The abundant richness of Einstein-Yang-Mills

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Outline

Black hole uniqueness theorem

Black hole uniqueness theorem
[Mazur/Bunting 1983 and earlier work]

Stationary, axisymmetric, asymptotically flat black hole solutions of the Einstein equations in vacuum or with electromagnetic field:

- Geometry is a member of the Kerr-Newman family;
- Geometry determined by mass, angular momentum and electric charge;
- Geometry determined by these global charges measureable at infinity.

Important results

Hawking's theorem [1973]

- A stationary, asymptotically flat, electrovac black hole must be static or axisymmetric;
- The event horizon of a stationary, axisymmetric, electrovac black hole has spherical topology.

Israel's theorem [1967-8]

Static electrovac black holes are spherically symmetric.

The classic 'no-hair' theorem

Carter/Robinson theorem [1971-4]

There do not exist regular, small perturbations of the stationary and axisymmetric black hole solutions of the Einstein-Maxwell equations preserving the boundary conditions of a regular event horizon and asymptotic flatness for fixed values of the mass, angular momentum and charge.

The only deformations of black holes that do exist are those obtained by a change of mass, angular momentum and charge.

Failure of uniqueness in Einstein-Maxwell theory

Negative cosmological constant

Event horizon topology no longer necessarily spherical - 'topological' black holes.

Higher dimensions

Event horizon topology no longer necessarily spherical - e.g. black rings.

Non-existence of various solitons

No gravitational solitons

The only globally regular, asymptotically flat, static vacuum solution to the Einstein equations with finite total energy is Minkowski space.

No EM solitons

No pure YM solitons

Deser [1976], Coleman [1979]

The model

Einstein-Yang-Mills theory with $\mathfrak{su}(N)$ gauge group

$$S = \frac{1}{2}\int d^4x \sqrt{-g}\left[R - 2\Lambda - \frac{1}{8\pi}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\right].$$

Field equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi T_{\mu\nu}; \\ D_\mu F^\mu_\nu = \nabla_\mu F^\mu_\nu + [A_\mu, F^\mu_\nu] &= 0. \end{aligned}$$

Stress-energy tensor

$$T_{\mu\nu} = 2\,\mathrm{Tr}\, F_{\mu\lambda}F^\lambda_\nu - \frac{1}{2}g_{\mu\nu}\mathrm{Tr}\, F_{\lambda\sigma}F^{\lambda\sigma}.$$

Spherically symmetric gauge field ansatz

Kunzle [1994] ansatz for $\mathfrak{su}(N)$ gauge potential

$$\mathcal{A}\,dt + \mathcal{B}\,dr + \frac{1}{2}\left(C - C^H\right)d\theta - \frac{i}{2}\left[\left(C + C^H\right)\sin\theta + D\cos\theta\right]d\phi.$$

- $\mathcal{A}$ and $\mathcal{B}$ are traceless $N \times N$ matrices.
- C is upper triangular:

$$C_{j,j+1} = \omega_j \exp(i\gamma_j).$$

- $D = \text{diag}\,\{N-1, N-3, \ldots, -N+3, -N+1\}$.

Equilibrium, static, purely magnetic gauge field

- $\mathcal{A} = \mathcal{B} = 0$;
- $\gamma_j = 0$; $\omega_j = \omega_j(r)$ for $j = 1, \ldots, N-1$.

Ansatz for $\mathfrak{su}(2)$

Slightly different ansatz in this case

$$a\tau_r\,dt + b\tau_r\,dr + \left[\nu\tau_\theta - (1+\omega)\,\tau_\phi\right]d\theta + \left[(1+\omega)\,\tau_\theta + \nu\tau_\phi\right]d\phi.$$

Equilibrium, static, purely magnetic gauge field

Ershov and Galtsov [1990]:

Non-trivial, static, spherically symmetric, asymptotically flat solutions must be purely magnetic:

- $a = b = \nu = 0$;
- $\omega = \omega(r)$.

Static, spherically symmetric, field equations

Metric:

$$ds^2 = -\mu(r)\exp(2\delta(r))\,dt^2 + [\mu(r)]^{-1}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right);$$

$$\mu(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}.$$

Field equations for $\mathfrak{su}(2)$

$$\begin{aligned} m' &= \mu\omega'^2 + \frac{\left(\omega^2-1\right)^2}{2r^2}; \\ \delta' &= \frac{2\omega'^2}{r}; \\ 0 &= r^2\mu\omega'' + \left[2m - \frac{2\Lambda r^3}{3} - \frac{\left(\omega^2-1\right)^2}{r}\right]\omega' + \left(1-\omega^2\right)\omega. \end{aligned}$$

Boundary conditions

Regular solitons

Regular origin

$$\omega(r) = 1 - br^2 + O(r^4);$$
$$m(r) = 2b^2 r^3 + O(r^5).$$

Black holes

Regular horizon at $r = r_h$:

$$\mu(r_h) = 0; \qquad \mu'(r_h) > 0.$$

Cosmological horizon at $r = r_c$ if $\Lambda > 0$.

Boundary conditions at infinity

Metric

Asymptotically flat/de Sitter/anti-de Sitter space:

$$m(r) \rightarrow M + O(r^{-1}); \qquad \omega(r) \rightarrow \omega_\infty + O(r^{-1}).$$

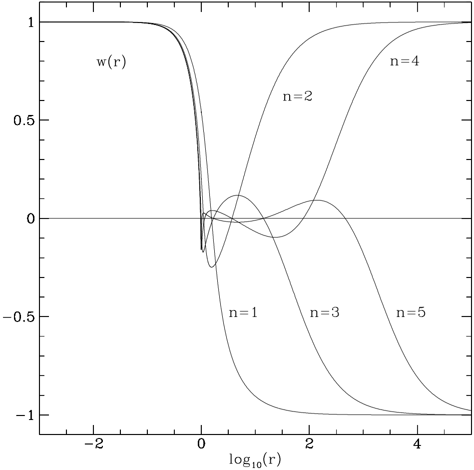
Asymptotically flat space

$$\omega_\infty = \pm 1 \text{ if } \Lambda = 0$$

No global magnetic charge.

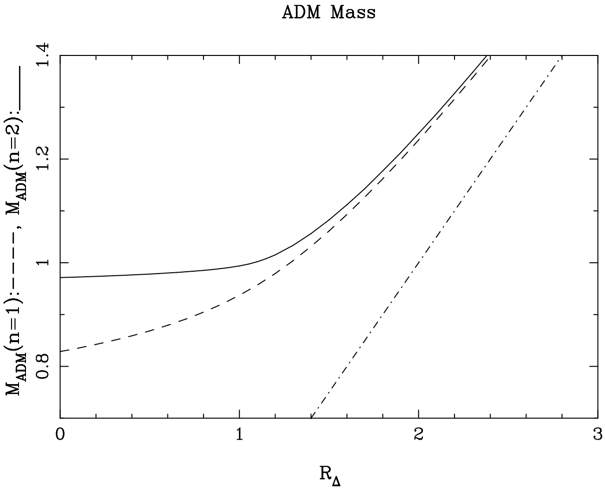
The Bartnik-McKinnon solitons [1988]

Gauge field functions [Volkov and Galtsov review 1999]:



Coloured black holes [Bizon 1990]

Phase space of solutions [Ashtekar, Corichi and Sudarsky 2001]:



General properties of the solutions

- Found numerically by the shooting method;
- Analytic proof of existence of solutions of the differential equation;
- Gauge function ω has at least one zero;
- Discrete families of solutions, indexed by the event horizon radius r_h and n, the number of zeros (nodes) of ω;
- Solutions can be generalized to $\mathfrak{su}(N)$;
- General proof that all these solutions are unstable [Brodbeck and Straumann 1994].

Israel's theorem

Theorem for Einstein-Maxwell black holes

Static electrovac black holes are spherically symmetric.

Counterexamples for EYM

Static, axisymmetric solitons and black holes have been found [Kleihaus and Kunz 1997].

Metric ansatz in isotropic co-ordinates:

$$ds^2 = -f\,dt^2 + \frac{m}{f}dr^2 + \frac{mr^2}{f}d\theta^2 + \frac{lr^2\sin^2\theta}{f}d\phi^2.$$

f, m, l, depend on r and θ.

General static $\mathfrak{su}(2)$ gauge potential ansatz

Generalized ansatz for the gauge potential

$$\frac{1}{2r}\left\{\tau_\phi^k\left[H_1\,dr+(1-H_2)\,r\,d\theta\right]\right.$$
$$\left.-k\left[\tau_r^k H_3+\tau_\theta^k(1-H_4)\right] r\sin\theta\,d\phi\right\}$$

k is a winding number (round the axis of symmetry),

$$\tau_r^k=\vec{\tau}.\left(\sin\theta\cos k\phi,\sin\theta\sin k\phi,\cos\theta\right),$$

where

$$\vec{\tau}=\left(\tau_x,\tau_y,\tau_z\right).$$

$k=1$ corresponds to spherical symmetry.

Rotating solutions

Rotating black holes

Kleihaus and Kunz [2001] also found rotating generalizations of the Kerr-Newman black holes:

- Indexed by winding number k and node number n;
- Carry no magnetic charge;
- All have non-zero electric charge Q.

Rotating solitons?

- Rotating solitons in EYMH found by Paturyan, Radu and Tchrakian [2005].
- Rotating solitons in EYM predicted perturbatively [Brodbeck *et al* 1997] but have not been found numerically.
- If they exist, they must be electrically charged and $J = 4\pi kQ$.

Including a cosmological constant

Positive cosmological constant

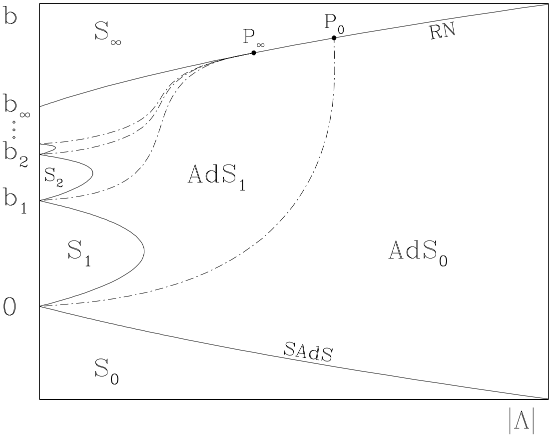
- Discrete families of solutions;
- Gauge field function ω has at least one zero;
- Solutions are unstable.
- Complete classification of solutions [Breitenlohner, Forgacs and Maison 2004].

Negative cosmological constant

- Continous families of solutions;
- Gauge field function ω can have no zeros;
- Solutions can carry electric charge;
- At least some solutions are stable.
  [EW 1999, Bjoraker and Hosotani 2000]

Phase space of solitons

[Breitenlohner, Maison and Lavrelashvili 2004]



Phase space of black holes - monotonic behaviour

$\Lambda = -0.01, \ r_h = 1$

$\omega(r_h)$	n
0.01	3
$\vdots$	$\vdots$
0.052155	3
0.052156	2
$\vdots$	$\vdots$
0.54526	2
0.545427	1
$\vdots$	$\vdots$
0.76819	1

[Jason Baxter, Marc Helbling and EW, in progress]

Phase space of black holes - non-monotonic behaviour

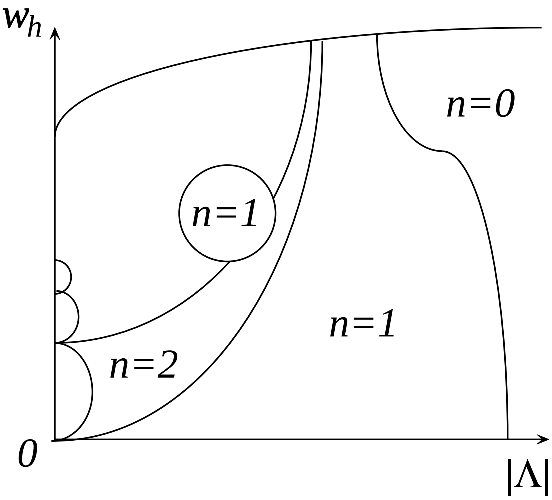
$\Lambda = -0.001, \; r_h = 1$

$\omega(r_h)$	n
0.060785	1
0.060790	1
0.060793	1
0.060794	1
0.060795	3
0.060800	3
0.060803	3
0.060804	1
0.060805	0
0.060806	1

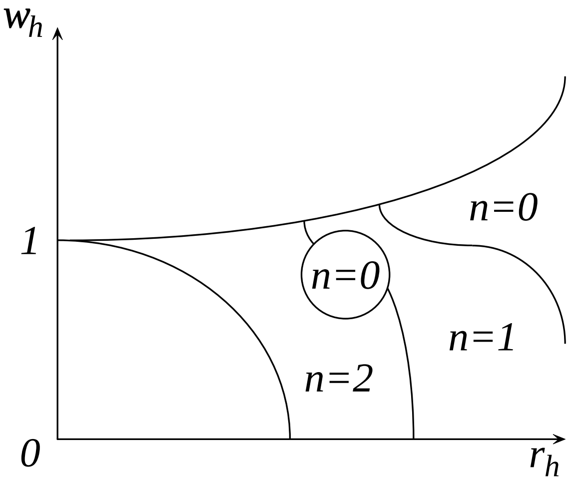
$\omega(r_h)$	n
0.060807	1
0.060808	1
0.060809	3
0.060810	2
0.060813	2
0.060814	1
0.060815	3
0.060816	2
0.060817	1
0.060820	1

[Jason Baxter, Marc Helbling and EW, in progress]

Sketch of possible black hole phase space for fixed r_h



Sketch of possible black hole phase space for fixed Λ



Stability analysis

$\mathfrak{su}(2)$ gauge field ansatz for spherically symmetric perturbations

$$a(t,r)\tau_r\, dt + b(t,r)\tau_r\, dr + [\nu(t,r)\tau_\theta - (1+\omega(t,r))\,\tau_\phi]\, d\theta + [(1+\omega(t,r))\,\tau_\theta + \nu(t,r)\tau_\phi]\, d\phi.$$

Gauge choice $a(t,r) \equiv 0$

Perturbed metric

$$ds^2 = -\mu(t,r)\exp\left[2\delta(t,r)\right]\, dt^2 + [\mu(t,r)]^{-1}\, dr^2 + r^2 d\Omega^2;$$

$$\mu(t,r) = 1 - \frac{2m(t,r)}{r} - \frac{\Lambda r^2}{3}.$$

Perturbation equation sectors

Perturbation equations then decouple into two sectors:

Sphaleronic sector

- b, ν - does not depend on detailed structure of solutions.
- If ω has no zeros, then there are no instabilities in this sector.

Gravitational sector

- ω, m, δ - does depend on detailed structure of solutions.
- If ω has no zeros, there are some solutions with no instabilities in this sector.

Stability for at least some solutions with ω nodeless extended to non-spherically symmetric perturbations in both sectors.
[Olivier Sarbach and EW 2001/02]

Topological EYM black holes in AdS

[van der Bij and Radu 2002]

Alternative metric ansatz

$$ds^2 = -\mu(r)\exp(2\delta(r))\,dt^2 + [\mu(r)]^{-1}\,dr^2 + r^2\left(d\theta^2 + \mathcal{F}^2(\theta)\,d\phi^2\right);$$

$$\mu(r) = K - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3};$$

$$\mathcal{F}(\theta) = \begin{cases} \sin\theta & K = 1 \\ 1 & K = 0 \\ \sinh\theta & K = -1. \end{cases}$$

- Gauge field ω cannot have any zeros.
- All $K = 0$ solutions are stable.
- $K = -1$ solutions are stable if $|\omega| > 1$ at $r \to \infty$.

Non-spherically symmetric solutions in AdS

Modified metric

$$ds^2 = -f\left(1-\frac{\Lambda r^2}{3}\right)dt^2 + \frac{m}{f}\left(1-\frac{\Lambda r^2}{3}\right)^{-1}dr^2 + \frac{mr^2}{f}d\theta^2 + \frac{lr^2\sin^2\theta}{f}\left(d\phi + \frac{\Omega}{r}dt\right)^2$$

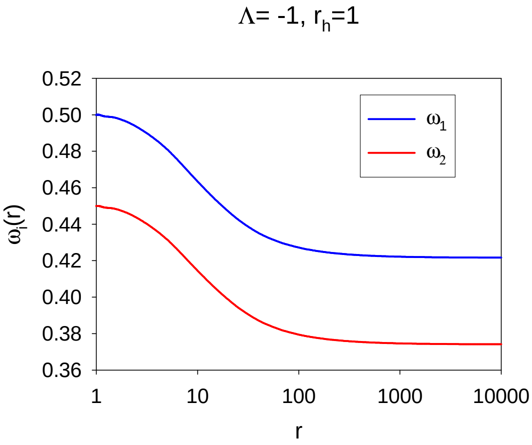
- Static solutions ($\Omega \equiv 0$):
  - Solitons [Radu 2001];
  - Black holes [Radu and EW 2004].
- Rotating solutions ($\Omega \neq 0$):
  - Rotating solitons [Radu 2002];
  - Rotating black holes not yet found numerically, but should exist!

$\mathfrak{su}(N)$ EYM solutions in AdS

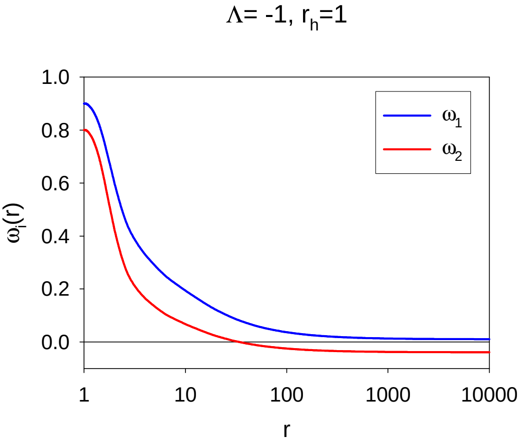
[Jason Baxter and EW, in progress]

- Complicated phase space - only purely magnetic solutions studied;
- $N-1$ gauge field degrees of freedom ω_i;
- Solutions in which all ω_i have no nodes;
- Solutions in which ω_i have different numbers of nodes;
- Stability analysis in progress.

$\mathfrak{su}(3)$ solution with both ω_i having no nodes



$\mathfrak{su}(3)$ solution with the ω_i having different numbers of nodes



Isolated horizons

Definition

A weakly isolated horizon is an expansion-free, null, three-dimensional submanifold Δ of a four-dimensional spacetime, equipped with an equivalence class of future-directed null normals which Lie-drag (certain components of) the intrinsic connection on Δ.

- Generalizes the notion of Killing horizon;
- Allows one to define horizon quantities (such as horizon mass M_Δ) entirely locally to Δ.

For the moment, work in asymptotically flat space only.

First law

Defining horizon energy

Horizon energy E_Δ defined via the first law:

$$\delta E_\Delta = \kappa_\Delta \delta a_\Delta + \Phi_\Delta \delta Q_\Delta$$

where

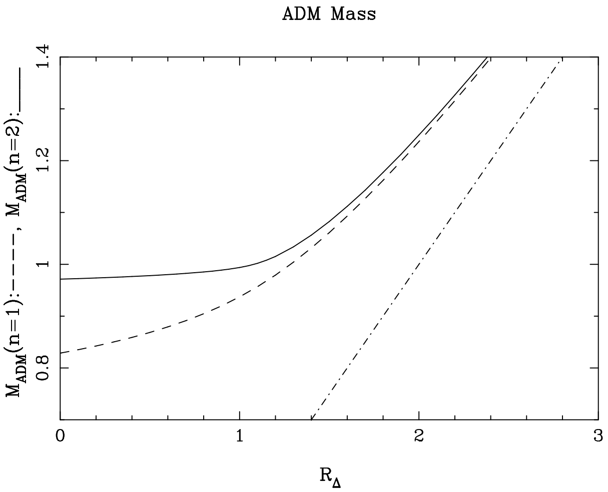
$$a_\Delta = 4\pi R_\Delta^2 = \text{horizon area}, \qquad Q_\Delta = \text{electric charge}$$

κ_Δ depends only on Q_Δ and a_Δ

For EYM black holes, E_Δ cannot be the ADM mass (because of the soliton solutions!).

Phase space of $\mathfrak{su}(2)$ EYM solutions

ADM masses of solutions:



[Ashtekar, Corichi and Sudarsky 2001]

Defining horizon mass

[Ashtekar, Corichi and Sudarsky 2001]

- Consider a one-parameter family of solutions with node number n.
- Surface gravity κ_Δ depends only on horizon area a_Δ along this branch of solutions.
- Define horizon mass:

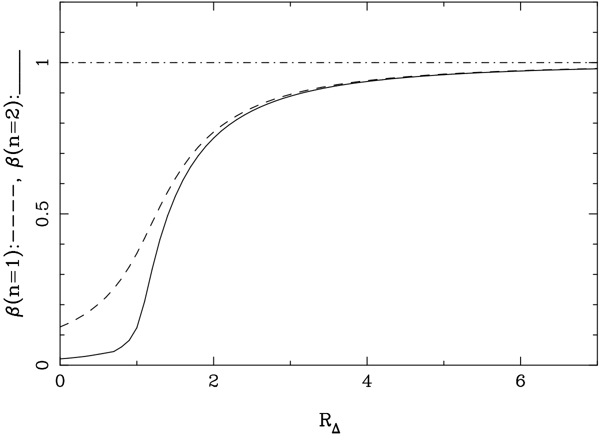
$$M_\Delta = \int_{r=0}^{R_\Delta} \beta(r)\, dr.$$

where

$$\beta\left(R_\Delta\right) = 8\pi R_\Delta \kappa_\Delta \left(R_\Delta\right).$$

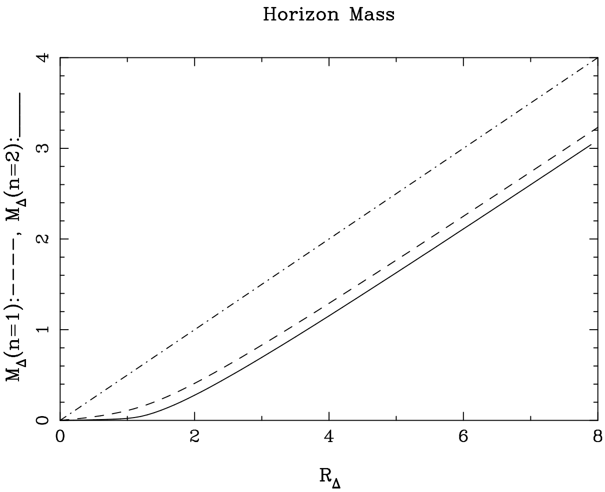
Plot of $\beta(R_\Delta)$

$\beta(R_\Delta)$ is a monotonically increasing function of R_Δ:



[Ashtekar, Corichi and Sudarsky 2001]

Horizon mass



[Ashtekar, Corichi and Sudarsky 2001]

Consequences of the definition of horizon mass

- Hamiltonian is constant along each branch of solutions, so

$$M_{sol} = M_{ADM} - M_{\Delta}$$

  Tested numerically by Corichi and Sudarsky [2000].

- Define binding energy $E_{binding} < 0$:

$$M_{ADM}\left(R_{\Delta}\right) = M_{sol} + M_{\Delta}^{0}\left(R_{\Delta}\right) + E_{binding},$$

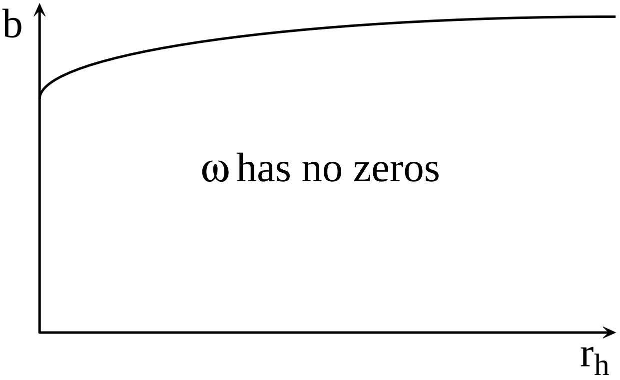
  $M_{\Delta}^{0}\left(R_{\Delta}\right)$ is the horizon mass of the Schwarzschild black hole of radius R_{Δ}.

- Decay to a Schwarzschild black hole of radius $R_{\Delta}^{final} > R_{\Delta}^{initial}$ is energetically preferred:

$$M_{\Delta}\left(R_{\Delta}^{initial}\right) + M_{sol} > M_{\Delta}^{0}\left(R_{\Delta}^{final}\right) > M_{\Delta}^{0}\left(R_{\Delta}^{initial}\right).$$

Isolated horizons for EYM in AdS?

Simplified phase space in large $|\Lambda|$ limit



A possible strategy for applying isolated horizons to EYM black holes in AdS

Choose foliation such that first law holds along each leaf:

$$\delta E_{\Delta} = \kappa_{\Delta} \delta a_{\Delta}$$

Use this to:

- Define horizon mass;
- Relate the masses of the solitons and black holes;
- Find ‘binding’ energy;
- Explain stability/instability.

[Olivier Sarbach and EW, in progress]

Summary

- EYM provided the first example of gravitational solitons and 'hairy' black holes;
- Almost every part of the uniqueness theorem for Einstein-Maxwell black holes has a counter-example in EYM;
- Solution space even more complicated in AdS;
- Isolated horizons approach can aid understanding of these solutions;
- EYM may have more surprises in store!