Phase Space for the ADM/RT Hamiltonian

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Newton Institute: Constraints Workshop

Outline

AIMS: Construct a Hilbert space setting (phase space $\mathcal{F}$) for

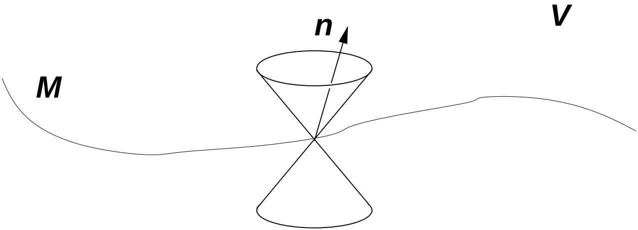
- Constraint submanifold $\mathcal{C} \subset \mathcal{F}$;
- Energy-momentum $\mathbb{P}$ of asymptotically flat initial data;
- Hamiltonian $\mathcal{H}$ evolution form of Einstein;
- Critical points of the total ADM energy.

APPLICATIONS:

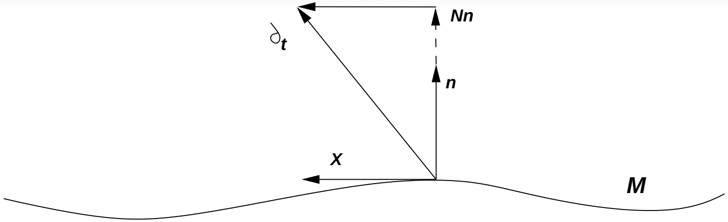
- Linearisation stability;
- Quasi-local mass and the stationary metric conjecture;
- Low regularity solutions of the Einstein equations.

Hypersurface Geometry

$$(M^3, g) \hookrightarrow (V^4, g^V) \text{ is a spacelike hypersurface}$$
$$\Leftrightarrow \quad g = g^V|_{TM} \text{ is Riemannian}$$
$$\Leftrightarrow \quad M \text{ has a future timelike unit normal } n$$



The geometry of M is described by the induced metric g and second fundamental form (*extrinsic curvature*) $K(X, Y) = g^V(\nabla^V_X n, Y)$.



In adapted coordinates (x, t) the time evolution vector is

$$\partial_t = Nn + X = Nn + X^i \partial_i,$$

where N is the *lapse*, $X = X^i \partial_i$ is the *shift* and the future unit normal is $n = N^{-1}(\partial_t - X)$. The spacetime metric is

$$g^V = -N^2 dt^2 + g_{ij}(dx^i + X^i dt)(dx^j + X^j dt),$$

and the extrinsic curvature is $K_{ij} = \frac{1}{2} N^{-1}(\partial_t g_{ij} - \mathcal{L}_X g_{ij})$.

Constraint Equations

The spacelike data (g, K) are not arbitrary if the Einstein equations

$$\mathrm{Ein}_{\alpha\beta} := \mathrm{Ric}^V_{\alpha\beta} - \tfrac{1}{2}\mathrm{R}^V g^V_{\alpha\beta} = 0$$

are imposed. The Gauß and Codazzi equations give

$$\begin{aligned} 2\,\mathrm{Ein}(n,n) &= \Phi_0(g,K) := R(g) - \|K\|^2 + (\mathrm{tr}_g K)^2 \\ 2\,\mathrm{Ein}(n,i) &= \Phi_i(g,K) := 2\nabla^j(K_{ij} - \mathrm{tr}_g K\, g_{ij}), \end{aligned}$$

so $\mathrm{Ein} = 0$ imposes constraints on admissible data (g, K).

Einstein evolution equations

The Einstein-Hilbert Lagrangian $\int_V R(g^V)\sqrt{g^V}$ becomes

$$\mathcal{L}_{EH} \simeq \int_V (\pi \bullet \partial_t g - \xi^\alpha \Phi_\alpha(g, \pi))$$

where $\pi^{ij} = (K^{ij} - \mathrm{tr}_g K g^{ij})\sqrt{g}$ is the *conjugate momentum* and $\xi = (N, X^i)$ is the lapse-shift vector. This leads to the ADM Hamiltonian

$$\mathcal{H}_{ADM}(g, \pi; \xi) = -\int_{\mathcal{M}} \xi^\alpha \Phi_\alpha(g, \pi),$$

and the corresponding Hamilton evolution equations are:

$$\frac{d}{dt}\begin{bmatrix} g \\ \pi \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} D\Phi(g, \pi)^* \xi.$$

Linearization Stability

Question: Given a solution to the linearized Einstein equations, is it tangent to a 1-parameter family of full solutions?

Possible Answer: Use the fundamental existence theorem of Choquet-Bruhat: If (g, K) satisfy the constraints, then there is a corresponding vacuum spacetime, unique up to diffeomorphism. Thus we need only show the solutions of the constraints form a manifold, because then any solution to the linearized constraints $D\Phi(g, K)(h, p) = 0$ is tangent to a curve $s \mapsto (g(s), \pi(s))$ of constraint solutions, which in turn generates the required 1-parameter family of solutions to the full Einstein equations.

Killing data and quadratic conditions

However, if (g, K) and $\xi \not\equiv 0$ satisfy $D\Phi^* \xi = 0$ and M is compact, then any variation $s \mapsto (g(s), \pi(s))$ satisfies

$$\int_M \xi^\alpha D^2 \Phi_\alpha((h,p),(h,p)) = 0$$

where $(h,p) = (g'(0), \pi'(0))$ and $\mathcal{C}$ has a quadratic singularity at (g, K). This leads also to a quadratic condition on solutions of the linearized equations, and linearization stability *fails*.
Moncrief showed that if $D\Phi^*(g,K)\xi = 0$ then ξ generates a Killing vector in the vacuum evolution of (g, K), and conversely, any spacetime symmetry yields a solution of $D\Phi^*(g,K)\xi = 0$.

A Suitable Phase Space

Let $\mathring{g}$ be a C^∞ background metric which is flat near infinity, and define the weighted Sobolev spaces H^k_δ by the norm

$$\|u\|^2_{k,\delta} = \sum_{j=0}^{k} \int_M |\mathring{\nabla}^j u|^2 (1+r)^{2(j-\delta)-3} dv_o \,.$$

The *phase space* $\mathcal{F} = \mathcal{G}^+ \times \mathcal{K}$ is

$$\mathcal{F} = \left\{ (g, K) : g - \mathring{g} \in H^2_{-1/2},\ g > 0,\ K \in H^1_{-3/2} \right\}.$$

By Sobolev-Morrey embedding, $g_{ij} \in C^{0,1/2}$ is Hölder continuous and decays as $g_{ij} - \mathring{g}_{ij} = o(r^{-1/2})$, and $K \in L^2(M)$.

$\Phi(g, K)$ is a smooth map

An extended exercise with Sobolev and Hölder inequalities (with weights) gives

Lemma

$\Phi : \mathcal{F} \to L^2_{-5/2}(\mathcal{T})$ *is bounded and smooth.*

This follows easily from the boundedness estimates:

$$\begin{aligned}
\|\Phi_0(g, K)\|_{2,-5/2} &\leq c\left(1 + \|g - \mathring{g}\|^2_{2,2,-1/2} + \|K\|^2_{1,2,-3/2}\right), \\
\|\Phi_i(g, K)\|_{2,-5/2} &\leq c\left(\|\mathring{\nabla} K\|_{2,-5/2} + \|\mathring{\nabla} g\|_{1,2,-3/2}\,\|K\|_{1,2,-3/2}\right).
\end{aligned}$$

The Constraint Set

Theorem

The set of all data satisfying the constraints

$$\mathcal{C} = \Phi^{-1}(0) = \{(g, K) \;:\; \Phi(g, K) = 0\}$$

is a Hilbert submanifold of $\mathcal{F}$.

Proof.

(sketch) The aim is to apply the Implicit Function Theorem. For this we must show $D\Phi : \mathcal{F} \to L^2_{-5/2}(\mathcal{T})$ is surjective and splits i.e. ker $D\Phi$ is closed and has a complementary subspace. Surjectivity is the hard part, and involves two major steps:

1. show $\mathrm{coker}\ D\Phi = \ker\ D\Phi^*$ is trivial;
2. show $D\Phi$ has closed range.

□

$D\Phi$ and $D\Phi^*$

$$D\Phi(g,K)(h,p) = \begin{bmatrix} \delta_g\delta_g - \Delta_g \mathrm{tr}_g + S - E & -2K \\ K * \nabla + 2\mathrm{div}_g(K - \mathrm{tr}_g K g) & 2\mathrm{div}_g \end{bmatrix} \begin{bmatrix} h \\ p \end{bmatrix},$$

$$D\Phi(g,\pi)^*(N,X) = \begin{bmatrix} \nabla^2 - g\Delta_g + S - E & \nabla K - K * \nabla \\ -2K & -\mathcal{L}_\bullet g \end{bmatrix} \begin{bmatrix} N \\ X \end{bmatrix},$$

where

$$S \sim K^2, \quad E = Ric - \tfrac{1}{2} R g$$

Step 1: $\ker D\Phi^*$ is trivial

$D\Phi^* : H^2_{-1/2} \to L^2_{-5/2} \times H^1_{-3/2}$ has leading term

$$D\Phi(g,\pi)^*(N,X) \sim \left(\nabla^2 N\ ,\ -\nabla_i X_j - \nabla_j X_i\right)$$

and the identity $X_{i|jk} = R_{ijkl}X^l + X_{(i|j)k} + X_{(i|k)j} - X_{(j|k)i}$ shows $D\Phi^*\xi = 0$ gives $\nabla^2\xi = F(\xi, \nabla\xi)$, which is highly overdetermined. Now $\xi = o(r^{-1/2})$ so in the C^2 case, $\xi \equiv 0$ follows easily by integrating along curves. An argument using sharp Sobolev inequalities eg. if $u \equiv 0$ in $B_{\eta R}$, $\eta < 1$, then

$$\|u\|_{2n/(n-2),B_R} \leq c\eta^{2-n}\|Du\|_{2,B_R},$$

handles $\xi \in H^2_{-1/2}$, and a bootstrapping argument handles weak solutions $\xi \in L^2_{-1/2}$.

Step 2: $D\Phi$ is surjective

Since $\operatorname{coker} D\Phi = \ker D\Phi^* = 0$, it suffices to show $D\Phi$ has closed range. The leading term of $D\Phi(h,p)$ is

$$D\Phi(g,\pi)(h,p) \sim \left(\nabla^i\nabla^j h_{ij} - \Delta h^i_i\ ,\ \nabla_j p^{ij}\right),$$

so we choose the variations $h_{ij} = y g_{ij}$, $p^{ij} = 2\nabla^{(i}Y^{j)} - \nabla_k Y^k g^{ij}$ and the leading term of $F(y,Y) := D\Phi(h,p)$ is

$$F(y,Y) \sim \left(-4\Delta y\ ,\ 2\Delta Y^i\right).$$

Weighted elliptic estimates show F has finite-dimensional kernel and cokernel, so $\operatorname{ran} D\Phi \supset \operatorname{ran} F$ is also closed.

Total Energy-Momentum

If $g_{ij} = \delta_{ij} + o(1)$ is "asymptotically flat" we define the ADM energy-momentum covector $\mathbb{P} = (E, p_i)$,

$$\begin{aligned} 16\pi E &= \oint_{S_\infty} (\partial_j g_{ij} - \partial_i g_{jj})\, dS^i \\ 8\pi p_i &= \oint_{S_\infty} \pi_{ij}\, dS^j \end{aligned}$$

These expressions are well-defined on $\mathcal{C}$:

Theorem

$\mathbb{P} = \mathbb{P}(g, \pi) \in C^\infty(\mathcal{C}, \mathbb{R}^{3,1})$.

Remarks:

- $\mathbb{P}$ is not well-defined on $\mathcal{F}$.
- This definition of $\mathbb{P}$ is independent of the choice of "structure at infinity"

Proof.

Write the boundary integral as a volume integral of a divergence, eg.

$$
\begin{aligned}
8\pi X^i_\infty \mathbb{P}_i(g,\pi) &= \int_M (X^i \mathring{g}_{ij} \mathring{\nabla}_k \pi^{jk} + \pi^j_i \mathring{\nabla}_j X^i) \\
&= \int_M (X^i \Phi_i(g,\pi) + \text{ terms in } L^1(M)),
\end{aligned}
$$

where $X^i \in X^i_\infty + H^2_{-1/2}$ and thus π^{ij}, $\mathring{\nabla} g$ and $\mathring{\nabla} Xi$ are all $H^1_{-3/2}$, so quadratic terms are $L^1(M)$. Similarly, use

$$
16\pi \xi^0_\infty \mathbb{P}_0(g,\pi) = \int_{\mathcal{M}} \left(\xi^0_\infty \mathcal{R}_o(g) + \mathring{\nabla}^i \xi^0_\infty \left(\mathring{\nabla}^j g_{ij} - \mathring{\nabla}_i \mathrm{tr}_{\mathring{g}} g \right) \sqrt{\mathring{g}} \right)
$$

where $\mathcal{R}_o(g) = \left(\mathring{\nabla}^{ij} g_{ij} - \Delta_o \mathrm{tr}_{\mathring{g}} g \right) \sqrt{\mathring{g}}$.

□

ADM Hamiltonian

Theorem

The ADM Hamiltonian

$$\mathcal{H}_{ADM}(g, \pi; \xi) = -\int_{\mathcal{M}} \xi^\alpha \Phi_\alpha(g, \pi),$$

with lapse-shift $\xi \in L^2_{-1/2}$ *defines a smooth map of Hilbert manifolds* $\mathcal{H}_{ADM} : \mathcal{F} \times L^2_{-1/2} \to \mathbb{R}$. *If* $\xi \in W^{2,2}_{-1/2}(\mathcal{T})$, *then for all* $(h, p) \in T_{(g,\pi)}\mathcal{F}$,

$$D_{(g,\pi)}\mathcal{H}_{ADM}(g, \pi; \xi)(h, p) = -\int_{\mathcal{M}} (h, p) \cdot D\Phi(g, \pi)^*(\xi).$$

Thus the ADM Hamiltonian with lapse-shift in H^2 and decaying $o(r^{-1/2})$ generates the correct evolution equations. But we want $\xi \to \xi_\infty \neq 0$...

Regularized Hamiltonian $\mathcal{H}$

The non-integrable parts of $\Phi_\alpha(g, K)$ are divergence terms which produce the total energy-momentum $\mathbb{P}$. This leads to the Regge-Teitelboim Hamiltonian

$$\mathcal{H}_{RT}(g, K; \xi) = 16\pi\xi_\infty^\alpha \mathbb{P}_\alpha(g, K) - \int_\mathcal{M} \xi^\alpha \Phi_\alpha(g, K),$$

which is well-defined on $\mathcal{C}$, for $\xi \in \xi_\infty + L^2_{-1/2}$. Converting $\mathbb{P}$ to divergences leads to the regularized Hamiltonian $\mathcal{H}(g, K; \xi)$, where the individual terms are each $L^1(M)$:

$\mathcal{H}(g, \pi; \xi) =$

$$\begin{aligned}
& \int_\mathcal{M} \left(\xi_\infty^0 - \xi^0\right) \Phi_0 + \int_\mathcal{M} \left(\xi_\infty^i - \xi^i\right) \Phi_i \\
+ & \int_\mathcal{M} \xi_\infty^0 \left(\mathcal{R}_o(g) - \Phi_0\right) + \int_\mathcal{M} \mathring{\nabla}^i \xi_\infty^0 \left(\mathring{\nabla}^j g_{ij} - \mathring{\nabla}_i \mathrm{tr}_{\mathring{g}} g\right) \sqrt{\mathring{g}} \\
+ & \int_\mathcal{M} \xi_\infty^i \left(\mathcal{P}_{o\,i}(\pi) - \Phi_i\right) + \int_\mathcal{M} 2\pi^{ij} \mathring{\nabla}_i \xi_{\infty j}.
\end{aligned}$$

Derivative of the regularized Hamiltonian

Theorem

The regularised Hamiltonian $\mathcal{H} : \mathcal{F} \times (\mathbb{R}^{3,1} + \mathcal{L}) \to \mathbb{R}$ *is is a smooth function on all* $\mathcal{F}$. *If* $\xi \in \xi_\infty + W^{2,2}_{-1/2}(\mathcal{T})$ *then for all* $(g, \pi) \in \mathcal{F}$ *and* $(h, p) \in T_{(g,\pi)}\mathcal{F}$ *we have*

$$D_{(g,\pi)}\mathcal{H}(g, \pi; \xi)(h, p) = -\int_{\mathcal{M}} (h, p) \cdot D\Phi(g, \pi)^* \xi.$$

Proof.

Show $\mathcal{H}$ is bounded by direct estimation of the terms in the definition of $\mathcal{H}$. □

Remark: This shows that $\mathcal{H}$ defines the Einstein evolution flow on $(H^4 \times H^3) \cap \mathcal{F}$.

Critical points of the mass

Theorem

Suppose $(g, \pi) \in \mathcal{F}$ *satisfies* $\Phi(g, \pi) = (\varepsilon, S_i) \in L^1(\mathcal{T}^* \otimes \Lambda^3)$, *let* $\xi_\infty \in \mathbb{R}^{3,1}$ *be a fixed future timelike vector and define the energy functional* $E \in C^\infty(\mathcal{C}(\varepsilon, S_i))$ *by*

$$E(g', \pi') = \xi_\infty^\alpha \mathbb{P}_\alpha(g', \pi'), \quad \forall\, (g', \pi') \in \mathcal{C}(\varepsilon, S_i).$$

Then the following two statements are equivalent:

(i) *For all* $(h, p) \in T_{(g,\pi)}\mathcal{C}(\varepsilon, S_i)$ *we have*

$$DE(g, \pi)(h, p) = 0;$$

(ii) *There is* $\xi \in \xi_\infty + W^{2,2}_{-1/2}(\mathcal{T})$ *satisfying*

$$D\Phi(g, \pi)^* \xi = 0.$$

Quasi-local mass and Stationary Metric Conjecture

Given a bounded region $(\Omega, g_\Omega, K_\Omega)$ we define the (quasi–local) energy

$$\begin{aligned} m_{QL}(\Omega) &= \inf\{m_{ADM}(M, g, K) : \\ &\qquad (M, g, K) \text{ extends } (\Omega, g_\Omega, K_\Omega), \\ &\qquad \text{and } (M, g, K) \text{ has no horizons}\} \end{aligned}$$

Conjecture: The infimum is realised, by data (M, g, K) which is *stationary* (ie. admits a timelike Killing vector) outside Ω.

Summary

- There is a natural Hilbert manifold structure for the phase space and constraint submanifold.
- Some of the ADM formal calculations can be interpreted literally: the mass is differentiable and has the expected critical point, and the Hamiltonian generates the correct equations of motion.

Some open questions

- Fix the symplectic form to be well-defined on $T\mathcal{F}$;
- Analyse the boundary term contributions;
- Extend to yet weaker regularity: $g - \mathring{g} \in W^{1,3}_{-1/2}$, $K \in L^2_{-3/2}$;
- Extend the lapse-shift asymptotics to include boosts and rotations, thereby defining angular momentum.

Acknowledgements

I'd like to thank Vince Moncrief first, for patient explanations of the ADM framework, too many years ago. Thanks also to Leon Simon, John Hutchinson and Rick Schoen for useful comments, and to the Australian Research Council for its support.