Quantum Geometry and Its Ramifications

Key feature of general relativity: gravity is encoded in geometry. There are no background fields —no background Riemannian geometry.

The central idea in Loop Quantum Gravity is to elevate this interplay between gravity Riemannian geometry to the quantum level, preserving the background independence. Matter and Geometry, both quantum from 'birth' $\Rightarrow$ Have to learn to do physics without a classical space-time geometry.

Goal of this talk: Explain basics of quantum Riemannian geometry and illustrate its ramifications. In particular, the continuum breaks down near singularities and quantum geometry effects lead to new vistas.

Only broad-brush Strokes. Mathematically oriented review:

AA+ Lewandowski, CQG 2004; gr-qc/0404018

Organization:

1. Elements of Quantum Riemannian Geometry
   - Underlying ideas
   - Breakdown of the continuum
2. Quantum black holes
   - Hints and challenges
   - Quantum geometry and entropy
   - Glimpses of the quantum space-time beyond singularity

1. Elements of Quantum Riemannian Geometry

• Geometry: Physical entity, as real as tables and chairs. Riemann 1854: Göttingen Address; Einstein 1915: General Relativity

• Matter has constituents. GEOMETRY??

'Atoms of Geometry'? Why then does the continuum picture work so well? Are there physical processes which convert Quanta of Geometry to Quanta of Matter and vice versa?

• A paradigm shift to address these issues

"The major question for anyone doing research in this field is: Of which mathematical type are the variables ...which permit the expression of physical properties of space... Only after that, which equations are satisfied by these variables?" Albert Einstein (1946); Autobiographical Notes.

Choice in General Relativity: Metric, $g_{\mu\nu}$. Directly determines Riemannian geometry: Geometrodynamics.

In all other interactions, by contrast, the basic variable is a Connection, i.e., a matrix valued vector potential A_a^i; Gauge theories: Connection-dynamics

Key New idea: 'Kinematic unification'. Cast GR also as a theory of connections. Import into GR techniques from gauge theories that do not refer to a background geometry.

Holonomies and Triads

• As in Wheeler's Geometrodynamics, Hamiltonian framework (a la Dirac): Good mathematical control for a non-perturbative, background independent treatment.
[Path Integral Approach: Spin Foams]

• Phase space: $\mathbf{\Gamma} \ni \ (A_a^i,\ E_i^a)$ on a 3-manifold M.
A_a^i: An SU(2) connection and E_i^a: a su(2)-valued vector density on M;

$$\mathbf{\Omega}\mid_{(A,E)} (\delta_1, \delta_2) = \int_M (\delta_1 A_a^i)(\delta_2 E_i^a) - (\delta_1 \longleftrightarrow \delta_2)\, d^3x$$

Phase space $(\mathbf{\Gamma}, \mathbf{\Omega})$ the same as in SU(2) YM theory.

• But Dual interpretation (SU(2) is the (double cover) of the group of rotations of orthonormal triads on M. A_a^i parallel transports chiral spinors.)
Due to this 'soldering': Relation to geometrodynamics
$E_i^a E^{bi} = \text{const}\ q\, q^{ab}$
$E_i^a\ \sim$ orthonormal triad (with density weight 1)
$A_a^i = \Gamma^i_a(E) - (\text{const}) k_a^i$
Classically, only a canonical transformation but can import new quantization techniques from gauge theories.

• 'Elementary' Phase Space Functions:
$\star$ Holonomies: $h_e := \mathcal{P}\ \exp - \int_e A$;
Configuration variables: Ring $\mathbf{R}$ of functions on $\mathcal{A}$ generated by matrix elements of holonomies.
$\star$ Momenta: Electric/Triad Fluxes: $E_{S,f} := \int_S f^i E_i^a$
Via Poisson brackets, Fluxes define derivations on $\mathbf{R}$: $\{F,\, E_{S,f}\} := \mathcal{L}_{X(S,f)} F,\ \forall F \in \mathbf{R}$.
Situations completely parallel to QM on manifolds.

Quantum Theory

• Construct the free $\star$ algebra generated by elementary variables and mod by relations. The task is to construct suitable representations of this algebra $\mathfrak{a}$.

• Surprising uniqueness result: 'The quantum algebra $\mathfrak{a}$ admits a unique irreducible diff(M) invariant (i.e., background independent) representation'. (Lewandowski, Okolow, Sahlmann, Thiemann; Fleischhack) Contrast with QFTs in Minkowski space-time. Background independence unexpectedly powerful.

Representation (AA, Baez, Lewandowski, Rovelli, Smolin, ...)

• Quantum Configuration space $\bar{\mathcal{A}}$: Gel'fand spectrum of the Abelian unital $C^\star$ algebra generated by the matrix elements of holonomies. ($\bar{\mathcal{A}}$ is the compact topological space, the $C^\star$ algebra of all continuous functions on which is isomorphic to the holonomy $C^\star$ algebra.)

• 'Controllable': $\bar{\mathcal{A}}$ consists of 'generalized' connections $\bar{A}$: $\bar{A}$ associates to each edge e an element $\bar{A}(e)$ of SU(2) such that (i) $\bar{A}(e_1) \cdot \bar{A}(e_2) = \bar{A}(e_1 \cdot e_2)$; and (ii) $\bar{A}(e^{-1}) = [\bar{A}e]^{-1}$.

• State Space: $\mathcal{H}_{\rm geom} = L^2(\bar{\mathcal{A}}, d\mu_o)$.
Spin-network decomposition: $\mathcal{H}_{\rm geom} = \bigoplus_{\alpha,\vec{j}} \mathcal{H}_{\alpha,\vec{j}}$
(Finite dim Hilbert spaces of spin systems.)
This greatly facilitates calculations.
'Typical states': Given a graph γ with n edges and a function F on $[\mathrm{SU}(2)]^{\rm n}$, set $\Psi_{\gamma,f}(\bar{A}) = F(\bar{A}(e_1), \ldots \bar{A}(e_n))$. By varying γ, F one obtains a dense subspace of $\mathcal{H}_{\rm geom}$.

• Operators: Holonomies and (smeared) triads are self-adjoint on $\mathcal{H}_{\rm geom}$. $\hat{h}_e$ act by multiplication (Gel'fand theory) $\hat{E}_{S,f}$ triads act by derivations. (Recall PBs define derivations.) Quantum Riemannian geometry constructed from triad operators.

Examples of Novel features

$\star$ All eigenvalues of geometric operators discrete. Eigenvalues not just equally-spaced but crowd in a rather sophisticated way. Geometry quantized in a very specific way. (Recall Hydrogen atom.)

$\star$ Fundamental excitations of geometry are 1-dimensional (holonomies). 'Polymer geometry'. Flux-lines of area, with evs $a_j = \text{const}\,\sqrt{j(j+1)}\ell_{\rm Pl}^2$. Continuum only a coarse-grained approximation; a fabric woven by quantum threads.

$\star$ Inherent non-commutativity: Areas of intersecting surfaces don't commute. Inequivalent to (at least the naive) quantum geometrodynamics.

$\star$ No operator corresponding to the connection itself. Mathematical Analogy: $\hat{U}(\lambda) := \widehat{\exp i\lambda x}$ well-defined but not $\hat{x}$. In QM, the von-Neumann uniqueness theorem no longer applicable $\Rightarrow$ New Quantum mechanics possible. Realized in mini-superspace reductions, e.g., of quantum cosmology.

2. Quantum Black Holes

• First law of BH Mechanics + Hawking's discovery that $T_{\rm BH} = \kappa\hbar/2\pi \;\; \Rightarrow$ for large BHs, $S_{\rm BH} = a_{\rm hor}/4\ell_{\rm Pl}^2$

• 1. Entropy. Why is the entropy proportional to area and not volume? For a $M_\odot$ black hole, we must have $\exp 10^{76}$ micro-states, a HUGE number even by standards of statistical mechanics. Where do these micro-states come from?
For gas in box, the microstates come from molecules; for a ferromagnet, from Heisenberg spins; Black hole ?
Cannot be gravitons: gravitational fields stationary.

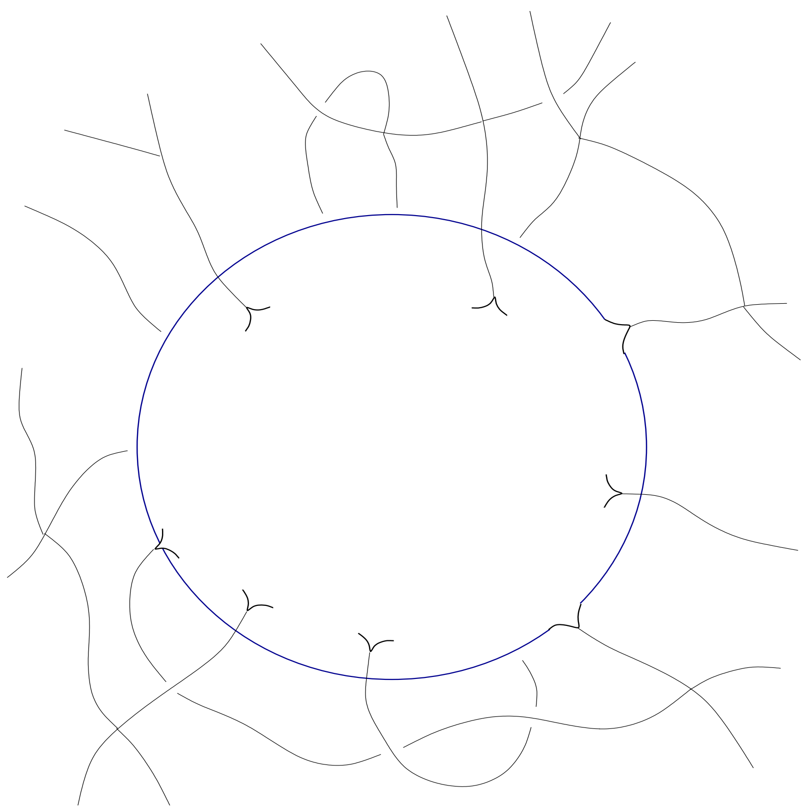
• To answer these questions, must go beyond the classical space-time approximation used in the Hawking effect. Must take in to account the quantum nature of gravity.

• Distinct approaches. In Loop Quantum Gravity, this entropy arises from the huge number of microstates of the quantum horizon geometry. 'Atoms' of geometry itself!

Quantum Geometry and Entropy

• Heuristics: Wheeler's It from Bit: Divide the horizon into elementary cells, each carrying area $\ell_{\rm Pl}^2$ and assign to each cell a 'Bit' i.e. 2 states.
Then, # of cells $n \sim a_o/\ell_{\rm Pl}^2$; No of states $\mathcal{N} \sim 2^n$; $S \sim \ln \mathcal{N} \sim n \ln 2 \sim a_o/\ell_{\rm Pl}^2$. Thus, $S \propto a_o/\ell_{\rm Pl}^2$.

• Argument made rigorous in quantum geometry. Many inaccuracies of the heuristic argument have to be overcome. Interesting mathematical structures (U(1) Chern-Simons theory; non-commutative torus, quantum $U(1)$, ...) .



Detailed treatment: (AA, Baez, Krasnov; Domagala, Lewandowski; Meissner; AA, Engle, Van Den Broeck, ...)

⋆ Black holes in equilibrium modelled by Isolated Horizons. Only the horizon is isolated —external space-time can be dynamical. Laws of BH mechanics continue to hold. BHs can be rotating, distorted by external matter, hair, etc.

⋆ Construct the phase space of general relativity consisting of space-times which admit an Isolated Horizon with a fixed set of multipoles as the inner boundary (a set of numbers which provide a diff invariant characterization of the classical horizon geometry.)

⋆ Quantize this sector of GR + matter. In the bulk: states are polymer-like —described by $\mathcal{H}_{\rm geom}$. But there are also surface states associated with the horizon. The two are intertwined by the quantum isolated horizon boundary conditions. Physically, the quantum horizon can fluctuate and so can the polymer excitations. But they must do so in tandem.

⋆ Microstates responsible for entropy arise from the intrinsic geometry of the quantum isolated horizon. Described by a topological theory: $U(1)$ Chern-Simons theory.

⋆ Area of elementary 'cells' const $\sqrt{j(j+1)}\ell_p^2$; j can be an *arbitrary* 1/2 integer. Cells don't have the same area.

⋆ Each cell has $(2j+1)$ states; not just 2.

Final Result

Careful counting of surface states (Domagala, Lewandowski, Meissner) shows that for large BHs, the leading contribution to the entropy is proportional to the horizon area as required by the Bekenstein-Hawking semi-classical formula. The sub-leading, quantum gravity correction has also been calculated (Meissner) .

[For simplicity of presentation, have not discussed the precise numerical coefficients. Related to unspecified proportionality constants —Barbero-Immirzi parameter.]

Quantum resolution of Singularities: First Steps

• The Big bang Singularity: Detailed, analytical and numerical treatment in mini-superspace models was discussed in this program. So, I will now consider the Schwarzschild singularity. (AA, Bojowald)

• Interior of the horizon of Schwarzschild solution is foliated by homogeneous slices $M = \mathbb{R} \times \mathbb{S}^2$ with 4 Killing fields. Kantowski-Sachs symmetry group. So, as a first step we can carry out a symmetry reduction and focus on the Kantowski-Sachs mini-superspace. Symmetry adapted coordinates (x, θ, ϕ); metric functions: (q_{xx}, r). 2-d conf space and 4-d phase space.

• Connection dynamics:

$$A = c_o \tau_3 dx + c_1 \tau_2 d\theta - c_1 \tau_1 \sin\theta d\varphi + \tau_3 \cos\theta d\varphi$$

$$E = p_o \tau_3 \sin\theta \partial/\partial x + p_1 \tau_2 \sin\theta \partial/\partial\theta - p_1 \tau_1 \partial/\partial\varphi$$

$$\mathbf{\Gamma} \ni \; (c_o, c_1;\, p_o, p_1) \equiv (-c_o, c_1;\, -p_o, p_1)$$

⋆ Orthonormal co-triad:

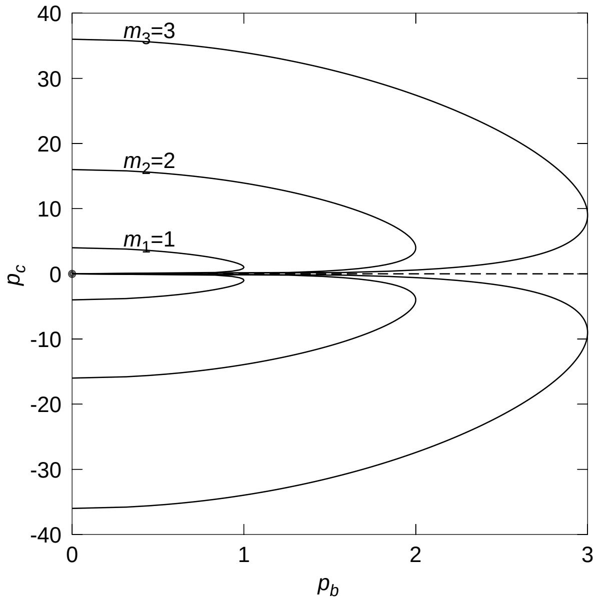
$\omega^{(x)} = f^{(x)} dx;\ \omega^{(\theta)} = f^{(\theta)} d\theta,\ \omega^{(\varphi)} = f^{(\varphi)} \sin\theta d\varphi$ where,

$f^{(x)} = \mathrm{sgn}\, \mathrm{p_o} \sqrt{|p_1/p_o|};\quad f^{(\theta)} = -f^{(\varphi)} = \mathrm{sgn}\, \mathrm{p_1} \sqrt{|p_o|}$

⋆ Scalar/Hamiltonian constraint:

$$C = -(p_o c_1 c_o + (c_1^2/2 + 4) p_1) = 0$$

• Mini-superspace $\mathcal{M} \ni \ (p_o, p_1)$



• Wheeler-DeWitt type quantization:
$\mathcal{H} = \{\Psi(p_o, p_1) \in L^2(R^2, d^2x);\ \Psi(p_o, p_1) = \Psi(p_o, -p_1)\}$
$\hat{p_o}, \hat{p_1}$ act by multiplication;
$\hat{c_o}, \hat{c_1}$ act by differentiation.

Then:
$\star\ \hat{\omega}^{(x)}$ is unbounded above; and
$\star$ WDW equation gives

$$\left(p_o \frac{\partial}{\partial p_o} \frac{\partial}{\partial p_1} + \frac{1}{2} \frac{\partial}{\partial p_1} p_1 \frac{\partial}{\partial p_1} - 4p_1\right) \Psi(p_o, p_1) = 0$$

Singularity effectively pushed to $t := \ln p_o = -\infty$.

New Quantum Mechanics

- Key strategy: Follow the full theory

$\star$ States: depend on c only through holonomies
$\Rightarrow$ Almost periodic functions of $c_i \equiv (c_o, c_1)$
$\star$ Operators: holonomies $\hat{h}_e(c_i)$ act by multiplication
Momentum fluxes $\sim \hat{p}_i = -i\hbar \frac{d}{dc_i}$

Full theory suggests: No operator $\hat{c}_i$ corresponding to c; i.e., $\hat{h}_e(c_i)$ not weakly continuous in e $\Rightarrow$ von Neumann's uniqueness theorem by-passed. New Quantum Mechanics possible.

- Differences from standard quantum mechanics:

$\star$ States: Built from holonomies: $\Psi(c) = \sum \alpha_j \, e^{ip_j c}$
where $p_j \in \mathbf{R}$; $\alpha_j \in \mathbf{C}$; $\sum_j |\alpha_j|^2 < \infty$
$\mathcal{H}_{\text{grav}} = L^2(\bar{\mathbf{R}}_{\text{Bohr}}, dp_o) \quad \neq L^2(\mathbf{R}, dc)$
$\star$ operators: $\hat{h}$, $\hat{p}$ well-defined. But No $\hat{c}$.
Spectrum of $\hat{p}$ is the real line with *discrete* topology; eigenstates e^{ipc} normalizable.

Now, the co-triad operator bounded above!
Denote eigenstates of $\hat{p}_o$, $\hat{p}_1$ by $|\, p_o, p_1 >$
$\hat{f}^{(x)}|p_o, p_1 >= \frac{1}{2}\ell_{\text{Pl}}|p_1|(\sqrt{|p_o+1|} - \sqrt{|p_o-1|})\,|p_o, p_1 >$
Has all the physically expected properties.

Wheeler DeWitt equation replaced by a Difference Equation; evolution well-defined across $p_o = 0$.

Summary and Outlook

• Loop Quantum Gravity is a background independent, non-perturbative approach to quantum gravity based on a specific theory of quantum Riemannian geometry. I sketched an outline of how this theory is constructed and applied.

• A key feature of this approach is that the interplay between geometry and gravity is elevated to a quantum level. Enables us to revisit many of the long standing questions of quantum gravity. I sketched two examples: Statistical mechanical derivation of the entropy of physically realistic black holes and resolution of singularities. More detailed discussion of the big-bang singularity were discussed earlier in the program. Together, these results suggest that the physical, quantum space-time does not end at singularities. New vistas open up.

• Quantum geometry has also led to significant new research in mathematics. (New calculi on infinite dimensional spaces of connections; graph theory; quantum groups, topological field theory; state sum models; spin foams; ...)

Challenges: Examples

Quantum Dynamics: Many ambiguities in the expression of the Quantum Hamiltonian constraint in the *full* theory. Physical meaning/implications are poorly understood. In particular, do not have a sufficient control on the semi-classical sector of the theory. (Concrete advances were made in situations where the ambiguities are under control or do not matter. Here, semi-classical sectors recovered.)

• Cosmology: Some long standing problems have been addressed in detail through detailed analytical and numerical investigations, but only in the context of homogeneous models. Challenge: Systematic development of perturbation theory to go beyond homogeneity and closer contact with phenomenology.