Nonsingular stationary metrics with a negative cosmological constant

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Abstract: We construct infinite dimensional families of non-singular stationary space times, solutions of the vacuum Einstein equations with a negative cosmological constant.

1 Introduction

We construct Lorentzian metrics ^{n+1}g in any space-dimension $n \geq 3$, with Killing vector $X = \partial/\partial t$. In adapted coordinates those metrics can be written as

$$^{n+1}g = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{g_{ij} dx^i dx^j}_{=g} , \quad (1)$$

$$\partial_t V = \partial_t \theta = \partial_t g = 0 . \quad (2)$$

Remarks : If $\Lambda = 0$ from A. Lichnerowicz 1955 (AF) and M. Anderson 2000 (complete), when there is no black hole, it must be Minkowski.
If $\Lambda < 0$, with M. Anderson and P. T. Chruściel (2002-2005) : infinite dimensional families of static vacuum space-times.

Theorem : Let $n = \dim M \geq 3$**, and consider a static Lorentzian Einstein metric of the form (1)(2) with** $V = \mathring{V}$, $g = \mathring{g}$**, and** $\theta = 0$**, such that the associated Riemannian metric** $\tilde{g} = \mathring{V}^2 d\varphi^2 + \mathring{g}$ **on** $\mathbb{S}^1 \times M$ **is smoothly compactifiable and non-degenerate. For every smooth** $\hat{\theta}$**, sufficiently close to zero in** $C^{k+2,\alpha}(\partial M, \mathcal{T}_1)$**, there exists a unique stationary vacuum metric (1)(2) such that, in local coordinates near the conformal boundary** ∂M**,**

$$V - \mathring{V} = O(\rho) \ , \quad \theta_i = \hat{\theta}_i + O(\rho), \quad g_{ij} - \mathring{g}_{ij} = O(1) \ . \qquad (3)$$

The vacuum Einstein equations for a metric satisfying (1)(2) read

$$\begin{cases} V(\nabla^*\nabla V + nV) = \frac{1}{4}|\lambda|^2_g \ , \\ \mathrm{Ric}(g) + ng - V^{-1}\mathrm{Hess}_g V = \frac{1}{2V^2}\lambda \circ \lambda \ , \\ \mathrm{div}(V\lambda) = 0 \ , \end{cases} \tag{4}$$

where

$$\lambda_{ij} = -V^2(\partial_i\theta_j - \partial_j\theta_i) \ , \quad (\lambda \circ \lambda)_{ij} = \lambda_i{}^k\lambda_{kj} \ .$$

2 The linearised equation

We first consider the operator :

$$\begin{pmatrix} V \\ g \end{pmatrix} \mapsto \begin{pmatrix} V(\nabla^*\nabla V + nV) \\ \mathrm{Ric}(g) + ng - V^{-1}\mathrm{Hess}_g V \end{pmatrix} .$$

The two parts of the linearisation of it at (V, g) **are**

$$\begin{aligned} p(W,h) &= V\big[(\nabla^*\nabla + 2n + V^{-1}\nabla^*\nabla V + V^{-2}|dV|^2)W \\ &\quad +V^{-1}\nabla_j V\nabla^j W - V^{-1}\nabla^j V\nabla^k V h_{kj} \\ &\quad +\langle \mathrm{Hess}_g V, h\rangle_g + \langle w, dV\rangle_g] \\ &=: l(W,h) + V\langle w, dV\rangle_g . \end{aligned}$$

and ...

$$
\begin{aligned}
P(W,h) \quad &= \quad \frac{1}{2}\Delta_L h_{ij} + n h_{ij} - \frac{1}{2}V^{-1}\nabla^k V \nabla_k h_{ij} \\
&\quad + \frac{1}{2}V^{-2}(\nabla_i V \nabla^k V h_{kj} + \nabla_j V \nabla^k V h_{ki}) \\
&\quad - \frac{1}{2}V^{-1}(\nabla_i \nabla^k V h_{kj} + \nabla_j \nabla^k V h_{ki}) \\
&\quad + 2V^{-2}W\,(\mathrm{Hess}_g V)_{ij} - 2V^{-3}\nabla_i V \nabla_j V W + (\mathrm{div}^* w)_{ij} \\
&=: \quad L(W,h) + \mathrm{div}^* w \, .
\end{aligned}
$$

where

$$
w_j = V^{-1}\nabla^k V h_{kj} + \nabla^k h_{kj} - \frac{1}{2}\nabla_j(\mathrm{Tr} h) - V^{-1}\nabla_j W - V^{-2}\nabla_j V W \, ,
$$

$$
(\mathrm{div} h)_i = -\nabla^k h_{ik}, \quad (\mathrm{div}^* w)_{ij} = \frac{1}{2}(\nabla_i w_j + \nabla_j w_i) \, ,
$$

Note : (l, L) **is elliptic.**

3 The modified equation

We will first solve :

$$\begin{cases} q(V,g) := V(\nabla^*\nabla V + nV + \langle \Omega, dV\rangle) - \frac{1}{4}|\lambda|_g^2 = 0 \;, \\ Q(V,g) := Ric(g) + ng - V^{-1}Hess_g V + div^*\Omega - \frac{1}{2V^2}\lambda\circ\lambda = 0 \;, \\ (\mathcal{Q} - 2(V^{-1}\nabla^*\nabla V + n))(\theta) := -V^{-3}div(V\lambda) + d\sigma = 0 \end{cases}$$

with

$$\begin{aligned} -\Omega_j \equiv -\Omega(V,g,\mathring{V},\mathring{g})_j \quad := \quad & \mathring{V}^{-1}\mathring{\nabla}^k\mathring{V}(g-\mathring{g})_{kj} + \mathring{\nabla}^k(g-\mathring{g})_{kj} \\ & -\frac{1}{2}\partial_j(Tr_{\mathring{g}}(g-\mathring{g})) \\ & -\mathring{V}^{-1}\partial_j(V-\mathring{V}) - \mathring{V}^{-2}\partial_j\mathring{V}(V-\mathring{V}) \;, \end{aligned}$$

and $\sigma = V^{-3}\nabla^i(V^3\theta_i)$. **In particular** $D(q,Q)(\mathring{V},\mathring{g}) = (l,L)$ **is elliptic and the same is true for** $\mathcal{Q}$**.**

4 Solutions to the modified equation gives solutions :

Suppose that θ **solves** $\mathrm{div}(V\lambda) + V^3 d\sigma = 0$**, then clearly** $\mathrm{div}[\mathrm{div}(V\lambda) + V^3 d\sigma] = \mathrm{div}(V^3 d\sigma) = 0$. **If** $\sigma = O(\rho^2)$ **then** $\sigma = 0$**. Now the divergence of** $E(g) := \mathrm{Ric}(g) - \frac{1}{2}R(g)g$ **equals**

$$\begin{aligned} \mathrm{div}E(g)_j &= -\frac{1}{2}[\nabla^k\nabla_k\Omega_j + V^{-1}\nabla^i V\nabla_i\Omega_j \\ &\quad -V^{-2}\nabla_j V\nabla^i V\Omega_i + R_{ij}\Omega^i - V^{-1}\nabla_j\nabla^i V\Omega_i] \\ &=: -\frac{1}{2}\mathcal{B}(\Omega)_j \ . \end{aligned}$$

The Bianchi identity $\mathrm{div}E(g) = 0$ **gives** $\mathcal{B}(\Omega) = 0$**. If** $\Omega = O(\rho)$ **then** $\Omega = 0$**.**

5 The construction

We consider $\mathring{V}^2 d\varphi^2 + \mathring{g}$**, a non degenarte AHES metric on** $\mathbb{S}^1 \times M$**. For any Riemannian metric** g **on** M**, close to** $\mathring{g}$ **in** $O(\rho^{-1})$**, for any function** V **on** M**, close to** $\mathring{V}$ **in** $O(\rho^0)$**, and for any** $\hat{\theta} \in C^{k+2,\alpha}(\partial M, \mathcal{T}_1)$**, there exists a unique solution**

$$\theta = \theta(\hat{\theta}, V, g) \in O(\rho^0)$$

to

$$\begin{cases} \mathrm{div}(V\lambda) + V^3 d\sigma \equiv V^3[-\mathcal{Q} + 2(V^{-1}\nabla^*\nabla V + n)]\theta = 0 \;, \\ \theta - \hat{\theta} \in O(\rho) \;, \end{cases}$$

where we recall that

$$\lambda_{ij} = -V^2(\partial_i\theta_j - \partial_j\theta_i) \text{ and } \sigma = V^{-3}\nabla^i(V^3\theta_i) \;.$$

Moreover, the map $(\hat{\theta}, V, g) \mapsto \theta - \hat{\theta}$ **is smooth. Now** $\sigma = O(\rho^2)$ **thus** $\sigma = 0$**. Let** $F(\hat{\theta}, V, g)$ **defined by**

$$\begin{pmatrix} V(\nabla^*\nabla V + nV + \langle \Omega(V, g, \mathring{V}, \mathring{g}), dV\rangle) - \frac{1}{4}|\lambda|_g^2 \\ \mathrm{Ric}(g) + ng - V^{-1}\mathrm{Hess}_g V + div^*\Omega(V, g, \mathring{V}, \mathring{g}) + \frac{1}{2V^2}\lambda \circ \lambda \end{pmatrix},$$

and

$$\mathcal{F}(\hat{\theta}, W, h) := F(\hat{\theta}, \mathring{V} + W, \mathring{g} + h),$$

then $\mathcal{F}$ **is smooth from** $C^{k+2,\alpha}(\partial M, \mathcal{T}_1) \times O(\rho^1) \times O(\rho^0)$ **to** $O(\rho^0) \times O(\rho^0)$

Proof of the theorem :

$$D_{(W,h)}\mathcal{F}(0,0,0) = D_{(V,g)}F(0,\mathring{V},\mathring{g}) = (l,L)\ ,$$

is an isomorphism. From the implicit function theorem the conclusion of Theorem are valid but for the modified equation. Now $\Omega = \Omega(V,g,\mathring{V},\mathring{g}) \in O(\rho^1)$ **and** $\mathcal{B}(\Omega) = 0$**, so** $\Omega = 0$**.**

6 An example of isomorphism

Theorem [J.M. Lee 2001] : Let $\mathbb{S}^1 \times M$ **be equipped with a non-degenerate asymptotically hyperbolic metric** $\widetilde{g}$**. For** $\delta \in (0, n)$ **the operator** $\widetilde{\Delta}_L + 2n$ **is an isomorphism from** $O(\rho^{\delta-2})$ **to** $O(\rho^{\delta-2})$**.**

If $\widetilde{g} = \mathring{V}^2 d\varphi^2 + \mathring{g}$**,** W, θ, h **does not depend on** φ**, let**

$$\widetilde{h} = 2\mathring{V}W d\varphi^2 + 2\mathring{V}^2 \theta_i d\varphi dx^i + h_{ij} dx^i dx^j \ .$$

Then

$$\begin{aligned} \widetilde{\Delta}_L \widetilde{h}_{00} + 2n\widetilde{h}_{00} &= 2l(W, h). \\ \widetilde{\Delta}_L \widetilde{h}_{i0} + 2n\widetilde{h}_{i0} &= V^2 \mathcal{Q}(\theta)_i \\ \widetilde{\Delta}_L \widetilde{h}_{ij} + 2n\widetilde{h}_{ij} &= 2L(W, h)_{ij};. \end{aligned}$$

Thus the Lee's theorem gives

Corollary : The operator (l, L) **is an isomorphism from** $O(\rho^{\delta-1}) \times O(\rho^{\delta-2})$ **to** $O(\rho^{\delta-2}) \times O(\rho^{\delta-2})$ **when** $\delta \in (0, n)$**.The operator** $\mathcal{Q}$ **is an isomorphism from** $O(\rho^{\delta})$ **to** $O(\rho^{\delta})$ **when** $\delta \in (0, n)$**.**