Constructing solutions of the constraint equations with sources

The Einstein-Scalar field system

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INI Global Problems in Mathematical Relativity
Einstein Constraint Equations

Based on joint work with Yvonne Choquet-Bruhat
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Initial data for the Cauchy problem in General Relativity

- To study solutions of the Einstein field equations

$$Ric(g) - \frac{1}{2}R(g)g = T$$

  dynamically, we often begin with solutions to the corresponding system of constraint equations.

- The initial data on an n-dimensional manifold Σ consists of
  - a Riemannian metric $\bar{\gamma}$
  - a symmetric 2-tensor $\bar{K}$
  - $\mathcal{F}$ a collection of initial data for the non-gravitational fields.

Einstein constraint equations

- In terms of this data $(\bar{\gamma}, \bar{K}, \mathcal{F})$, the Einstein constraint equations are

$$\begin{aligned} \mathrm{div}_{\bar{\gamma}} \bar{K} - \nabla(\mathrm{tr}\, \bar{K}) &= J(\bar{\gamma}, \mathcal{F}) && \text{(Momentum constraint)} \\ R(\bar{\gamma}) - |\bar{K}|^2_{\bar{\gamma}} + (\mathrm{tr}\, \bar{K})^2 &= 2\rho(\bar{\gamma}, \mathcal{F}) && \text{(Hamiltonian constraint)} \\ C(\bar{\gamma}, \mathcal{F}) &= 0 && \text{(Non-gravitational constraints)} \end{aligned}$$

This is a highly underdetermined system of equations.

- For vacuum data ($\rho = 0 = J$ and no non-gravitational constraints) in $3 + 1$ dimensions this is 4 equations for 12 unknowns.
- This observation foreshadows a surprising degree of flexibility in constructing solutions (cf. Corvino, Chruściel-Delay, Corvino-Schoen, Chruściel-Isenberg-P.).

Einstein-scalar fields

Einstein-scalar field action is:

$$\mathcal{S}(g, \Psi) = \int_M [R(g) - \frac{1}{2}|\nabla\Psi|_g^2 - V(\Psi)] d\eta_g,$$

From this we obtain the Einstein-scalar field equations:

$$G_{\alpha\beta} = T_{\alpha\beta} = \nabla_\alpha\Psi\nabla_\beta\Psi - \tfrac{1}{2}g_{\alpha\beta}\nabla_\mu\Psi\nabla^\mu\Psi - g_{\alpha\beta}V(\Psi)$$

$$\nabla_\mu\nabla^\mu\Psi = V'(\Psi).$$

(where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R(g)\, g_{\alpha\beta}$ is the Einstein curvature tensor).

Einstein-scalar field Constraint Equations

Initial data on Σ^n consists of

$\bar{\gamma}$ (the spatial metric)

$\bar{K}$ (the second fundamental form, or extrinsic curvature)

$\bar{\psi}$ (the scalar field restricted to Σ)

$\bar{\pi}$ (the normalized time derivative of Ψ restricted to Σ).

The constraint equations are then

$$\begin{aligned} \mathrm{div}_{\bar{\gamma}} \bar{K} - \nabla(\mathrm{tr}\, \bar{K}) &= -\bar{\pi} \nabla \bar{\psi} \\ R(\bar{\gamma}) - |\bar{K}|^2_{\bar{\gamma}} + (\mathrm{tr}\, \bar{K})^2 &= \bar{\pi}^2 + |\nabla \bar{\psi}|^2_{\bar{\gamma}} + 2V(\bar{\psi}). \end{aligned}$$

(Note: the scalar field does not introduce any new constraint equations.)

The conformal method (aprés Lichnerowicz, Choquet-Bruhat and York)

Split the initial data into two parts

- "conformal data": regard as being freely chosen.
- "determined data": found by solving the constraint equations, reformulated as a determined system of elliptic PDE.

General Criteria: For constant mean curvature (CMC) initial data, where $\tau = \mathrm{tr}_{\bar{\gamma}} \bar{K}$ is constant, we want the equations to be "semi-decoupled":

- First solve the nongravitational constraints.
- Then solve the conformally formulated momentum constraint.
- These solutions enter into the conformally formulated Hamiltonian constraint, which we solve for the remaining piece of determined data.

conformal and determined data (vacuum case)

For the gravitational (vacuum) data, the free "conformal data" consists of

- γ, a Riemannian metric on Σ, representing a chosen *conformal class of metrics* $[\gamma] = \{\tilde{\gamma} = \theta^{\frac{4}{n-2}}\gamma : \theta > 0\}$.
- $\sigma = \sigma_{ab}$, a symmetric tensor which is divergence-free and trace-free w.r.t. γ (σ is a transverse-traceless or TT-tensor).
- τ, a scalar function representing the mean curvature of the Cauchy surface Σ in the resulting spacetime.

The "determined data" consists of

- ϕ, a positive function
- $W = W^a$, a vector field

Reconstructed data (vacuum case)

Use (ϕ, W) to reconstruct an initial data set $(\bar{\gamma}, \bar{K})$ from the conformal data set (γ, σ, τ) via:

$$\begin{aligned} \bar{\gamma} &= \phi^{\frac{4}{n-2}}\gamma \\ \bar{K} &= \phi^{-2}(\sigma + \mathcal{D}W) + \frac{\tau}{n}\phi^{\frac{4}{n-2}}\gamma \end{aligned}$$

here the operator $\mathcal{D}$ is the conformal Killing operator relative to γ.

$(\bar{\gamma}, \bar{K})$ satisfy the vacuum constraint equations if and only if (ϕ, W) satisfy

$$\begin{cases} \operatorname{div}(\mathcal{D}W) = \frac{n}{n-1}\phi^{\frac{2n}{n-2}}\nabla\tau \\ \\ c_n^{-1}\Delta_\gamma\phi - R(\gamma)\phi + \left(|\sigma + \mathcal{D}W|_\gamma^2\right)\phi^{-\frac{3n-2}{n-2}} - \dfrac{n-1}{n}\tau^2\phi^{\frac{n+2}{n-2}} = 0 \end{cases}$$

where $c_n = \frac{n-2}{4(n-1)}$.

Conformally formulated momentum constraint equation for the Einstein-scalar field system

Under the conformal rescaling $\bar{\gamma} = \phi^{\frac{4}{n-2}}\gamma$, we rescale the scalar field initial data as follows:

$$\bar{\psi} = \psi \qquad \text{and} \qquad \bar{\pi} = \phi^{\frac{2n}{n-2}}\pi.$$

The momentum constraint then becomes

$$\boxed{\mathrm{div}_\gamma(\mathcal{D}W) = \frac{n-1}{n}\phi^{\frac{2n}{n-2}}\nabla\tau - \pi\nabla\psi.}$$

- When τ is constant, the conformally formulated momentum constraint equation does not involve the conformal factor ϕ.
- $\mathrm{div}_\gamma \circ \mathcal{D}$ is a self-adjoint, second order, elliptic operator. On a compact manifold, $\ker(\mathrm{div}_\gamma \circ \mathcal{D}) = \{\text{conformal Killing vector fields}\}$. (If there are no conformal Killing vector fields, this equation has a unique solution for any choice of (ϕ, τ, ψ, π).)

Conformally formulated Hamiltonian constraint equation for the Einstein-scalar field system

Let

$$\mathcal{R}_{\gamma,\psi} = c_n \left(R(\gamma) - |\nabla\psi|^2_\gamma \right), \qquad \mathcal{A}_{\gamma,W,\pi} = c_n \left(|\sigma + \mathcal{D}W|^2_\gamma + \pi^2 \right)$$

and

$$\mathcal{B}_{\tau,\psi} = c_n \left(\frac{n-1}{n}\tau^2 - 4V(\psi) \right).$$

The Hamiltonian constraint equation for the Einstein-scalar conformal data $(\gamma, \sigma, \tau, \psi, \pi)$ (with a given vector field W satisfying the conformally formulated momentum constraint equation) is

$$\boxed{\Delta_\gamma \phi - \mathcal{R}_{\gamma,\psi}\, \phi + \mathcal{A}_{\gamma,W,\pi}\, \phi^{-\frac{3n-2}{n-2}} - \mathcal{B}_{\tau,\psi}\, \phi^{\frac{n+2}{n-2}} = 0.}$$

This is the Einstein-scalar field Lichnerowicz equation.

Analysis of the Einstein-scalar field Lichnerowicz equation

This equation differs from other matter/field Lichnerowicz equations (e.g. for vacuum, Maxwell, Yang-Mills, fluids) in two very significant ways:

- coefficient of linear term is $\mathcal{R}_{\gamma,\psi} = c_n \left(R(\gamma) - |\nabla\psi|^2_\gamma \right)$ vs. $R(\gamma)$.
- $\mathcal{B}_{\tau,\psi} = c_n \left(\frac{n-1}{n}\tau^2 - 4V(\psi) \right)$ may not, in general, have a fixed sign.

However

- The Lichnerowicz equation is conformally invariant: set

$$\tilde{\gamma} = \theta^{\frac{4}{n-2}}\gamma \qquad \tilde{\sigma} = \theta^{-2}\sigma \qquad \tilde{\tau} = \tau$$
$$\tilde{\psi} = \psi \qquad \tilde{\pi} = \theta^{\frac{2n}{n-2}}\pi$$

  ϕ solution w.r.t. $(\gamma, \sigma, \tau, \psi, \pi) \Leftrightarrow \frac{\phi}{\theta}$ solution w.r.t. $(\tilde{\gamma}, \tilde{\sigma}, \tilde{\tau}, \tilde{\psi}, \tilde{\pi})$.
- $\mathcal{A} \equiv 0 \Rightarrow$ a solution to the Lichnerowicz equation corresponds to a solution of the prescribed scalar curvature-scalar field equation

$$\mathcal{R}_{\tilde{\gamma},\psi} = -\mathcal{B}_{\tau,\psi}.$$

The Yamabe-scalar field conformal invariant

Definition

The Yamabe-scalar field conformal invariant is defined by

$$\mathcal{Y}_\psi([\gamma]) = \inf_{u \in H^1(\Sigma)} \frac{c_n^{-1} \int_\Sigma [|\nabla u|_\gamma^2 + c_n \left(R(\gamma) - |\nabla \psi|_\gamma^2\right) u^2] \, d\eta_\gamma}{\left(\int_\Sigma u^{\frac{2n}{n-2}} \, d\eta_\gamma\right)^{\frac{n-2}{n}}}.$$

- Hölder's inequality $\Rightarrow \mathcal{Y}_\psi([\gamma]) > -\infty$.
- $\mathcal{Y}_\psi([\gamma])$ is independent of the choice of background metric in the conformal class used to define it. It therefore defines an invariant of the conformal class and scalar field.

The conformal information from $\mathcal{Y}_\psi([\gamma])$

Define the *conformal scalar-field Laplace operator* $L_{\gamma,\psi}$ by

$$L_{\gamma,\psi} u = \Delta_\gamma u - c_n \left(R(\gamma) - |\nabla\psi|^2_\gamma \right) u.$$

(scalar-field analog of the conformal Laplace operator).

Proposition

The following conditions are equivalent:

(i) $\mathcal{Y}_\psi([\gamma]) > 0$ *(respectively* $= 0, < 0$*).*

(ii) *There exists a metric* $\tilde{\gamma} \in [\gamma]$ *which satisfies* $(R(\tilde{\gamma}) - |\tilde{\nabla}\psi|^2_{\tilde{\gamma}}) > 0$ *everywhere on* Σ *(respectively* $= 0, < 0$*).*

(iii) *For any metric* $\tilde{\gamma} \in [\gamma]$*, the first eigenvalue,* λ_1*, of the self-adjoint, elliptic operator* $-L_{\tilde{\gamma},\psi}$ *is positive (respectively zero, negative).*

Solving the Lichnerowicz equation I

On a compact manifold our main results are as follows

	$\mathcal{B}_{\tau,\psi} < 0$	$\mathcal{B}_{\tau,\psi} \leq 0$	$\mathcal{B}_{\tau,\psi} \equiv 0$	$\mathcal{B}_{\tau,\psi} \geq 0$	$\mathcal{B}_{\tau,\psi} > 0$
$\mathcal{Y}_\psi([\gamma]) < 0$	N	N	N	N&S Cond.	Y
$\mathcal{Y}_\psi([\gamma]) = 0$	N	N	Y	N	N
$\mathcal{Y}_\psi([\gamma]) > 0$	PR	PR	N	N	N

Table 1: Results for $\mathcal{A}_{\gamma,W,\pi} \equiv 0$ and $\mathcal{B}_{\tau,\psi}$ of determined sign.

- Y The Lichnerowicz equation can be solved for that class of conformal data
- N The Lichnerowicz equation has no positive solution
- N&S There is a necessary and sufficient condition which needs to be checked
- PR We have partial results

Solving the Lichnerowicz equation II

On a compact manifold our main results are as follows

	$\mathcal{B}_{\tau,\psi} < 0$	$\mathcal{B}_{\tau,\psi} \leq 0$	$\mathcal{B}_{\tau,\psi} \equiv 0$	$\mathcal{B}_{\tau,\psi} \geq 0$	$\mathcal{B}_{\tau,\psi} > 0$
$\mathcal{Y}_\psi([\gamma]) < 0$	N	N	N	N&S Cond.	Y
$\mathcal{Y}_\psi([\gamma]) = 0$	N	N	N	Y	Y
$\mathcal{Y}_\psi([\gamma]) > 0$	PR	NR	Y	Y	Y

Table 2: Results for $\mathcal{A}_{\gamma,W,\pi} \not\equiv 0$ and $\mathcal{B}_{\tau,\psi}$ of determined sign.

- Y The Lichnerowicz equation can be solved for that class of conformal data.
- N The Lichnerowicz equation has no positive solution.
- N&S There is a necessary and sufficient condition which needs to be checked.
- PR We have partial results.
- NR We have no results indicating existence or non-existence.

Remarks on the Proofs I: Non-existence

We assume (via conformal invariance and Proposition 1) that $\mathrm{sign}(\mathcal{R}_{\gamma,\psi}) = \mathrm{sign}(\mathcal{Y}_\psi([\gamma]))$ and write the Lichnerowicz equation as

$$\Delta_\gamma \phi = \mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi)$$

where

$$\mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi) = \mathcal{R}_{\gamma,\psi}\,\phi - \mathcal{A}_{\gamma,W,\pi}\,\phi^{-\frac{3n-2}{n-2}} + \mathcal{B}_{\tau,\psi}\,\phi^{\frac{n+2}{n-2}}.$$

- All of the "N" entries in Tables 1 & 2 correspond to the situation where if ϕ were a positive solution on Σ then either $\mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi) \leq 0$ or $\mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi) \geq 0$ (but not identically zero). Integration then leads to an immediate contradiction.

Remarks on the Proofs II: Existence

We again assume that $\mathrm{sign}(\mathcal{R}_{\gamma,\psi}) = \mathrm{sign}(\mathcal{Y}_{\psi}([\gamma]))$. All the "Y" existence results are obtained by the method of sub- and super-solutions.

- Y's correspond to where one can directly find constant sub- and super-solutions.
- Y's correspond to where we first conformally transform the data via the positive solution to an well chosen linear equation, and then find constant sub- and super-solutions.
- The $\mathcal{A}_{\gamma,W,\pi} \not\equiv 0$ case (with $\mathcal{Y}_{\psi}([\gamma]) < 0$ and $\mathcal{B}_{\tau,\psi} \geq 0$) listed as "N&S Cond." may be reduced to the $\mathcal{A}_{\gamma,W,\pi} \equiv 0$ case, where this is the prescribed scalar curvature-scalar field problem

$$\mathcal{R}_{\tilde{\gamma},\psi} = -\mathcal{B}_{\tau,\psi}.$$

  The necessary and sufficient condition for solving this problem in the pure scalar curvature case is due to A. Rauzy. We believe that his argument generalizes to our setting.

Remarks and Questions

- We obtain similar results when (Σ, γ) is asymptotically flat.
- Our constructions allow for rough (low regularity) initial data (cf. work of Y. Choquet-Bruhat and D. Maxwell).
- For existence cases we can also (usually) show the uniqueness of the solution within the conformal class.
- Results can be extended to other related models, e.g. Einstein-Maxwell-scalar fields.
- Conformal gluing for Einstein-scalar field data ... away from the support of $V(\psi)$ (cf. Isenberg-Maxwell-P.)?
- Are there other (not esoteric) Einstein-matter/field models for which the basic questions of existence and non-existence of the constraint equations are not well understood?