The twistor theory of the Ernst equations

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Credits: R.S.Ward, N.M.J.Woodhouse, J.Fletcher, K.Y.Tee, based on classic work on Ernst equations (Ehlers, Ernst, Geroch, Belinsky, Zakharov, Breitenlohner, Maison, Kramer, Neugebauer, Cosgrove, Alekseev, Hauser, Hoenselaar, Griffiths, . . .).
References: Mason & Woodhouse (1988), Fletcher & Woodhouse (1991), 'Integrability, self-duality and twistor theory', Mason & Woodhouse, OUP, (1996).

Twistor theory

Basic aim: find 1–1 correspondences

$$\left\{\begin{array}{l}\text{Solutions to physical equations:}\\ \text{Yang-Mills, Einstein} \ldots\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Deformed complex struc-}\\ \text{tures on twistor space}\end{array}\right\}$$

Hope: twistor space is correct geometric arena for physics.

Twistor correspondences for self-dual Yang-Mills & Einstein equations suggests the larger programme.

Integrable systems

Spin off: Symmetry reduction of self-duality equations $\rightsquigarrow$ many/most integrable systems.

Twistor correspondences reduce to 'prove' complete integrability.

Programme:

(1) Classify integrable systems that arise as reductions of the self-duality equations.

(2) Derive theory of reduced equations from twistor correspondences.

Stationary axisymmetric self-dual Yang-Mills

Yang's form of Self-dual Yang-Mills on $\mathbb{R}^4$ with coords $(u, v) = (z + it, \rho\,\mathbf{e}^{i\theta})$, metric $\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}\bar{u} + \mathrm{d}v\mathrm{d}\bar{v}$ is:

$$\frac{\partial}{\partial \bar{u}}\left(J^{-1}\frac{\partial J}{\partial u}\right) + \frac{\partial}{\partial \bar{v}}\left(J^{-1}\frac{\partial J}{\partial v}\right)$$

where $J = J(u, v, \bar{u}, \bar{v})$ is a Hermitian matrix function.
t and θ-independence $\rightsquigarrow$

$$\partial_\rho(\rho J^{-1}\partial_\rho J) + \partial_z(\rho J^{-1}\partial_z J) = 0\,.$$

The reduced vacuum equations
Ward's reduction (1983)

Metric:

$$\mathrm{d}s^2 = \pm \mathbf{e}^{2k}(\mathrm{d}\rho^2 + \mathrm{d}z^2) \pm J_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

$$k = k(\rho, z)\,, \quad J_{ij} = J_{ij}(\rho, z).$$

$\partial/\partial x^i$, $i = 1, \dots n-2$ are 2-surface orthogonal Killing vectors.

The Vacuum field equations

$$\rho^2 = \det J$$

$$4i\frac{\partial k}{\partial \zeta} = -\rho\,\mathrm{tr}\,((J^{-1}\partial_\zeta J)^2) - \frac{1}{\rho}\,, \quad \zeta = z + i\rho.$$

$$\partial_\rho(\rho J^{-1}\partial_\rho J) + \partial_z(\rho J^{-1}\partial_z J) = 0$$

We focus on last equation and $n = 4$.

Pick one Killing vector X, set $X^\flat = g(X,\ \ )$ and

$$f = g(X,X)\,, \quad \mathrm{d}\psi = \ {}^*(X^\flat \wedge \mathrm{d}X^\flat)$$

Then define

$$K = \frac{1}{f}\begin{pmatrix} f^2+\psi^2 & -\psi \\ -\psi & 1 \end{pmatrix}$$

Vacuum equations are

$$\partial_\rho(\rho K^{-1}\partial_\rho K) + \partial_z(\rho K^{-1}\partial_z K) = 0\,.$$

($\exists$ 3×3 reductions of both types for Electrovacuum solutions.)

The Lax pair

- Let $U \subset H = \{(z,\rho) \in \mathbb{R}^2 | \, \rho \geq 0\}$ and set $\zeta = z + i\rho$.
- For $(\zeta, \lambda) \in U \times \mathbb{CP}^1$ define

$$\begin{aligned} L &:= \partial_\zeta + \frac{i\lambda(\lambda - i)}{2\rho(\lambda + i)}\partial_\lambda + \frac{i}{\lambda + i} J^{-1}\partial_\zeta J \,, \\ \tilde{L} &:= \partial_{\bar{\zeta}} - \frac{i\lambda(\lambda + i)}{2\rho(\lambda - i)}\partial_\lambda - \frac{i}{\lambda - i} J^{-1}\partial_{\bar{\zeta}} J \end{aligned}$$

- Then

$$[L, \tilde{L}] = 0 \qquad \Leftrightarrow \qquad \partial_\rho(\rho J^{-1}\partial_\rho J) + \partial_z(\rho J^{-1}\partial_z J) = 0$$

Twistor space

For a given $U \subset H$ define the reduced twistor space to be

$$\mathbb{T}(U) = U \times \mathbb{CP}^1 / \{l, \tilde{l}\}$$

where $\{l, \tilde{l}\}$ is the distribution spanned by

$$l = \partial_\zeta + \frac{i\lambda(\lambda - i)}{2\rho(\lambda + i)}\partial_\lambda \,, \qquad \tilde{l} = \partial_{\bar{\zeta}} - \frac{i\lambda(\lambda + i)}{2\rho(\lambda - i)}\partial_\lambda \,.$$

Points of $\mathbb{T}(U)$ are the leaves of $\{l, \tilde{l}\}$ given by constant

$$\gamma = z + \frac{\rho}{2}(\frac{1}{\lambda} - \lambda)$$

$\mathbb{T}(U)$ is a non-Hausdorff Riemann surface with holomorphic coordinate γ (or $1/\gamma$) so $\gamma : \mathbb{T}(U) \to \mathbb{CP}^1$.

Reduced Ward correspondence

Thus we have $p : U \times \mathbb{CP}^1 \to \mathbb{T}(U)$ and $\mathbb{T}(U)$ is a covering of $\mathbb{CP}^1$.

Theorem

Solutions to the stationary axisymmetric SDYM equations on U are in 1:1 correspondence with holomorphic vector bundles $E \to \mathbb{T}(U)$ such that p^*E is trivial over each $\mathbb{CP}^1$ in $U \times \mathbb{CP}^1$.

(Condition is generically satisfied for small enough U.)
$E_\gamma =$ solutions Ψ to $L\Psi = \tilde{L}\Psi = 0$ on leaf of constant γ.

Theorem

Suppose

- $\bar{E} =$ *pull-back of* E^* *by complex conjugation* $\gamma \to \bar{\gamma}$
- *interchange of sheets of* $\mathbb{T}(U) \to \mathbb{CP}^1$ *sends* $E \to E^*$,

then J is real and symmetric and is a solution of the reduced vacuum equations.

The twistor data
The general complexified case

- Assume U connected, simply connected & disjoint from $\rho = 0$.
- Let $V \subset \mathbb{CP}^1$ be the 'glued down region' in $\mathbb{T}(U)$.
  Definition of $\gamma \Leftrightarrow$
  $$\frac{\gamma - (z + i\rho)}{\gamma - (z - i\rho)} = \left(\frac{\lambda + i}{\lambda - i}\right)^2 .$$
  $\Rightarrow$
  $$V = \{\gamma = z \pm i\rho \,|\, (z, \rho) \in U\} \equiv U \cup \tilde{U} .$$
- Twistor data can be normalized w.r.t. choice of $(z_0, \rho_0) \in U$.
- Twistor data is a pair of $\mathrm{SL}(2, \mathbb{C})$ valued functions
  $$P(\sqrt{\gamma - (z_0 + i\rho_0)}) \text{ on } U , \quad \text{and} \quad \tilde{P}(\sqrt{\gamma - (z_0 - i\rho_0)}) \text{ on } \tilde{U} ,$$
  subject to certain involutions.
- Metric constructed via 'Riemann-Hilbert' problem in λ-plane.

Gowdy/colliding plane waves

K.Y.Tee, D.Phil. thesis

- For Gowdy/colliding plane waves, $\tau := i\rho \in \mathbb{R}$ is timelike, z, x^i spacelike.
- $\tau = 0$ is generically singular, in past/future for Gowdy, future for colliding plane waves.
- Choice of $(z_0, \tau_0) \in U$ is natural for characteristic IVP for colliding plane waves.
- P and $\tilde{P}$ are naturally real functions of the real variables $\zeta = z + \tau$, $\tilde{\zeta} = z - \tau$ respectively.
- P and $\tilde{P}$ are equivalent to the characteristic data on $\tilde{\zeta}_0 =$const., $\zeta_0 =$const. respectively.

Axis simple case
The Ward ansatze

Assume glued down region V is connected & simply connected.

- $\mathbb{T}(U) = \mathbb{CP}^1_0$ glued to $\mathbb{CP}^1_\infty$ over V.
- There is a canonical normalization of twistor data:

$$E|_{\mathbb{CP}^1_0} \cong E|_{\mathbb{CP}^1_\infty} \cong \mathcal{O}(p) \oplus \mathcal{O}(q), \qquad (\mathcal{O}(p) = \mathbb{C}\text{-line bundle, } c_1 = p)$$

  Symmetric $P(\gamma) : V \to \mathrm{SL}(2, \mathbb{C})$ patches $E|_{\mathbb{CP}^1_0}$ to $E|_{\mathbb{CP}^1_\infty}$.
- Metric is obtained from *Riemann-Hilbert problem* in λ-plane

$$G_0(z, \rho, \lambda) = \begin{pmatrix} \frac{\rho^p}{\lambda^p} & 0 \\ 0 & \frac{\rho^q}{\lambda^q} \end{pmatrix} P(\gamma) \begin{pmatrix} (-\rho\lambda)^p & 0 \\ 0 & (-\rho\lambda)^q \end{pmatrix} G_\infty(z, \rho, \lambda)\,,$$

  $\gamma = z + \frac{\rho}{2}(\frac{1}{\lambda} - \lambda)$, G_0 holomorphic on $|\lambda| \leq 1$, G_∞ on $|\lambda| \geq 1$.

$$J(z, \rho) = G_0(z, \rho, 0) G_\infty(z, \rho, \infty)^{-1}.$$

- P is obtained from finite order ρ-expansion of J at $\rho = 0$.

Cyindrical symmetry
Woodhouse (1989)

Here, ρ is space-like, $\rho = 0$ is axis, and $t = iz \in \mathbb{R}$ is time.

- Ward ansatze applies with V being (nhd of) real axis.
- Ward's metric reduction requires $p = 1$, $q = 0$,
  L.Witten's Ernst reduction requires $p = q = 0$.
- $\rho = 0$ is not a natural Cauchy surface.
- However, the $P(\gamma)$ of the Ward ansatze has a canonical representation as a path-ordered exponential of cauchy data at $t = 0$.

Stationary axisymmetric case

James Fletcher's thesis (1990)

Here $(x^1, x^2) = (t, \phi)$, ϕ defined mod 2π, space-like, t time.

- Ward ansatze applies, $V = U \cup \bar{U}$ where U is region on which solution exists.
- Ward's metric reduction requires $p = 1$, $q = 0$,
  L.Witten's Ernst reduction requires $p = q = 0$.
- Fletcher characterizes the intersections of the axis and horizons in terms of the twistor data (simple pole in P).
- Proves the stationary-axisymmetric black-hole uniqueness theorem using extension of Liouville's theorem on twistor data.

Geroch group

Hidden symmetries, M. & Woodhouse (1988)

Geroch group is group of hidden symmetries.

- Generated by interplay of L. Witten's Ernst reduction, K, and Ward's metric reduction, J.
- Map $J \to K$ corresponds to 'twisting' (conjugating) twistor data by
$$\begin{pmatrix} 0 & 1 \\ \gamma^{-1} & 0 \end{pmatrix} .$$
- Generates action of loop group $g(\gamma) : S^1 \to \mathrm{SL}(2, \mathbb{C})$, $S^1 = \{\gamma, \, |\gamma| = 1\}$.
- Acts by twisting bundle.
- Can classify orbits in general case & prove transitivity in axis regular case.

Summary

- Twistor theory provides a geometric unifying framework for integrable systems approaches to the Ernst equations.
- Can be adapted to all the standard applications.

- **Further projects:**
  - Project with A. Gray & M. Singer on $++++$ signature case aims at classification of toric Ricci-flat 4-manifolds.
  - Work with D. Calderbank.
    Deformations of $\mathbb{T}(U) \leftrightarrow$ solutions to 'spinor-vortex' equation: Riemann surface Σ, metric g, Dirac op. D and spinor field Ψ,

$$D\Psi = -3\bar{\Psi}\,, R = 4 - 2|\Psi|^2$$

$\rightsquigarrow$ toric anti-self-dual conformal structures.

The end