periodic functions of c
$\star$ Operators: holonomies $\hat{h}_e(c)$ act by multiplication
Momentum fluxes $\sim \hat{p} = -i\hbar \frac{d}{dc}$

Full theory suggests: No operator $\hat{c}$ corresponding to c; i.e., $\hat{h}_e(c)$ not weakly continuous in $e \sim \mu \Rightarrow$ von Neumann's uniqueness theorem by-passed. New Quantum Mechanics possible. (AA, Bojowald, Lewandowski)

- Differences from standard quantum mechanics:

$\star$ States: Built from holonomies: $\Psi(c) = \sum \alpha_j \, e^{i\mu_j c}$
where $\mu_j \in \mathbf{R}$; $\alpha_j \in \mathbf{C}$; $\sum_j |\alpha_j|^2 < \infty$

$\mathcal{H}_{\rm grav} = L^2(\bar{\mathbf{R}}_{\rm Bohr}, d\mu_o) \neq L^2(\mathbf{R}, dc)$

$\star$ operators: $\hat{h}$, $\hat{p}$ well-defined. But No $\hat{c}$.
Spectrum of $\hat{p}$ is the real line with *discrete* topology; eigenstates $e^{i\mu c}$ normalizable.

- Structure mimics that of the full theory.

Quantum Dynamics

- Quantum Einstein's Equation

Expand the quantum state in terms of eigenstates of geometry (i.e., $\hat{p}$) and matter fields (i.e., $\hat{\phi}$) :

$$|\Psi>=\sum_{\mu,\phi}\psi(\mu,\phi)|\mu,\phi>$$

Then the Wheeler DeWitt equation is replaced by a difference equation:

$$C^+_\mu\,\psi(\mu+4\mu_o,\,\phi)+C^o_\mu\,\psi(\mu,\phi)+C^-_\mu\,\psi(\mu-4\mu_o)(\phi)=\gamma\ell_P^2\;\hat{H}_\phi\psi(\mu,\phi)$$

Fundamentally a constraint equation. Selects physical states. However, can also be thought of as an 'evolution equation' in discrete steps $4\mu_o$ (determined by the lowest eigenvalue of the area operator).
Precise reduction to the WDW equation for large μ.

- Is the singularity resolved by quantum dynamics? A priori not obvious.

Example of Potential Danger: The evolution may stop if the coefficients $C^\pm_\mu$ vanishes for some μ.

Coefficients $C^\pm_\mu, C^o_\mu$ such that the evolution does NOT stop! Can evolve right through $\mu=0$. Singularity is resolved. IMPLICATIONS??

Issues Which Were Open Until Recently

(also in the Wheeler-DeWitt theory)

Is there a coherent and complete mathematical theory?

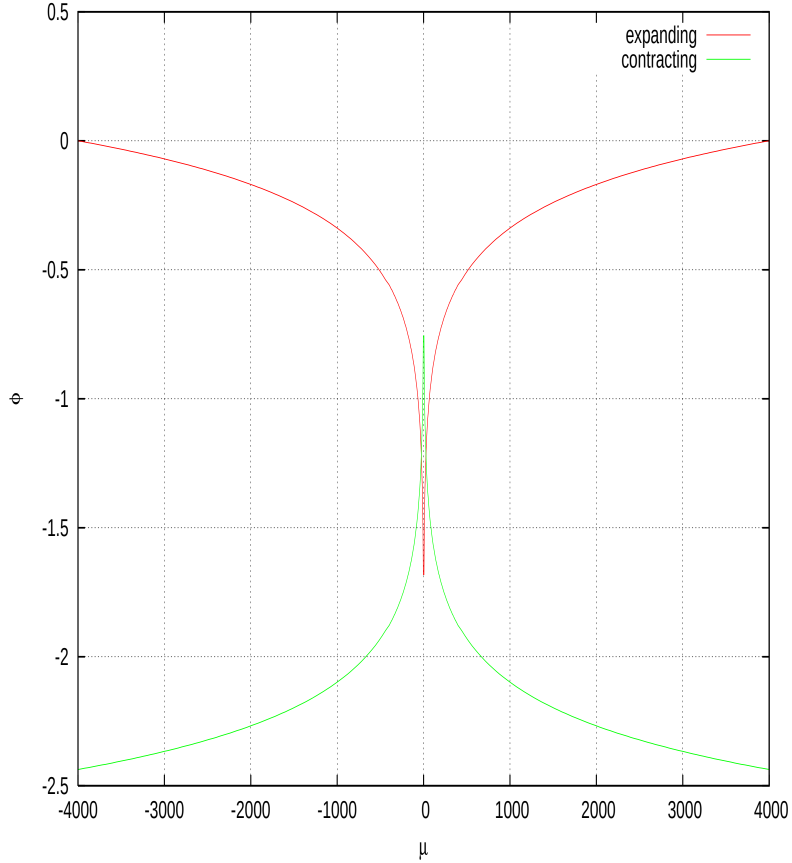
- Physical Hilbert space? Inner product on the Hilbert space? Is $\mathcal{H}_{\rm Phys}$ separable? What are the Dirac observables? Can you use them to construct semi-classical states? Interpretational issues, Problem of time.
- Domain of validity of the semi-classical theory? Well controlled approximations: Going beyond the WKB approximation.
- Other side of the Big-Bang? Quantum foam or another semi-classical universe?

These questions now answered in detail in the simplest model using a mixture of analytical and numerical methods. (AA, Pawlowski, Singh). Numerical quantum cosmology was systematically launched in January 2005; absolutely essential for progress.

3. A Detailed Model

To obtain detailed predictions: Simple Example: Gravity coupled to a massless scalar field ϕ. Complete analytical and numerical treatment possible. Provides a foundation for more complicated models. Incorporation of a potential for ϕ and anisotropies is conceptually straightforward.

($c, p \sim \mu; \phi, p_\phi$, subject to 1 constraint & p_ϕ const of motion.)



Basic Strategy

• Analytical Issues: Use ϕ as time 't'; then the quantum scalar constraint takes the form:

$$\partial_t^2 \Psi(\mu, t) = -\Theta\, \Psi(\mu, t)$$

where Θ is a self-adjoint difference operator independent of ϕ:

$$\Theta\, \Psi(\mu, \phi) = \; C_\mu^+\, \psi(\mu + 4\mu_o,\, \phi) + C_\mu^o\, \psi(\mu, \phi) + C_\mu^-\, \psi(\mu - 4\mu_o)(\phi)$$

Construct the Hilbert space as in the Klein-Gordon theory in static space-time. Physical states: Solve ($\star$) and the symmetry condition $\Psi(\mu, \phi) = \Psi(-\mu, \phi)$. Can construct semi-classical states. (Same structure for solutions of the WDW Eq. Θ: Differential operator.)

• Use numerical methods to solve the Quantum Constraint. Several subtleties had to be carefully handled.

- Illustration of the procedure: KG example

⋆ Equation:

$$\partial_t^2 \Psi(x,t) = -\Theta\,\Psi(x,t) = \partial_x^2 \Psi(x,t)$$

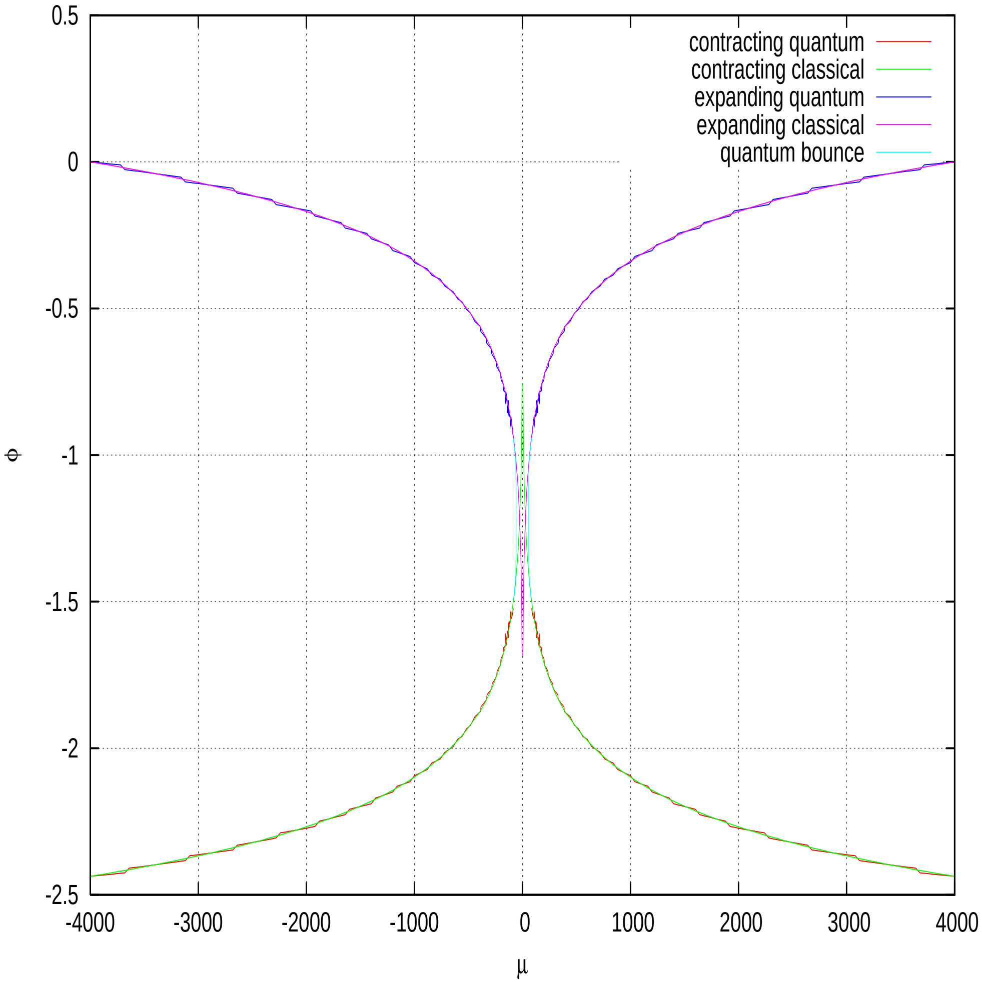
⋆ Positive frequency solutions: $-i\partial_t \Psi = \sqrt{\Theta}\,\Psi$
$\Rightarrow \Psi(x,t) = \exp[i\sqrt{\Theta}\,t]\,\Psi(x,0)$.

⋆ Inner product (+ve freq solns: Non-local!)

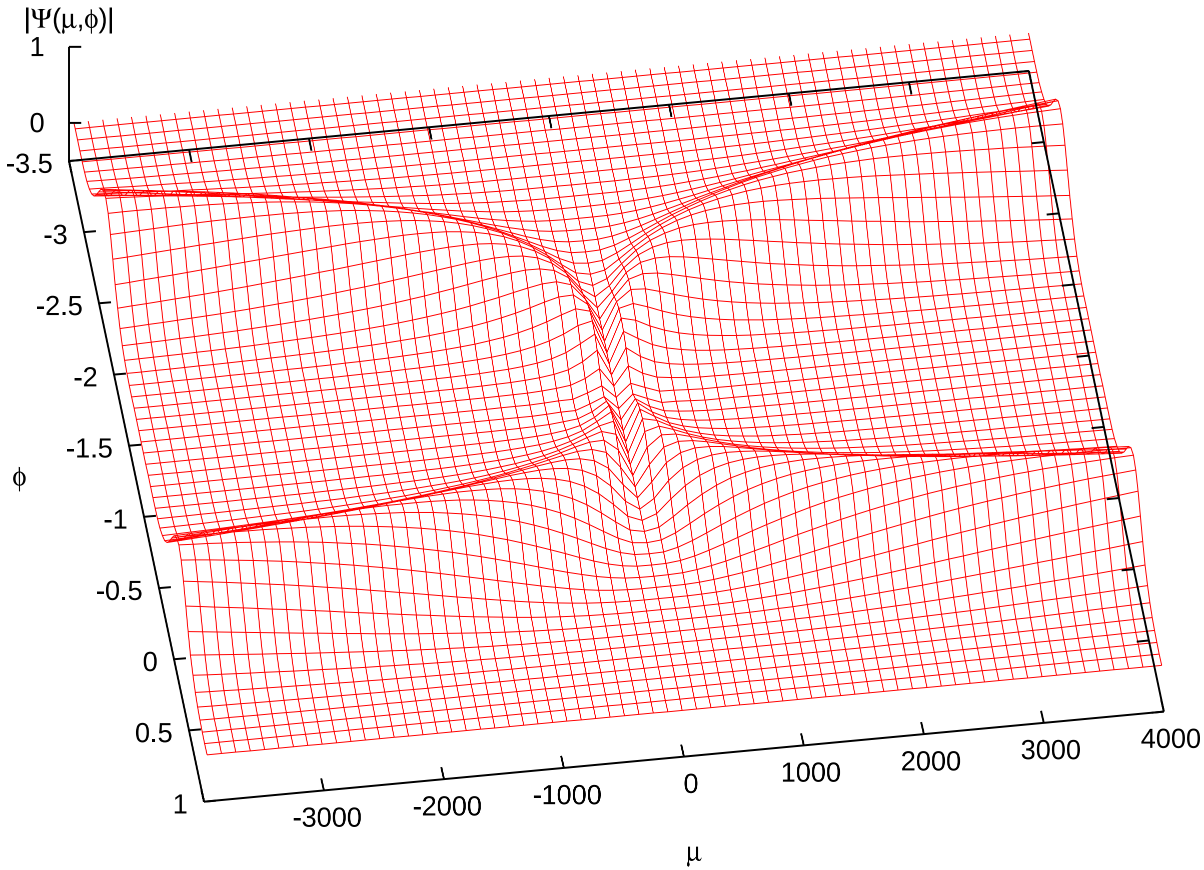
$$||\Psi||^2 = \int_{t=t_o} dx\,\mathrm{Im}\,\bar{\Psi}\partial_t\,\Psi = \int_{t=t_o} dx\,\bar{\Psi}\,\Theta^{1/2}\,\Psi$$

⋆ Dirac observables:
Momentum operator $\hat{p} = -i\partial_x \Psi$
'Position' operator: $\hat{X}_\tau\,\Psi = \exp[i\sqrt{\Theta}(t-\tau)]\,x\Psi(x,\tau)$

⋆ Semi-Classical states ∼ Coherent states.



$$\partial_\phi^2 \Psi(\mu, \phi) = -\Theta \Psi \qquad \Psi(\mu, \phi) = \Psi(-\mu, \phi)$$



$$\partial_\phi^2 \Psi(\mu,\phi) = -\Theta\Psi \qquad \Psi(\mu,\phi) = \Psi(-\mu,\phi)$$

Results

Assume that the quantum state semi-classical at late times (e.g., NOW) and evolve backwards. Then:

• The state remains semi-classical till *very* early times! Till $\rho \sim \rho_{\rm Pl}$. $\Rightarrow$ Space-time can be taken to be classical during the inflationary era.

• In the deep Planck regime, semi-classicality fails. But quantum evolution is well-defined through the classical singularity, and is deterministic. No new principle needed.

• In the past of the deep Planck regime, again semi-classicality. Quantum geometry 'bridges' an infinite expanding branch with an infinite contracting branch. The universe bounces. Unlike in other approaches with bounces, unambiguous evolution across the 'bridge', provided by quantum Einstein equation.

• No unphysical matter. Notion of semi-classicality precise ($\sim$ Coherent States). Unlike in WKB methods, fluctuations under full control.

Summary and Outlook

• In Loop Quantum Gravity, the interplay between geometry and physics is elevated to quantum level. Physics does not end at classical singularities. Many long standing questions answered in simple models.

• In closed models, 'difficulties' pointed out by Green and Unruh disappear in a more complete treatment. Space-like black hole singularities also resolved. (AA, Bojowald, Husain, Pawlowski, Singh, Varadarajan, Winkler)

• Emerging picture: Quantum space-time significantly larger than in GR. Vast new regions bridged by quantum geometry. Quantum evolution is deterministic. A possible space-time resolution of the information loss issue.

Future Directions

- Inhomogeneous perturbations in cosmology. (Deterministic framework exists but calculations yet to be undertaken.)
- Systematic derivation of quantum cosmology from full loop quantum gravity. (Brunneman, Engle)
- Black hole singularities resulting from a full blown gravitational collapse in 4-dimensions.