Null data expansions of asymptotically flat, static, vacuum solutions.

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Abstract

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Null data expansions of asymptotically flat, static, vacuum solutions.

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AIM

Consider time reflection symmetric, asymptotically flat, vacuum initial data. Try to understand the relation

asymptoptic smoothness of solutions at null infinity

$\leftrightarrow$

asymptotic staticity of initial data

For this purpose it is necessesary to get control on the asymptotically flat, static, vacuum solutions.

O. Reula (1989), P. Miao (2003):
Characterize asymptotically flat, static, vacuum solutions in terms of outer boundary value problems.

Not useful for my purpose. Need characterization in terms of asymptotic fields.

Try to control just the 'germ at space-like infinity' of the solutions.

R. Geroch (1970):
Introduces multipole moments for static vacuum fields.

Convergence of corresponding expansions ?

TIME SYMMETRIC ASYMPT. FLAT VACUUM DATA

Time symmetric initial data

$(\tilde{S}, \tilde{h}_{ab})$ **Riemannian manifold, 3-dimensional,** C^∞**, n.n. complete, satisfying vacuum constraint:** $R[\tilde{h}] = 0$

Conformally smooth at space-like infinity **(→ asymptotically flat)**

$\exists\ C^\infty$ **manifold** $S \sim \tilde{S} \cup \{i\}$ **and** $\Omega \in C^\infty(\tilde{S}) \cap C^2(S)$, $\quad \Omega > 0$ **on** $\tilde{S}$**, so that**

– $h_{ab} = \Omega^2\, \tilde{h}_{ab}$ **extends to a smooth metric on** S**,**

– $\Omega(i) = 0$**,** $D_a\,\Omega(i) = 0$**,** $D_a D_b\,\Omega(i) = -2\,h_{ab} \qquad (sign\, h = (-,-,-))$**.**

Vacuum constraints near i **in terms of** h_{ab}**,** Ω**:**

$$(\Delta_h - \frac{1}{8}\,R[h])\,\frac{1}{\sqrt{\Omega}} = 4\,\pi\,\delta_i.$$

Imply in h**-normal coordinates** x^α **with** $x^\alpha(i) = 0$**,** $h_{\alpha\beta}(i) = -\delta_{\alpha\beta}$

$$\frac{1}{\sqrt{\Omega}} = \frac{U}{|x|} + W \quad \textbf{near} \quad i,$$

$$U, W \in C^\infty, \quad (\Delta_h - \frac{1}{8}\,R[h])\,W = 0, \quad U(i) = 1, \quad W(i) = \frac{m}{2} \overset{!}{>} 0.$$

The conformal freedom

$$h_{ab} \to h'_{ab} = \theta^4\,h_{ab}, \quad \Omega \to \Omega' = \theta\,\Omega, \qquad \theta \in C^\infty(S), \quad \theta > 0, \quad \theta(i) = 1,$$

used with $\theta = \frac{2}{m}\,W$ **it gives near** i **a** *'preferred gauge'* **in which**

$$\frac{1}{\sqrt{\Omega'}} = \frac{U'}{|x'|} + \frac{m}{2}, \qquad R[h'] = 0.$$

STATIC ASYMPTOTICALLY FLAT VACUUM DATA

Static metric in adapted coordinates

$$\tilde{g} = v^2\, d\,t^2 + \tilde{h}, \quad v = v(x^a), \quad \tilde{h} = \tilde{h}_{ac}(x^e)\, dx^a\, dx^c.$$

Vacuum field equations reduce *static vacuum field equations*

$$R_{ab}[\tilde{h}] = \frac{1}{v}\, \tilde{D}_a\, \tilde{D}_b\, v, \qquad \Delta_{\tilde{h}}\, v = 0,$$

a quasilinear, gauge-elliptic system of PDE's.

Assume: the solutions are asymptotically flat,
$v \to 1$ **a space-like infinity.**

H. Müller zum Hagen (1970): **the solutions are real analytic**

R. Geroch (1970) suggests, R. Beig and W. Simon (1980) show (cf. D. Kennefick, N. O'Murchadha (1995)):

The solutions with ADM mass $m > 0$ are conformally smooth, in fact real analytic, at space-like infinity in suitable coordinates and conformal scalings.

Turns out: in h-normal coordinates x^α as above

$$h_{ab}, \quad \zeta = \left(\frac{2}{m}\,\frac{1-v}{1+v}\right)^2 \quad \text{are} \quad C^\omega \quad \text{at} \quad i, \quad \zeta = O(|x|^2),$$

in the *preferred gauge*, **which is achieved with**

$$\Omega = \frac{(1-v)^2}{m^2} \qquad \text{so that} \qquad \frac{1}{\sqrt{\Omega}} = \frac{1}{\sqrt{\zeta}} + \frac{m}{2}, \qquad R[h] = 0.$$

We shall assume in the following this gauge.

THE CONFORMAL STATIC VACUUM EQUATIONS I

With

$$h_{ab}, \quad \zeta, \quad \mu = \frac{m^2}{4}, \quad s = \frac{1}{3}\Delta_h \zeta, \quad s_{ab} = R_{ab}[h] \quad (\textbf{trace free}),$$

we can rewrite the static vacuum equations and get *conformal static equations*

$$D_a D_b \zeta - s\, h_{ab} + \zeta\,(1 - \mu\,\zeta)\, s_{ab} = 0,$$

$$D_a s + (1 - \mu\,\zeta)\, s_{ab}\, D^b \zeta = 0,$$

$$(1 - \mu\,\zeta)\, D_{[c} s_{a]b} - \mu\,(2\, D_{[c}\zeta\, s_{a]b} + D^d \zeta\, s_{d[c}\, h_{a]b}) = 0.$$

The conditions on Ω translate into

$$\zeta(i) = 0, \quad D_a\zeta(i) = 0, \quad s(i) = -2.$$

For our application it will be convenient to rewrite these equations in the *space spinor frame formalism* **as equations for the unknown**

$$u = (e^{\alpha}{}_{AB},\ \Gamma_{ABCD},\ \zeta,\ \zeta_{AB},\ s,\ s_{ABCD}),$$

with

- $e_{AB} = e_{(AB)} = e^{\alpha}{}_{AB}\partial_{x^\alpha}$ **a frame field satisfying**

$$h(e_{AB}, e_{CD}) = h_{ABCD} \equiv -\epsilon_{A(C}\epsilon_{D)B},$$

  so that $e_{01} \perp e_{00}, e_{11}$ **and** $h(e_{00}, e_{00}) = h(e_{11}, e_{11}) = 0$**,**
- $\Gamma_{ABCD} = \Gamma_{(AB)(CD)}$ **the connection coefficients,**
- $\zeta_{AB} \equiv e_{AB}(\zeta)$**,**
- $s_{ABCD} = s_{(ABCD)}$ **the (trace free) Ricci tensor.**

THE CONFORMAL STATIC VACUUM EQUATIONS II

The equations are then given by

the first structural equation

$$[e_{AB}, e_{CD}] = 2\,\Gamma_{AB}{}^{H}{}_{(C}\,e_{D)H} - 2\,\Gamma_{CD}{}^{H}{}_{(A}\,e_{B)H},$$

the second structural equation

$$\begin{gathered} e_{CD}(\Gamma_{EFAB}) - e_{EF}(\Gamma_{CDAB}) + \Gamma_{EF}{}^{K}{}_{C}\,\Gamma_{KDAB} + \Gamma_{EF}{}^{K}{}_{D}\,\Gamma_{CKAB} \\ -\Gamma_{CD}{}^{K}{}_{E}\,\Gamma_{KFAB} - \Gamma_{CD}{}^{K}{}_{F}\,\Gamma_{EKAB} + \Gamma_{EF}{}^{K}{}_{B}\,\Gamma_{CDAK} - \Gamma_{CD}{}^{K}{}_{B}\,\Gamma_{EFAK} \\ = \frac{1}{2}\left\{s_{ABCE}\,\epsilon_{DF} + s_{ABDF}\,\epsilon_{CE}\right\}, \end{gathered}$$

which replace the second order operator $R_{ab} = R_{ab}[h]$

and, as before,

$$D_{AB}\,\zeta_{CD} - s\,h_{ABCD} + \zeta\,(1 - \mu\,\zeta)\,s_{ABCD} = 0,$$

$$D_{AB}\,s + (1 - \mu\,\zeta)\,s_{ABCD}\,\zeta^{CD} = 0,$$

$$D_{A}{}^{E}\,s_{BCDE} - \frac{2\,\mu}{1 - \mu\,\zeta}\,\zeta_{(A}{}^{E}\,s_{BCD)E} = 0,$$

the fields ζ, ζ_{AB} and s must satisfy

$$\zeta(i) = 0, \;\; \zeta_{AB}(i) = D_{AB}\zeta(i) = 0, \;\; s(i) = -2.$$

– **The systems are quasilinear, overdetermined gauge elliptic**

– **The ellipticity implies the analyticity**

– **Analogue of the 4-dimensional conformal field equations**

STATIC MULTIPOLES

R. Geroch (1970) **considers near** i **the sequence of tensor fields**

$$P = \Omega^{-1/2}\,(1 - v), \quad P_a = D_a P \quad \textbf{and, recursively,}$$

$$P_{a_1 \ldots a_{p+1}} = \mathcal{C}(D_{a_{p+1}} P_{a_1 \ldots a_p} - \frac{p\,(2\,p - 1)}{2}\, R_{a_1 a_2}\, P_{a_3 \ldots a_{p+1}}).$$

$$\mathcal{C} \; = \; \text{'take symmetric trace free part of'}$$

The Ricci term ensures the 'appropriate' behaviour under rescalings.

The multipole moments are then defined as

$$\nu = P(i), \qquad \nu_{a_1 \ldots a_p} = P_{a_1 \ldots a_p}(i), \quad p = 1, 2, \ldots$$

In our gauge and formalism they take the form

$$\nu = m, \quad \nu_{AB} = 0, \quad \textbf{and, for} \quad p = 0, 1, 2, \ldots,$$

$$\nu_{A_1 B_1 \ldots A_{p+2} B_{p+2}} = -\frac{m}{2}\, D_{(A_p B_p} \ldots D_{A_1 B_1} s_{A_{p+1} B_{p+1} A_{p+2} B_{p+2})}(i) + l.o.,$$

l.o. $=$ *terms in* s_{ABCD} *and its derivatives of lower order*.

– **Not a standard PDE problem for a quasilinear system to determine solutions from data given at a point,**

– **the** *l.o.* **terms become increasingly non-linear with** p**.**

The map

$$\mathcal{D}_{mp} \equiv \{\nu_{A_1 B_1 \ldots A_{p+2} B_{p+2}} |\; p \geq 0\}$$

$$\xrightarrow{1:1}$$

$$\mathcal{D}_n \equiv \{D_{(A_p B_p} \ldots D_{A_1 B_1}\, s_{ABCD)}(i) |\; p \geq 0\}$$

suggests:

Study in the preferred gauge the expansions of the field in terms of **m** *and the data* $\mathcal{D}_n$, **if possible.**

THE EXACT SET OF FIELDS ARGUMENT

Assume the data $\mathcal{D}_n$ to be given in a frame field $e*_{AB}$ which is parallely propagated along the geodesics through i, write

$$\mathcal{D}_n^* \equiv \{D^*_{(A_pB_p} \ldots D^*_{A_1B_1} s^*_{ABCD)}(i) |\ p \geq 0\},$$

assume the *reality conditions* **to hold at all orders.**

The '*exact sets of fields argument' of Penrose* **allows us, by using the the conformal static field equations, to express all contractions in the decompositions**

$$D^*_{A_pB_p} \ldots s^*_{ABCD}(i) = D^*_{(A_1B_1} \ldots s^*_{ABCD)}(i)\ +\ contractions \times \epsilon' s.$$

in terms of $\mathcal{D}_n^*$. This yields a unique formal expansion of the fields in terms of the data $\mathcal{D}_n^*$.

R. Beig and W. Simon (1980) **obtain the same result by using real tensor arguments.**

Lemma 1:
The data $\mathcal{D}_n^*$ determine a unique real formal expansion type solution to the conformal static field equations.

It is the purpose of this talk to analyse the convergence problem under general assumptions. (*For static axisymmetric space-times, cf. H. Weyl (1917) … T. Bäckdahl, M. Herberthson (2005)*)

– **Formalize the exact sets of fields argument ?**

– **Derive estimates on the resulting coefficients ?**

– **Impose estimates on the 'free' coefficients in $\mathcal{D}_n^*$?**

A GEOMETRIC PROBLEM

Need to calculate coefficients in a setting which allows us to derive estimates $\rightarrow$

Express the fields in h-normal coordinates x^α *and extend them by analyticity near* i *into the complex domain*.

Differential geometric formulas and concepts extend.

At each point we get a null cone of complex tangent vectors which at points of the real slice are transverse to it.

$\mathcal{N}_i = \{\delta_{\alpha\beta}\, x^\alpha\, x^\beta = 0\}$ **describes near** i **the cone generated by the complex null geodesics through** i. *Observe connectivities*.

Let $x(u)$ **be a null geodesic through** i **with tangent vector field** $\iota^A\,\iota^B$. **The analytic function**

$$s_0(u) = \iota^A\,\iota^B\,\iota^C\,\iota^D\, s_{ABCD}(x(u)),$$

then has Taylor expansion

$$s_0 = \sum_{p=0}^{\infty} \frac{1}{p!}\, u^p\, \frac{d^p}{du^p}\, s_0(0),$$

with

$$\frac{d^p}{du^p}\, s_0(0) = (\iota^E\,\iota^F\, D_{EF})^p\, (s_{ABCD}\,\iota^A\,\iota^B\,\iota^C\,\iota^D)(0)$$
$$= \iota^{A_1}\,\iota^{B_1} \ldots \iota^C\,\iota^D\, D_{(A_1B_1} \ldots D_{A_pB_p} s_{ABCD)}(i).$$

Suggests:

Analyse the 'characteristic initial value problem' for the conformal static field equations with arbitrarily prescribed **null data** $s_0 \sim \mathcal{D}^*_n$ *on the complex cone* $\mathcal{N}_i$.

- **Somewhat unusual problem for a quasilinear elliptic system,**
- **the main problem arises from the singularity of our initial 'hypersurface'** $\mathcal{N}_i$ **at its vertex** i.

SOME GEOMETRIC STRUCTURES

We choose a gauge based on a nested family of cones which share a common generator γ.

We will have to adapt the coordinates and a frame field to this family.

To organize and facilitate the discussion of the resulting singularities, we make use of the following geometric notions:

– **bundle of normalized spin frames** $SU(S) \xrightarrow{\pi} S$, **group** $SU(2)$,
resp. its holomorphic extension $SL(S_c) \xrightarrow{\pi} S_c$, **group** $SL(2, C)$,

– **the** *vertical* **vector fields** Z_α: **if** $\alpha \in sl(2, C)$ **and** $e \in SL(S_c)$
then $Z_\alpha(e) = \partial_v (e \cdot exp(v\,\alpha))|_{v=0}$,

– **the** $sl(2, C)$**-valued connection form** $\omega^A{}_B$ **on** $SL(S_c)$

– **the** *horizontal* **vector fields** H_{AB}, **projection** $T_e(\pi)H_{AB} = \iota_{(A}\,\iota_{B)}$,
their integral curves project onto geodesics and define parallely transported frame fields along these geodesics

– **a holomorphic spinor field on** S_c **defines a holomorphic spinor-valued function** $\psi_{A_1 \ldots A_j}$ **on** $SL(S_c,)$,
we write $\psi_k = \psi_{(A_1 \ldots A_j)_k}$, $k = 0, \ldots, j$,

$$(\ldots\ldots)_k \;=\; \text{'set } k \text{ indices equal to } 1 \text{ the rest equal to } 0\text{'}$$

GAUGE CONDITIONS

- **Choose a spin frame $e^* = (\iota^*_A)_{A=0,1} \in SU(S)$ over i with $\iota^*_{(A}\,\iota^*_{B)} = e^*_{AB}$.**

  The curve

  $$C \ni v \to (\iota_A(v)) = (\iota^*_B\, s^B{}_A(v)) \in SL(S_c),$$

  with

  $$s(v) = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = exp(v\,\alpha), \qquad \alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in sl(2, C).$$

  defines a *vertical*, 1-dimensional, holomorphic submanifold, I through e^*. The corresponding family of frames

  $$e_{AB}(v) = e^*_{CD}\, s^C{}_A(v)\, s^D{}_B(v),$$

  at i is given explicitly by

  $$e_{00}(v) = e^*_{00} + 2\,v\,e^*_{01} + v^2\,e^*_{11}, \qquad e_{01}(v) = e^*_{01} + v\,e^*_{11}, \qquad e_{11}(v) = e^*_{11}.$$

- **Push I forward *near* I with the flow of H_{11} to obtain a holomorphic 2-manifold U_0 containing I.**

  Denote by w the parameter on the integral curves of H_{11} which vanishes on I.

  Assume $v = const.$ on these integral curves.

- **Push U_0 forward *near* U_0 with the flow of H_{00} to obtain a holomorphic 3-manifold $\hat{S}$ containing U_0.**

  Denote by u the parameter on the integral curves of H_{00} which vanishes on U_0.

  Assume $v = const., w = const.$ on these integral curves.

PROPERTIES OF THE GAUGE

– $z^1 = u$, $z^2 = v$, $z^3 = w$ **define holomorphic coordinates on** $\hat{S}$,

– **the curves** $w \to (u = 0, v = v_0, w)$ **project onto the null geodesic** $\gamma(w)$ **with tangent vector** e^*_{11} **at** $i = \gamma(0)$,

– **the set** $W_0 = \{w = 0\}$ **projects onto** $\mathcal{N}_i \setminus \gamma$, ...

– **the projection** π **induces on** $\hat{S} \setminus U_0$ **a biholomorphic diffeomorphism (***close to I***)**,

– **denote by** e_{AB} **the unique frame field on** $\hat{S} \setminus U_0$ **which projects at** $e = (\iota_A) \in \hat{S} \setminus U_0$ **onto** $\iota_{(A}\,\iota_{B)}$,

the degeneray of the projection of $\hat{S}$ **at** U_0 **is reflected by the frame coefficients**

$$< e_{AB}, dz^a > = e^a{}_{AB} = e^{*a}{}_{AB} + \hat{e}^a{}_{AB}$$

which have a *singular 'flat'* **part**

$$e^{*a}{}_{AB} = \delta^a_1\,\epsilon_A{}^0\,\epsilon_B{}^0 + \frac{1}{u}\,\delta^a_2\,\epsilon_{(A}{}^0\,\epsilon_{B)}{}^1 + \delta^a_3\,\epsilon_A{}^1\,\epsilon_B{}^1 \quad \textbf{and}$$

holomorphic $\hat{e}^a{}_{AB}$ **with** $\hat{e}^a{}_{00} = 0$, $\hat{e}^3{}_{AB} = 0$, $\hat{e}^a{}_{AB} = O(u)$ **as** $u \to 0$,

– **the connection coefficients** $\Gamma_{ABCD} \equiv < \omega_{CD}, e_{AB} >$ **satisfy**

$$\Gamma_{CD\,AB} = \Gamma^*_{CD\,AB} + \hat{\Gamma}_{CD\,AB},$$

with *singular 'flat'* **part**

$$\Gamma^*_{AB\,CD} = -\frac{1}{u}\,\epsilon_{(A}{}^0\,\epsilon_{B)}{}^1\,\epsilon_C{}^0\,\epsilon_D{}^0 \quad \textbf{and}$$

holomorphic $\hat{\Gamma}_{CD\,AB}$ **with** $\hat{\Gamma}_{00\,AB} = 0$, $\hat{\Gamma}_{CD\,AB} = O(u)$ **as** $u \to 0$.

– **holomorphic spinor-valued functions on** $SL(S_c)$ **induce holomorphic functions of** u, v, w **on** $\hat{S}$.

TENSORIALITY AND EXPANSION TYPE

Lemma 2

– *'Tensoriality condition': The components* $s_k = s_{(ABCD)_k}$ *of the Ricci spinor satisfy*

$$\partial_v \, s_k = (4-k) \, s_{k+1}, \quad k = 0, \ldots, 4. \quad \textbf{on} \quad U_0.$$

– *'Expansion type 4 - k': the functions* $s_k(u, v, w)$ *on* $\hat{S}$ *have at,* $(u, v, w) = (0, 0, 0)$ *Taylor expansions of the form*

$$s_k = \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{2\,m+4-k} s_{k,\,p,m,n} \, u^m \, v^m \, w^p.$$

On $SL(S_c):$ $Z_\alpha \, s_k = (4{-}k) \, s_{k+1}$ **and thus** $Z_\alpha^{4-k+1} \, s_k = 0.$

A horizontal field $H_x = x^{ab} \, H_{AB}$**,** $x^{AB} = const.$ **satisfies**

$$[Z_\alpha, H_x] = H_{\alpha \cdot x}.$$

This implies with some constants $a_{n,m}$ **and some operators** A**,** H_z

$$Z_\alpha^n \, H_{00}^m = a_{n,m} \, H_z^m \, Z^{n-2\,m} + A \, Z^{n-2\,m+1}, \quad m, n = 0, 1, 2, \ldots \; .$$

The lemma follows because $Z_\alpha = \partial_v$ **on** U_0 **and** $H_{00} = \partial_u$ **on** $\hat{S}$**.** $\square$

In fact, the null datum $s_0(u, v)$ **on** W_0 **has in terms of the data**

$$\mathcal{D}_n^* \equiv \{ D^*_{(A_1 B_1} \ldots D^*_{A_p B_p} s^*_{ABCD)}(i) | \; p \geq 0 \},$$

the expansion

$$s_0 = \sum_{m=0}^{\infty} \frac{1}{m\,!} \, u^m \, s^{A_1}{}_0 \, s^{B_1}{}_0 \ldots \, s^{D}{}_0 \, D^*_{(A_1 B_1} \ldots D^*_{A_m B_m} \, s^*_{ABCD)}(i)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{2\,m+4} \psi_{m,n} \; u^m \; v^n,$$

with

$$\psi_{m,n} = \frac{1}{m\,!} \binom{2\,m+4}{n} D^*_{(A_1 B_1} \ldots D^*_{A_m B_m} \, s^*_{ABCD)_n}(i), \quad 0 \leq n \leq 2\,m+4.$$

THE STATIC EQUATIONS IN OUR GAUGE I

In our gauge the conformal static equations imply the ∂_u*-equations*

$$\partial_u e_{CD} = -2\,\Gamma_{CD}{}^{H}{}_{0}\, e_{0H},$$

$$\partial_u \Gamma_{EFAB} + \Gamma_{EF}{}^{K}{}_{0}\,\Gamma_{K0AB} + \Gamma_{EF}{}^{K}{}_{0}\,\Gamma_{0KAB} = s_{AB0(E}\,\epsilon_{F)0}$$

$$0 = \partial_u\,\zeta_{CD} - s\,h_{00CD} + \zeta\,(1-\mu\,\zeta)\,s_{00CD},$$

$$0 = \partial_u\,s + (1-\mu\,\zeta)\,s_{00CD}\,\zeta^{CD},$$

$$\partial_u\,s_{BCD1} - D_{01}\,s_{BCD0} = -\frac{2\,\mu}{1-\mu\,\zeta}\;\zeta_{(0}{}^{E}\,s_{BCD)E}.$$

– $e^{w}{}_{01} = 0$, **the equations are interior to the 'cones'** $\{w = const\}$,

– **with** s_{0000} **given on** $\{w = const\}$**, one gets a hierachy of singular ODE's for the other unknowns; first few subsystems:**

$$\partial_u\hat{\Gamma}_{0100} + \frac{2}{u}\,\hat{\Gamma}_{0100} = 2\,\hat{\Gamma}^2_{0100} + \frac{1}{2}\,s_0,$$

$$\partial_u\hat{e}^2{}_{01} + \frac{1}{u}\,\hat{e}^2{}_{01} = \frac{1}{u}\,\hat{\Gamma}_{0100} + 2\,\hat{\Gamma}_{0100}\,\hat{e}^2{}_{01},$$

$$\partial_u\hat{e}^1{}_{01} + \frac{1}{u}\,\hat{e}^1{}_{01} = -2\,\hat{\Gamma}_{0101} + 2\,\hat{\Gamma}_{0100}\,\hat{e}^1{}_{01},$$

$$\partial_u\hat{\Gamma}_{0101} + \frac{1}{u}\,\hat{\Gamma}_{0101} = 2\,\hat{\Gamma}_{0100}\,\hat{\Gamma}_{0101} + \frac{1}{2}\,s_1,$$

$$0 = \partial_u\zeta - \zeta_{00},$$

$$0 = \partial_u\,\zeta_{00} + \zeta\,(1-\mu\,\zeta)\,s_0,$$

$$0 = \partial_u\,\zeta_{01} + \zeta\,(1-\mu\,\zeta)\,s_1,$$

$$\partial_u\,s_1 - \hat{e}^1{}_{01}\partial_u s_0 - \hat{e}^2{}_{01}\partial_v s_0 - \frac{1}{2\,u}(\partial_v s_0 - 4\,s_1)$$
$$= -4\,\hat{\Gamma}_{0101}\,s_0 + 4\,\hat{\Gamma}_{0100}\,s_1 - \frac{2\,\mu}{(1-\mu\,\zeta)}\;\{s_0\,\zeta_{01} - s_1\,\zeta_{00}\}\,,$$

– **the singular terms come with a coefficient of the 'right' sign**

– **the** s_k **equations remember/enforce the tensoriality condition**

– **the remaining subsystems are similar**

THE STATIC EQUATIONS IN OUR GAUGE II

The regular $\partial_w s_0$*-equation*

$$\partial_w s_0 + \hat{e}^1{}_{11}\, \partial_u s_0 + \hat{e}^2{}_{11}\, \partial_v s_0 - \partial_u s_2$$
$$= 4\, \hat{\Gamma}_{1101}\, s_0 + 4\, \hat{\Gamma}_{1100}\, s_1 + \frac{\mu}{(1-\mu\,\zeta)}\, \{s_0\, \zeta_{11} + 2\, s_1\, \zeta_{01} - 3\, s_2\, \zeta_{00}\}\,,$$

is obtained by adding two s_{ABCD}**-equations.**

It serves to propagate s_0 **(in the** C^ω **context) off** $\{w = const.\}$**.**

Determining the fields on $U_0 = \{u = 0\}$**:**

– **Holds**

$$\hat{e}^a{}_{AB} = 0, \quad \hat{\Gamma}_{ABCD} = 0 \quad \textbf{on} \quad U_0,$$

– **with**

$$\zeta(i) = 0, \;\; \zeta_{AB}(i) = 0,$$

the equations

$$D_{11}\, \zeta = \zeta_{11}, \qquad D_{11}\, \zeta_{CD} = s\, h_{11CD} - \zeta\, (1-\mu\,\zeta)\, s_{11CD},$$

imply

$$\zeta = 0, \;\; \zeta_{01} = 0, \;\; \zeta_{11} = 0 \quad \textbf{on} \quad U_0,$$

– **with**

$$\zeta_{00}(i) = 0, \;\; s(i) = -2,$$

the equations

$$\partial_w\, s = D_{11}\, s = -(1-\mu\,\zeta)\, s_{11CD}\, \zeta^{CD} = -\, s_{1111}\, \zeta_{00} \quad \textbf{on} \quad U_0,$$
$$\partial_w\, \zeta_{00} = D_{11}\, \zeta_{00} = s - \zeta\, (1-\mu\,\zeta)\, s_{0011} = s \quad \textbf{on} \quad U_0,$$

serve to determine

$$\partial^p_w\, s, \;\; \partial^p_w\, \zeta_{00} \quad \textbf{on} \quad I,$$

– **the formal derivatives of the tensoriality conditions**

$$\partial_v\, \partial^p_w\, s_k = (4-k)\, \partial^p_w\, s_{k+1},$$

serve with the $\partial_w\, s_0$**-equation to determine**

$$\partial^p_w\, s_k \quad \textbf{on} \quad I.$$

THE ITERATIVE PROCEDURE

– **Prescribe the data**

$$s_0(u,v,0) = s_0^*(u,v) \equiv \sum_{m=0}^{\infty} \sum_{n=0}^{2m+4} \psi_{m,n}\, u^m\, v^n \quad \textbf{on} \quad W_0 = \{w = 0\},$$

with

$$\psi_{m,n} = \frac{1}{m!} \binom{2m+4}{n} D^*_{(A_1 B_1} \ldots D^*_{A_m B_m}\, s^*_{ABCD)_n}(i),$$

– **determine as described above the unknown**

$$X = (\hat{e}^a{}_{AB},\ \hat{\Gamma}_{ABCD},\ \zeta,\ \zeta_{AB},\ s,\ s_1,\ s_2,\ s_3,\ s_4) \quad \textbf{on} \quad I,$$

– *solve* **the ∂_u-equations for X on W_0,**

– **use the $\partial_w\, s_0$-equation to** *determine* $\partial_w s_0$ **on W_0,**

– *determine* $\partial_w\, X$ **on I,**

– **apply ∂_w formally to the ∂_u-equations, derive equations for $\partial_w\, X$,** *solve* **these on W_0,**

– **apply ∂_w formally to the $\partial_w s_0$-equation to** *determine* $\partial_w^2\, s_0$ **on W_0,**

– **etc.**

Lemma 3:

– *The procedure above determines a unique formal expansion of the fields (X, s_0) at O in terms of u, v, w,*

– *as a consequence of the equations the expansion for s_k is of type $4 - k$,*

– *if the expansion converges it defines in fact a solution of the complete set of conformal static field equations.*

THEOREM

Suppose the datum

$$s_0^*(u,v) = \sum_{m=0}^{\infty} \sum_{n=0}^{2m+4} \psi_{m,n}\, u^m\, v^n,$$

defines on some open nbhd of $O = \{u = 0, v = 0, w = 0\}$ **in** $W_0 = \{w = 0\}$ **a holomorphic function so that there exist positive constants**

$$c_0^*,\ r_0,\ \rho_0 < \frac{1}{2},$$

with respect to which its derivatives satisfy the estimates

$$|\partial_u^m\, \partial_v^n\, s_0^*(0,0)| \leq c_0^*\, \frac{r_0^m\, m!\, \rho_0^n\, n!}{(m+1)^2\,(n+1)^2}, \quad m, n \geq 0.$$

Then the coefficients of the formal expansion obtained by our procedure satisfy estimates of the following form: There exist positive constants

$$r \geq r_0,\ \rho = \rho_0,\ c_{\hat{e}^a_{AB}},\ c_{\hat{\Gamma}_{ABCD}},\ c_\zeta,\ c_{\zeta_i},\ c_{\hat{s}},\ c_k,$$

so that

$$|\partial_u^m\, \partial_v^n\, \partial_w^p\, f(O)| \leq c_f\, \frac{r^{m+p-1}\,(m+p)!\, \rho^n\, n!}{(m+1)^2\,(n+1)^2\,(p+1)^2},$$

where f **stands for any of the functions**

$$\hat{e}^a_{AB},\ \hat{\Gamma}_{ABCD},\ \zeta,\ \zeta_i,\ \hat{s} = s + 2,$$

and

$$|\partial_u^m\, \partial_v^n\, \partial_w^p\, s_k(O)| \leq c_k\, \frac{r^{m+p}\,(m+p)!\, \rho^n\, n!}{(m+1)^2\,(n+1)^2\,(p+1)^2},$$

for $m, n, p = 0, 1, 2, \ldots$.

REMARKS I

- **If $s_0^*(u,v)$ is holomorphic on $P = \{|u| < 1/r_1, \ |v| < 1/\rho_1\}$, with some $r_1, \rho_1 > 0$, one has the Cauchy estimates**

$$|\partial_u^m \, \partial_v^n \, s_0^*(O)| \leq m! \; n! \; r_1^m \; \rho_1^n \; sup_P \, |s_0^*|.$$

They imply the estimates *required* **in the theorem with**

$$r_0 = e^2 \, r_1, \;\; \rho_0 = e^2 \, \rho_1, \;\; c_0^* = sup_P \, |s_0^*|.$$

- **The expansion types (***ignored in the statement of the theorem***) are important for deriving the estimates for the unknowns.**

- **We obtain a holomorphic solution. The formal expansion**

$$\sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{2m+4-k} \frac{1}{n! \, m! \, p!} \partial_u^m \, \partial_v^n \, \partial_w^p \, s_k(O) \, v^n \, u^m \, w^p,$$

for s_k is majorized e.g. with suitable constant c by

$$c \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+p}{m} (\rho \, v)^n \, (r \, u)^m \, (r \, w)^p$$

$$= \frac{c}{(1 - \rho \, v) \, (1 - r \, u - r \, v)} \quad \textbf{for} \quad |v| < 1/\rho, \;\; |u + w| < 1/r.$$

- **Because of the expansion types, the estimates required of s_0^* will for $0 < \alpha < 1$ remain valid under the replacements**

$$\rho_0 \rightarrow \alpha \, \rho_0, \quad r_0 \rightarrow \alpha^{-2} \, r_0, \quad c_0^* \rightarrow \alpha^{-4} \, c_0^*.$$

The requirement $\rho_0 < \frac{1}{2}$ does not impose a serious restriction. The estimates asserted for the functions s_k, f remain valid under the replacements

$$\rho \rightarrow \alpha \, \rho, \quad r \rightarrow \alpha^{-2} \, r, \quad c_k \rightarrow \alpha^{-(4-k)} \, c_k, \quad c_f \rightarrow \alpha^{-k_f} \, c_f.$$

The domain of convergence with respect to v can thus be extended by restricting that with respect to $u + w$.

REMARKS II

- **The domain of the solution to be determined by $\mathcal{D}_n^*$ cannot be covered by the solution of the theorem.**

- **The remaining gauge freedom in our construction is given by transitions of the form**

  $$e^*_{AB} \rightarrow e^t_{AB} = t^C{}_A \, t^D{}_B \, e^*CD \quad \textbf{at} \quad i \quad \textbf{with} \quad t \in SU(2).$$

  The corresponding transformation of the data

  $$\mathcal{D}_n^* \rightarrow \mathcal{D}_n^t \equiv \{ t^{C_1}{}_{A_1} \, t^{D_1}{}_{B_1} \ldots t^H{}_D \, D^*_{(C_1 D_1} \ldots D^*_{A_p B_p} s^*_{EFGH)}(i) | \; p \geq 0 \},$$

  does not affect the *type* of the estimates imposed on s_0^*.

- **Solutions corresponding to $\mathcal{D}_n^t$ for suitable $t \in SU(2)$ can be used to cover the prospective solution on a *punctured* normal neighbourhood of i. In normal coordinates at i this extends to a unique holomorphic solution near i.**

- **The formal expansion determined *in* this solutions from the null data $\mathcal{D}_n$ by the 'exact sets of fields argument' coincides with the one considered in Lemma 1. The solution we have constructed is thus the holomorphic extension of a unique real solution near i to the conformal static field equations.**