Periodic Instantons & Monopoles in Gauge Theory (and Gravity)

- Dirac monopole $\underline{B} = \underline{\nabla} \times \underline{A} = \underline{\nabla} V$, $V = \frac{k}{r}$

  smooth SU(2) monopole (Yang-Mills-Higgs)

  smooth grav monopole (Kaluza-Klein)

$*$ $ds^2 = -dt^2 + V^{-1}(dw - \underline{A} \cdot d\underline{r})^2 + V\, d\underline{r} \cdot d\underline{r}$

- $V = 1 + \frac{k}{r}$ [self-dual Taub-Nut instanton]
- w has period $4\pi k$

- Finite monopole chain $V = 1 + k \sum_{p=-N}^{N} \frac{1}{\sqrt{\rho^2 + (z-p)^2}}$

$(\rho^2 = x^2 + y^2)$

Limit $N \to \infty$: periodic in z

"Kaluza-Klein vortex"

Regularize:

$$V = \alpha + \frac{k}{r} + k \sum_{p \neq 0} \left[\frac{1}{\sqrt{\rho^2 + (z-p)^2}} - \frac{1}{|p|} \right]$$

Periodic in z, smooth except at $\rho = 0$, $z \in \mathbb{Z}$.

* $ds^2 = -dt^2 + V^{-1}(d\varpi - \underline{A}\cdot d\underline{r})^2 + V\, d\underline{r}\cdot d\underline{r}$

- Moduli space for type IIA string near a conifold singularity [Ooguri & Vafa, PRL 1996]
- Kaluza-Klein vortices [Onemli & Tekin, JHEP 2001]

But: curvature singularity ($V=0$) on timelike hypercylinder around z-axis.

*Question: Global structure, horizons? Is there a Kaluza-Klein vortex, in some sense?

- BPS monopoles (static)
- SU(2) gauge field F_{jk}, adjoint Higgs Φ,

$$D_j \Phi = \partial_j \Phi + [A_j, \Phi],$$

$$\boxed{D_j \Phi = \tfrac{1}{2}\,\varepsilon_{jkl} F_{kl}}$$ integrable

- Smooth topological solitons: N-monopoles.
- Nahm data: three $N\times N$ matrices $T_j(s)$ satisfying $T_j' = \tfrac{1}{2}\varepsilon_{jkl}[T_k, T_l]$.

$$\{T_j(s)\} \longleftrightarrow \{A_j(\underline{r}), \Phi(\underline{r})\}$$

- N-monopole string: $T_1(s) = f_1(s)\,\Sigma_1$ etc
  - * $f_j(s)$: specific Jacobi elliptic functions
  - * Σ_j: N-dim irrep of $SU(2)$

modulus $\updownarrow$ separation

[For pictures, see Dunne & Khemani, JPA 2005]

- What about $N \to \infty$ limit?
  - * Calorons in (2+1)-dim YMH theory

  [Phase transition... Dunne et al, JHEP 2001]

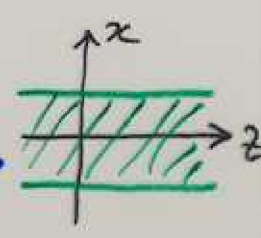
  - * Branes... [Cherkis & Kapustin, CMP 2001]

- Dirac chain has $V \sim \log \rho$ as $\rho \to \infty$, energy density $\sim \frac{1}{\rho^2}$: use these as BCs.

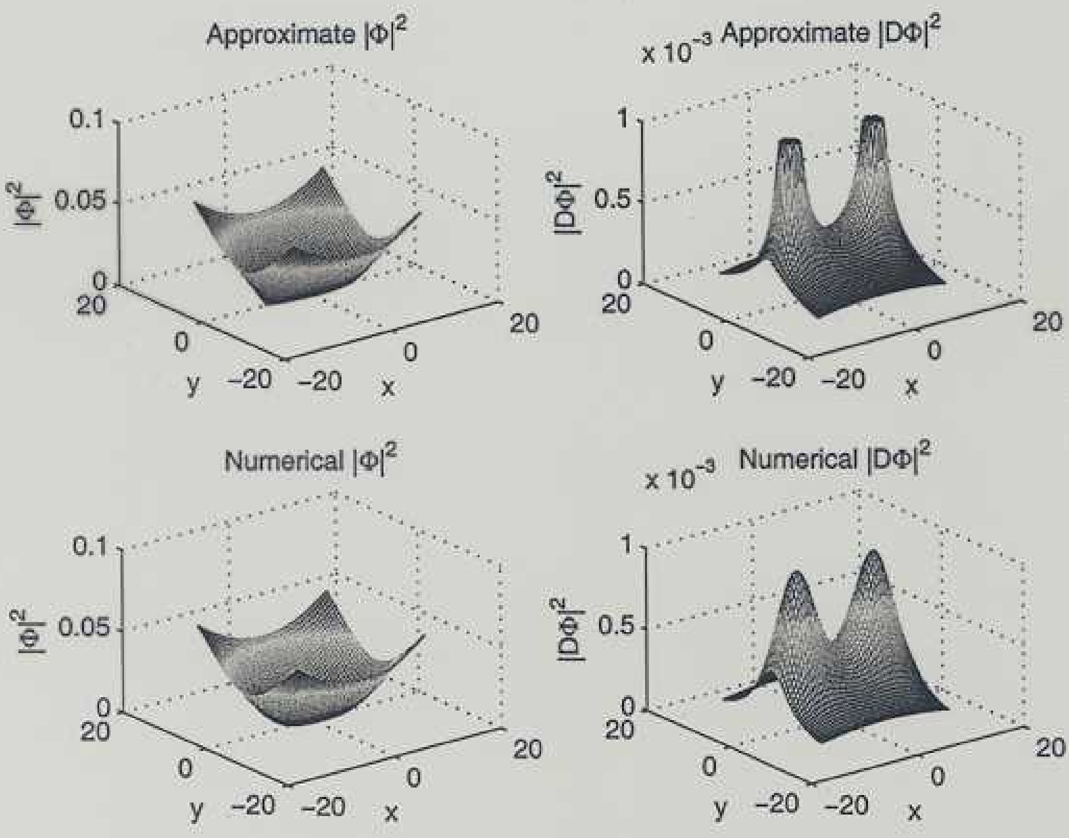
Open question: are there finite-action calorons?

- Nahm data: solution of Hitchin eqns on $\mathbb{R} \times S^1$. Simplest case explicit, one parameter $C \sim$ (monopole size)/(z-period)

- For $0 < C \ll 1$, get  as expected.

- For $C \gg 1$, get a "strip" of width $2C$. 

$c = 8$



Approximate |Φ|^2
x 10^{-3} Approximate |DΦ|^2
Numerical |Φ|^2
x 10^{-3} Numerical |DΦ|^2
|Φ|^2
|DΦ|^2
0.1
0.05
0
1
0.5
20
0
-20
y
x

Nahm Transform

- Self-dual gauge fields in 4D $(++++)$

d open coords ℓ periodic coords Indep of $4-d-\ell$ coords	$\longleftrightarrow$	$4-d-\ell$ open coords ℓ periodic coords Indep of d coords

$(4,0)$ instanton in $\mathbb{R}^4$ $\overset{\checkmark}{\longleftrightarrow}$ $(0,0)$ ADHM data

* $(3,0)$ monopole in $\mathbb{R}^3$ $\overset{\checkmark}{\longleftrightarrow}$ $(1,0)$ Nahm on $\mathbb{R}$

$(2,0)$ Hitchin equs on $\mathbb{R}^2$ $\longleftrightarrow$ $(2,0)$ Hitchin on $\mathbb{R}^2$

$(3,1)$ caloron on $\mathbb{R}^3 \times S^1$ $\overset{\text{almost}}{\longleftrightarrow}$ $(0,1)$ Nahm on S^1

* $(2,1)$ monopole on $\mathbb{R}^2 \times S^1$ $\overset{\checkmark}{\longleftrightarrow}$ $(1,1)$ Hitchin on $\mathbb{R} \times S^1$

$(2,2)$ instanton on $\mathbb{R}^2 \times T^2$ $\overset{\checkmark}{\longleftrightarrow}$ $(0,2)$ Hitchin on T^2

* $(1,2)$ monopole on $\mathbb{R} \times T^2$ $\longleftrightarrow$ $(1,2)$ Monopole on $\mathbb{R} \times T^2$

$(1,3)$ instanton on $\mathbb{R} \times T^3$ $\longleftrightarrow$ $(0,3)$ Monopole on T^3

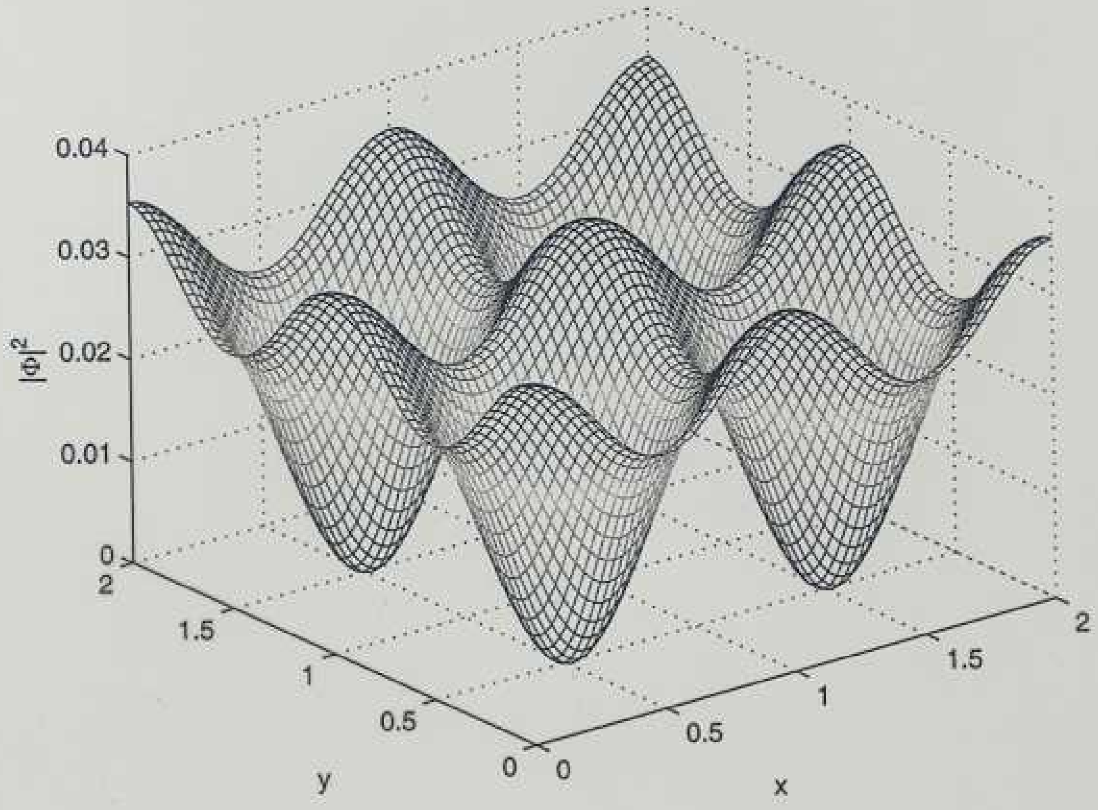
$(0,4)$ instanton on T^4 $\overset{\checkmark}{\longleftrightarrow}$ $(0,4)$ instanton on T^4

$$\boxed{\begin{array}{l} \text{Rank} \longleftrightarrow \text{Charge} \\ \text{Charge} \longleftrightarrow \text{Rank} \end{array}}$$

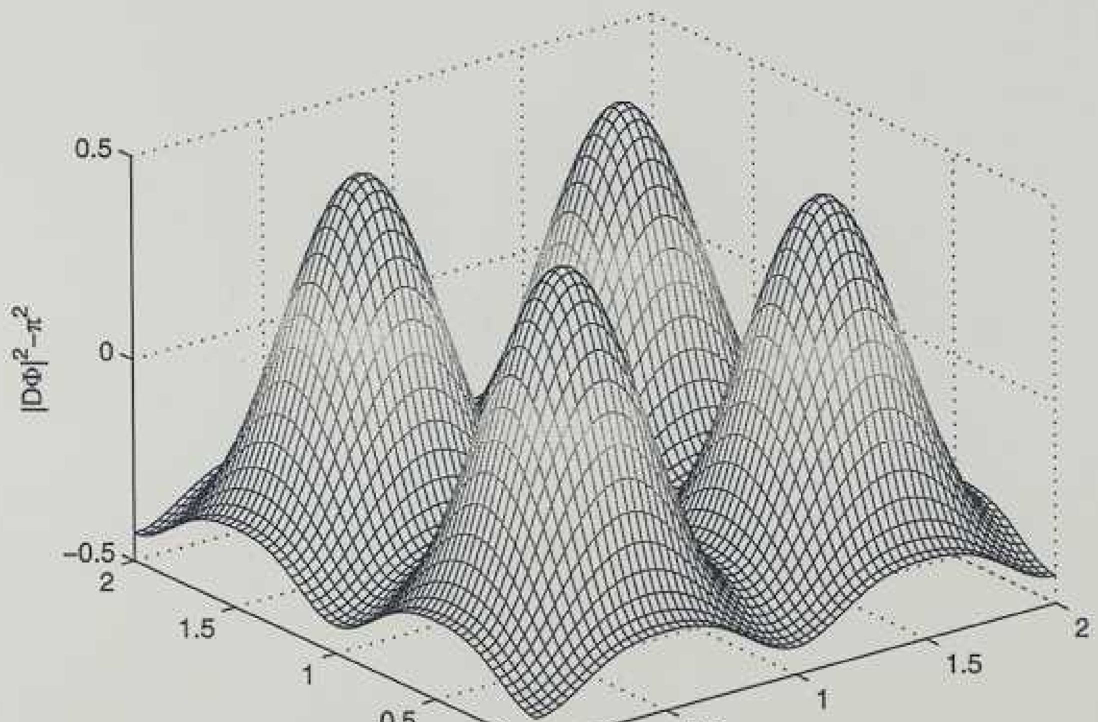
- Monopole sheet: periodic in x & y
- Homogeneous $U(1)$ solution:

$$\underline{A} = \tfrac{1}{2} B(-y, x, 0), \quad \Phi = Bz \quad (B = 2\pi N)$$

- Embed in $SU(2)$ ($\times\, i\sigma^3$) and perturb in σ^1 & σ^2 directions. Normalizable modes: θ-functions localized around $z = 0$.
- Questions: is there an actual $SU(2)$ solution, and what is its Nahm transform?



0.04
0.03
0.02
0.01
0
$|\Phi|^2$
2
1.5
1
0.5
0
y
0
0.5
1
1.5
2
x



0.5
0
−0.5
$|D\Phi|^2-\pi^2$
2
1.5
1
0.5
0
y
0
0.5
1
1.5
2
x