Numerical construction of initial data

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Outline

1. The five conformal methods
2. Astrophysical binary black hole initial-data
3. Numerics
4. Non-uniqueness in the constraints

The five conformal methods

Initial data for GR

- Initial data consists of **metric g_{ij}** and **extrinsic curvature K_{ij}** on one hypersurface Σ.

- It must **satisfy the constraints**

$$R + \tau^2 - K_{ij}K^{ij} = 0$$

$$\nabla_j \left(K^{ij} - g^{ij}\tau \right) = 0$$

- **Strategy:** Split g_{ij} and K_{ij} into smaller pieces, some **freely specifiable**, the rest **completely determined**.

Choose free data $\Rightarrow$ **Solve** elliptic equations $\Rightarrow$ **Assemble** g_{ij}, K_{ij}

- **Long history:** Lichnerowicz, Choquet-Bruhat, York, O'Murchadha
  Recently, York (1999), HP & York (2003)

Conformal method

- Conformal metric $g_{ij} = \phi^4 \tilde{g}_{ij}$

  $\Rightarrow$ relationships between $(\Sigma, g_{ij}, \nabla_i)$ and $(\Sigma, \tilde{g}_{ij}, \tilde{\nabla}_i)$

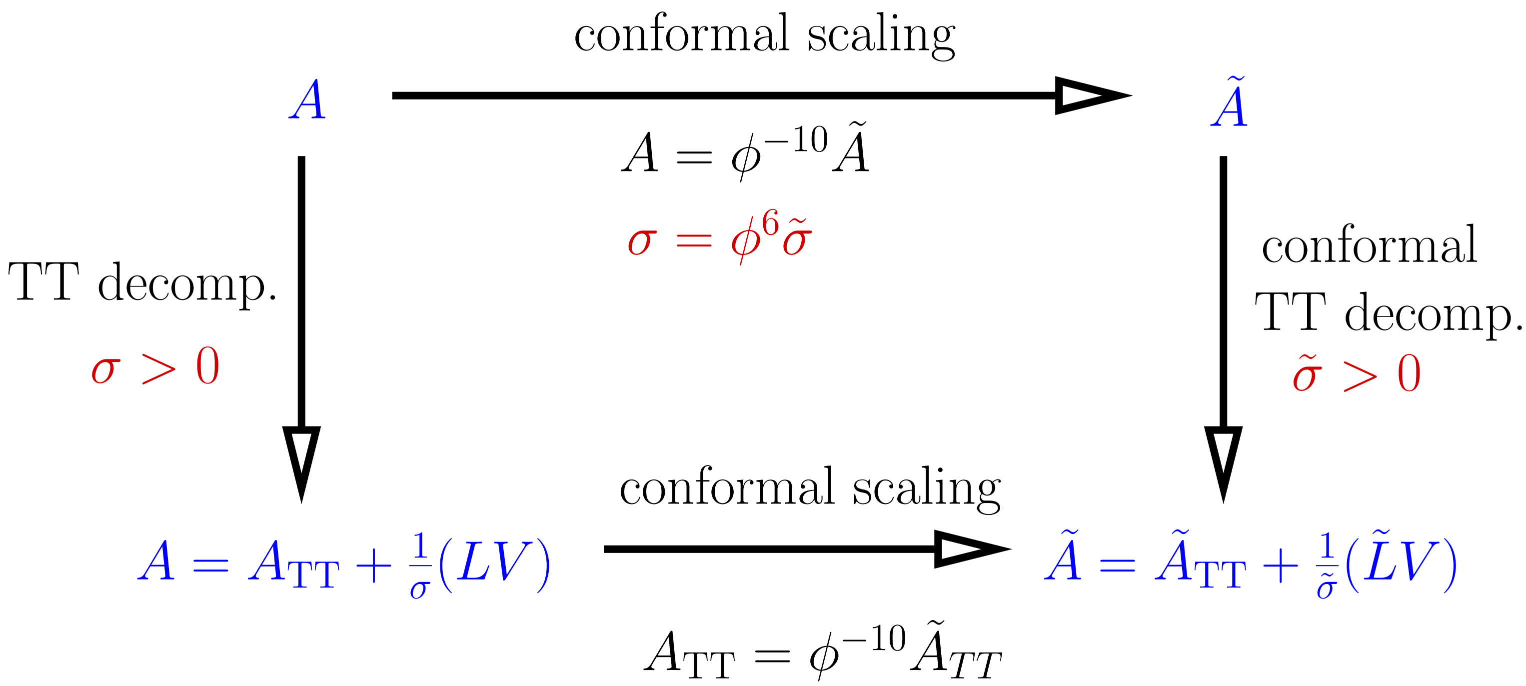
- Split K_{ij} into trace and trace-free parts, $K_{ij} = A_{ij} + \frac{1}{3}\tilde{g}_{ij}\tau$

  N.B. this talk never assumes CMC

- **Two alternatives** to deal with A_{ij}

  1. extrinsic curvature decomposition $A = A_{TT} + \frac{1}{\sigma}(\mathbb{L}V)$
  2. conformal thin sandwich equations (no decomposition required)

Extrinsic curvature decomposition (HP & York 2003)



- $\nabla A_{\rm TT} = 0 = \mathrm{Tr} A$
  $(\mathbb{L}V)_{ij} = 2\nabla_{(i}V_{j)} - \frac{2}{3}g_{ij}\nabla_k V^k$
- $\sigma = \phi^6\tilde{\sigma}$
- $\tilde{\nabla}\tilde{A}_{\rm TT} = 0 = \tilde{\mathrm{Tr}}\tilde{A}$
  $(\tilde{\mathbb{L}}V)_{ij} = 2\tilde{\nabla}_{(i}V_{j)} - \frac{2}{3}\tilde{g}_{ij}\tilde{\nabla}_k V^k$

- **Weighted TT-decomposition commutes with conformal scalings**

  cf. "old" method without σ: no commutation $\Rightarrow$ *TWO* variants

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Elliptic equations

Substitute decomposition and conformal scalings into constraints

$$\tilde\nabla^2\phi - \frac{1}{8}\tilde R\phi - \frac{1}{12}\phi^5\tau^2 + \frac{1}{8}\phi^{-7}\tilde A_{ij}\tilde A^{ij} = 0 \qquad \tilde A^{ij} = \tilde A^{ij}_{TT} + \frac{1}{\tilde\sigma}(\tilde{\mathbb{L}}V)^{ij}$$

$$\tilde\nabla_j\left(\frac{1}{\tilde\sigma}(\tilde{\mathbb{L}}V)^{ij}\right) - \frac{2}{3}\phi^6\tilde\nabla_i\tau = 0$$

Very similar to "old" conformal equations: Coupled quasi-linear elliptic PDEs on $(\Sigma, \tilde g_{ij})$

Free data:

1. $\tilde g_{ij}$, τ, $\tilde A^{ij}_{\rm TT}$ and $\tilde\sigma$
2. topology of Σ, boundary conditions on ϕ and V^i

Extrinsic curvature decomposition (Hamiltonian viewpoint)

"old" versions

York, O'Murchadha, 70's

– *two* inequivalent formulations

– only one conformally invariant

– only the other simplifies for CMC

Final version w/ weight-function (*EC*)

HP, York 2003

+ Eqns. decouple for CMC

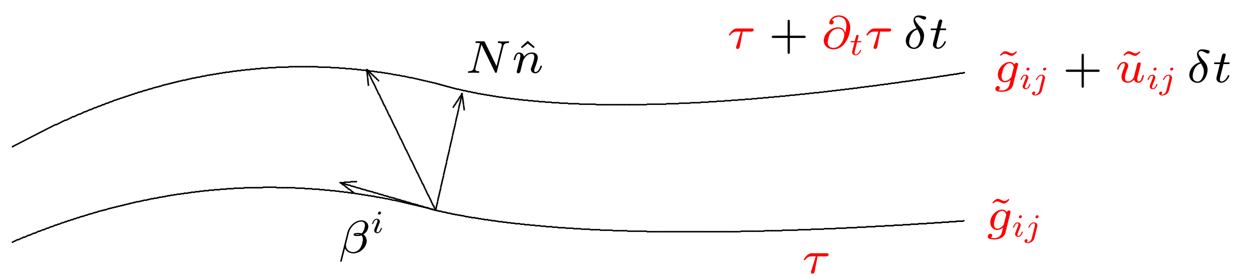
+ fully conformally invariant

+ Kerr has $A^{ij}_{\rm TT} = 0$

Conformal thin sandwich (Lagrangian viewpoint)

Conformal thin sandwich (CTS)

York 1999 – No decomposition; "just" say which free data you want



Specify these free data:

1. Standard CTS

   $\tilde{g}_{ij}, \tilde{u}_{ij} \equiv \partial_t\tilde{g}_{ij}, \tau, \tilde{N}$

2. Extended XCTS

   $\tilde{g}_{ij}, \tilde{u}_{ij} \equiv \partial_t\tilde{g}_{ij}, \tau, \partial_t\tau$

Elliptic equations follow:

$$\tilde{\nabla}^2\phi - \frac{1}{8}\tilde{R}\phi - \frac{1}{12}\tau^2\phi^4 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} = 0$$

$$\tilde{\nabla}_j\left(\frac{1}{2\tilde{N}}(\tilde{\mathbb{L}}\beta)^{ij}\right) - \frac{2}{3}\phi^6\tilde{\nabla}^i\tau - \tilde{\nabla}_j\left(\frac{1}{2\tilde{N}}\tilde{u}^{ij}\right) = 0$$

$$\tilde{\nabla}^2(\tilde{N}\phi^7) - \tilde{N}\phi^7\left(\frac{1}{8}\tilde{R} + \frac{5}{12}\tau^2\phi^4 + \frac{7}{8}\tilde{A}_{ij}\tilde{A}^{ij}\right) = -\phi^5(\partial_t - \beta^k\partial_k)\tau$$

Extrinsic curvature decomposition (Hamiltonian viewpoint)

Conformal thin sandwich (Lagrangian viewpoint)

"old" versions

York, O'Murchadha, 70's

– *two* inequivalent formulations

– only one conformally invariant

– only the other simplifies for CMC

– **neither equivalent to CTS**

— Extended system (XCTS) —

HP, York 2003

Five eqns. with free data

$$(\tilde{g}_{ij}, \partial_t \tilde{g}_{ij}; \tau, \partial_t \tau)$$

+ symmetrical free data $(q, \partial_t q)$

– complicated mathematical structure

Final version w/ weight-function (*EC*)

HP, York 2003

+ Eqns. decouple for CMC

+ fully conformally invariant

+ Kerr has $A^{ij}_{\mathrm{TT}} = 0$

+ equivalent to CTS

— Standard system (CTS) —

York, 1999

Four eqns. with free data

$$(\tilde{g}_{ij}, \partial_t \tilde{g}_{ij}, \tau, \tilde{N})$$

Conformal lapse $\tilde{N} = \phi^{-6} N$

+ equivalent to EC

Astrophysical binary black hole initial data

Binary black hole coalescence is a prime scientific target of GW detectors

- GEO 600
- LIGO (at design sensitivity)
- TAMA 300
- Virgo

Ligo Hanford site

Numerical simulations need **initial data**

Astrophysically realistic BBH initial data

Q1: Which formalism gives best control of properties of resulting ID?

Q2: How to choose free data and boundary conditions for a binary black hole in circular orbits?

Extrinsic curvature decomposition (Hamiltonian viewpoint)

"old" versions

York, O'Murchadha, 70's

– *two* inequivalent formulations

– only one conformally invariant

– only the other simplifies for CMC

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Final version w/ weight-function (*EC*)

HP, York 2003

+ Eqns. decouple for CMC

+ fully conformally invariant

+ Kerr has $A^{ij}_{\rm TT}=0$

+ equivalent to CTS

– Choice of $\tilde A^{ij}_{\rm TT}$ and $\tilde\sigma$ difficult

(HP, Cook, Teukolsky, 2002)

Conformal thin sandwich (Lagrangian viewpoint)

— Extended system (XCTS) —

HP, York 2003

Five eqns. with free data

$$(\tilde g_{ij}, \partial_t\tilde g_{ij}; \tau, \partial_t\tau)$$

+ symmetrical free data $(q, \partial_t q)$

– complicated mathematical structure

+ Time-derivatives more intuitive

+ often natural choice exists

+ obtain N, β^i

— Standard system (CTS) —

York, 1999

Four eqns. with free data

$$(\tilde g_{ij}, \partial_t\tilde g_{ij}, \tau, \tilde N)$$

Conformal lapse $\tilde N = \phi^{-6}N$

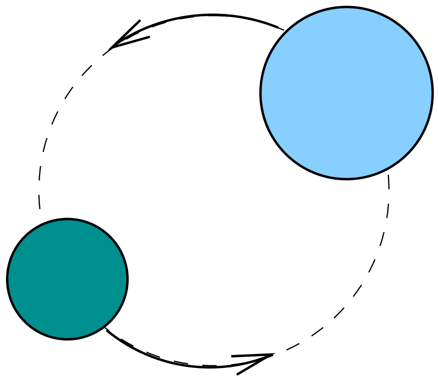
+ equivalent to EC

+ obtain N, β^i

– Choice of $\tilde N$ difficult

Quasi-equilibrium method

Basic idea:
Approx. time-independence in corotating frame
Approx. helical Killing vector
(both concepts essentially equivalent,
both useful depending on context)



History:

- Wilson & Matthews, 1985: Binary neutron stars
- Gourgoulhon, Grandclement & Bonazzola, 2002a,b: isometry BCs
- Cook & HP, 2002, 2003, 2004: Isolated horizon BCs
  Improvements over GGB:
  constraints satisfied, general spins possible, lapse positive, allows general $\tilde{g}_{ij}$, τ

Quasi-equilibrium method (the easy pieces)

- **Time-independence in corotating frame**

  $\Rightarrow$ natural choice: vanishing time derivatives

- **Conformal thin sandwich formalism**
  1. $\partial_t \tilde{g}_{ij} = 0 = \partial_t \tau$
  2. $\tilde{g}_{ij}$ and τ still undetermined

- **Boundary conditions at infinity**

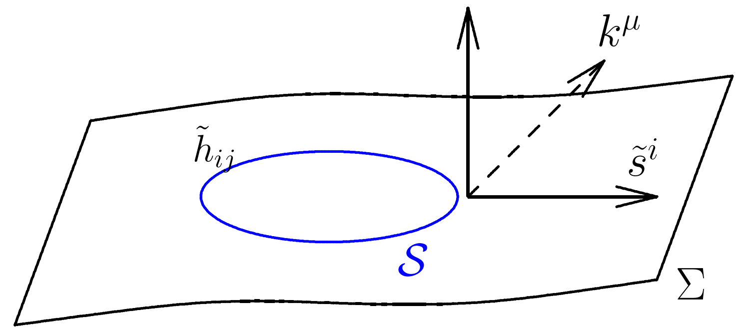
$$\phi = 1$$

$$\beta^i = (\vec{\Omega}_{\rm orbital} \times \vec{r})^i$$

$$N = 1$$

- **New contribution**: *inner boundary conditions* (next slide...)

Quasi-equilibrium excision boundary conditions



- **Excise** topological spheres $B_{1,2}$
- **Require**
  1. $\partial B_{1,2}$ be apparent horizon(s)
  2. The AH's remain stationary in evolution
  3. Shear of k^μ vanishes on $\partial B_{1,2}$ (isolated horizon)

  $\Rightarrow \mathcal{L}_k\theta = 0 \Rightarrow$ **AH moves along k^μ and $M_{\rm AH}$ initially constant**

- **Rewrite** in conformal variables $\Rightarrow$

$$\tilde{s}^k\tilde{\nabla}_k \ln\phi = -\frac{1}{4}\tilde{h}^{ij}\tilde{\nabla}_i\tilde{s}_j + \frac{1}{6}\phi^2\tau - \frac{\phi^2}{8N}\tilde{s}_i\tilde{s}_j(\tilde{\mathbb{L}}\beta)^{ij} \qquad \text{on } \partial B_{1,2}$$

$$\beta^i = \phi^2 N\tilde{s}^i + \beta^i_{||} \qquad \text{on } \partial B_{1,2}S$$

Boundary conditions for ϕ and β^i

Summary of equations

Computational domain $\Omega = \mathbf{R}^3 - B_1 - B_2$

$$\tilde\nabla^2\phi - \frac{1}{8}\tilde R\phi - \frac{1}{12}\tau^2\phi^4 + \frac{1}{8}\tilde A_{ij}\tilde A^{ij} = 0$$

$$\tilde\nabla_j\left(\frac{1}{2\tilde N}(\tilde{\mathbb{L}}\beta)^{ij}\right) - \frac{2}{3}\phi^6\tilde\nabla^i\tau = 0$$

$$\tilde\nabla^2(\tilde N\phi^7) - \tilde N\phi^7\left(\frac{1}{8}\tilde R + \frac{5}{12}\tau^2\phi^4 + \frac{7}{8}\tilde A_{ij}\tilde A^{ij}\right) - \phi^5\beta^k\partial_k\tau = 0$$

Boundary conditions

$$\tilde s^k\tilde\nabla_k \ln\phi = -\frac{1}{4}\tilde h^{ij}\tilde\nabla_i\tilde s_j + \frac{1}{6}\phi^2\tau - \frac{\phi^2}{8N}\tilde s_i\tilde s_j(\tilde{\mathbb{L}}\beta)^{ij} \quad \text{on } \partial B_{1,2}$$

$$\beta^i = \phi^2 N\tilde s^i + \beta^i_{\parallel} \quad \text{on } \partial B_{1,2}$$

$$\tilde s^k\tilde\nabla_k(\tilde N\phi^7) = 0 \quad \text{on } \partial B_{1,2}$$

$$\phi = 1,\ \ \beta^i = (\vec\Omega_{\rm orbital}\times\vec r)^i,\ \ N = 1 \quad \text{at } \infty$$

Numerics

Pseudospectral Collocation Method

- Nonlinear PDE $\mathcal{L}f = 0$
- Expand solution

$$f(x) \approx f^{(N)}(x) \equiv \sum_{n=0}^{N-1} \tilde{f}_n \phi_n(x).$$

$\phi_n(x)$ Chebyshev, spherical harmonics, ...

- Derivatives known *analytically*

$$\frac{df^{(N)}(x)}{dx} = \sum_{n=0}^{N-1} \tilde{f}_n \frac{d\phi_n(x)}{dx}$$

- Determine spectral coefficients $\tilde{f}_0, \ldots, \tilde{f}_N$, from

$$(\mathcal{L}f^{(N)})(x_i) = 0$$

for given collocation points $x_0, \ldots, x_N$.

- Use Newton-Raphson for nonlinear $\mathcal{L}$
  Solve linear systems with preconditioned Krylov-subspace methods

Domain decomposition

Cover computational domain with subdomains each having a separate set of basis functions and spectral coefficients.

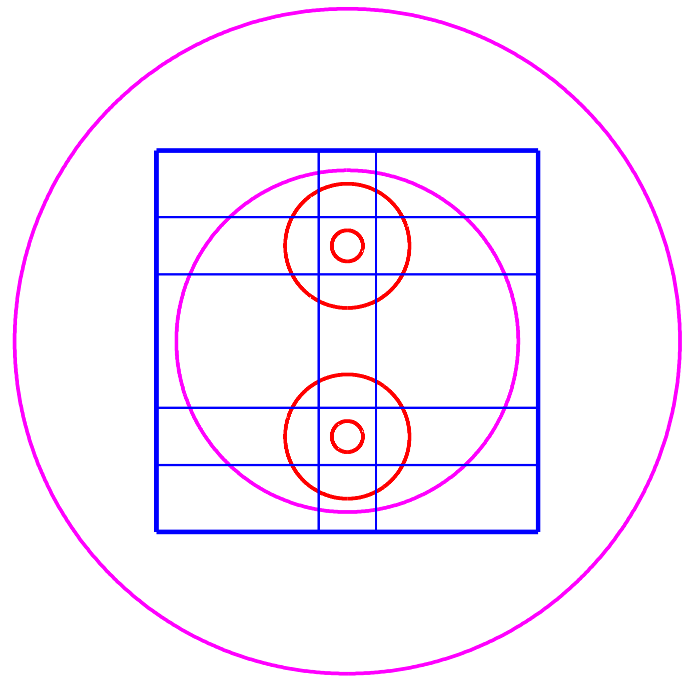
Implemented shapes:

- Blocks (incl. cubed sphere)
- Spherical shells
- Cylinders
- Overlapping and/or touching

Advantages

- flexible topology
- flexible resolution
  - Number of points
  - Mappings
- Excise singularities

At subdomain boundaries match function value and derivative.

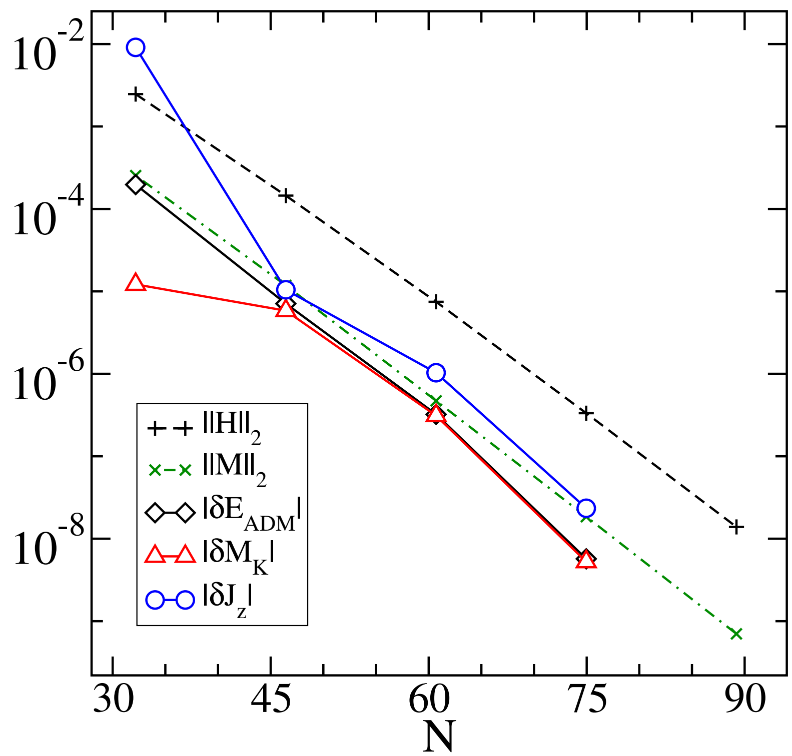


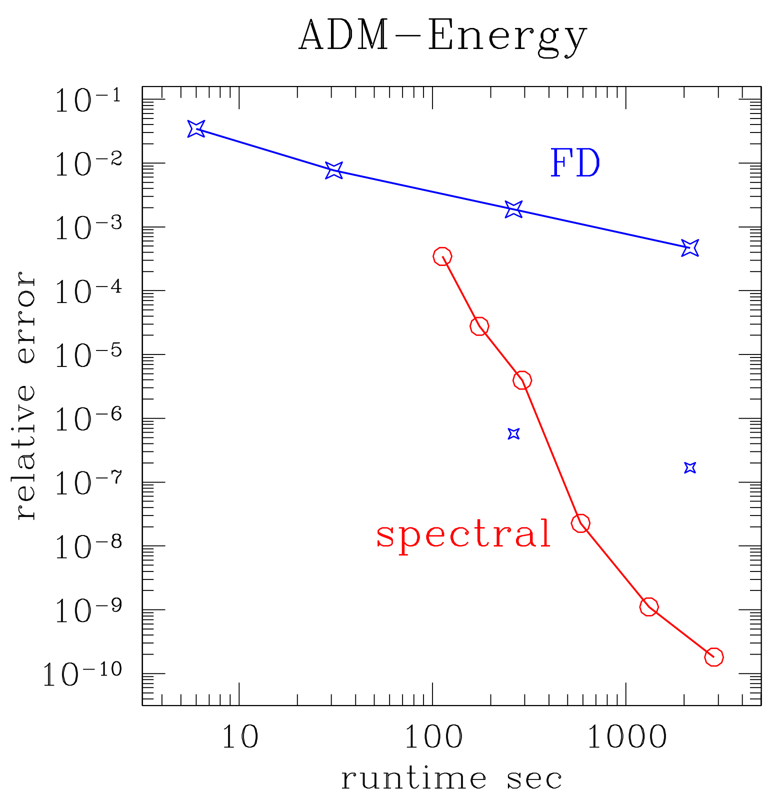
2 inner spherical shells (centers excised)
43 blocks
1 outer spherical shell

46 subdomains total

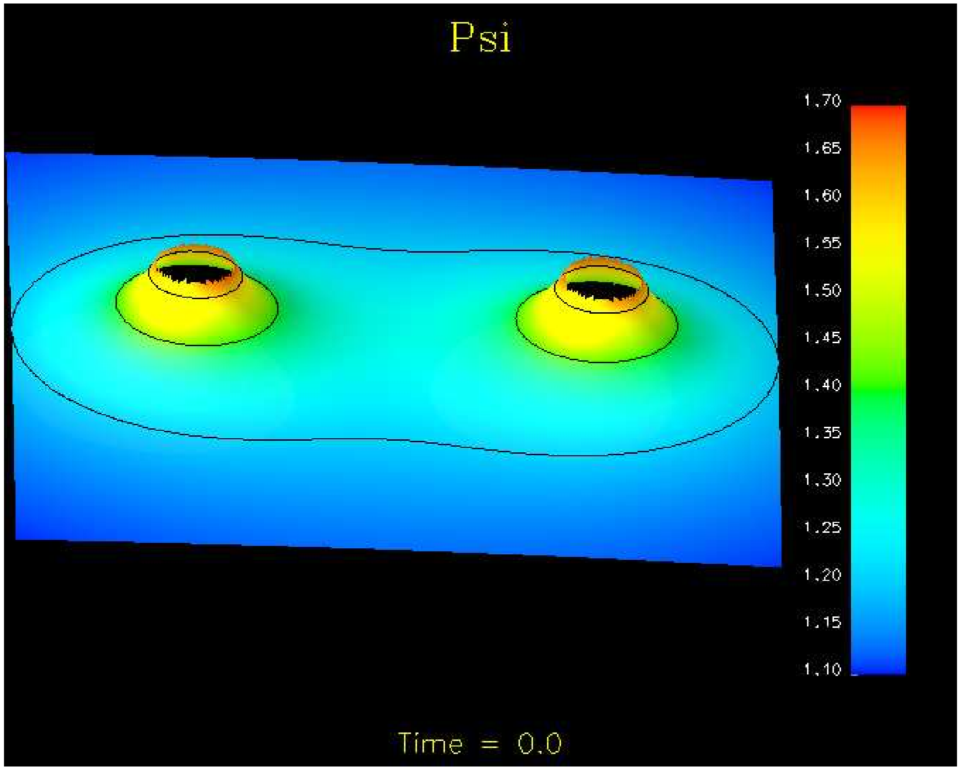
Smooth solutions ⇒ exponential convergence

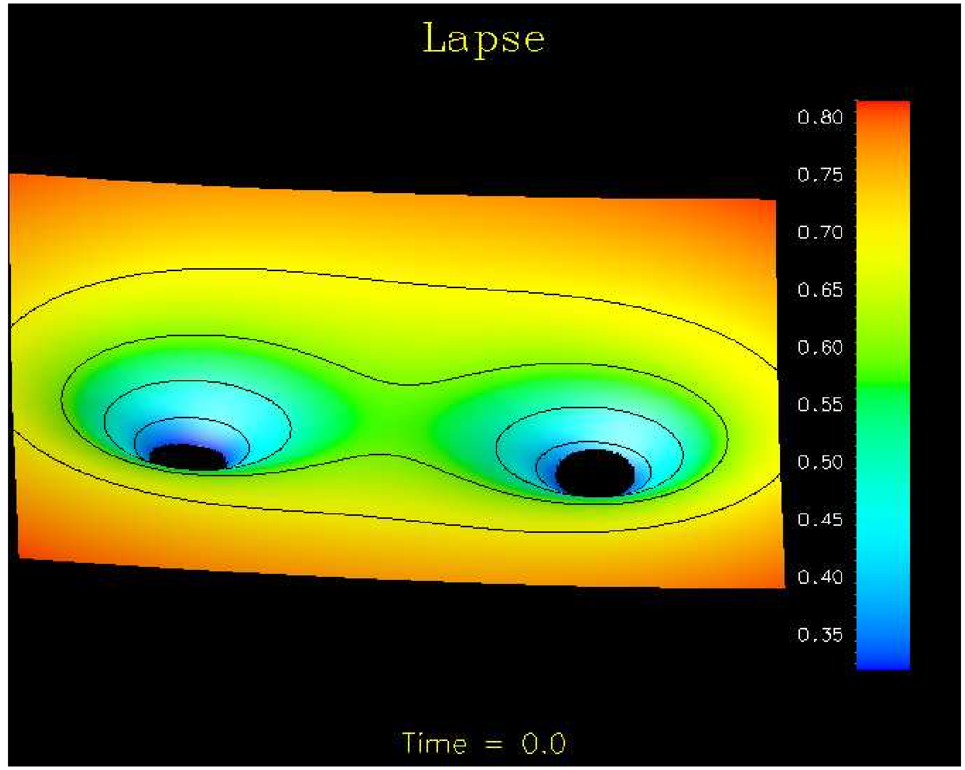
- *Superior accuracy:* Numerical errors ≪ physical effects
- *Superior efficiency:* Large parameter studies
- *Domain decomposition:* Nontrivial topologies & Multiple length-scales





Binary black hole solutions I

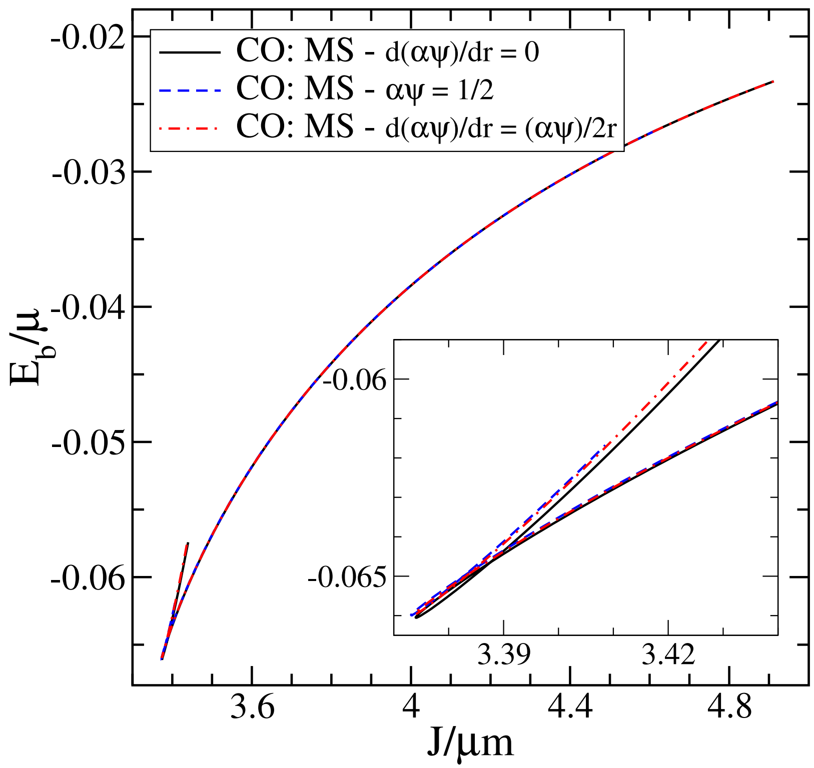




Lapse positive through horizon

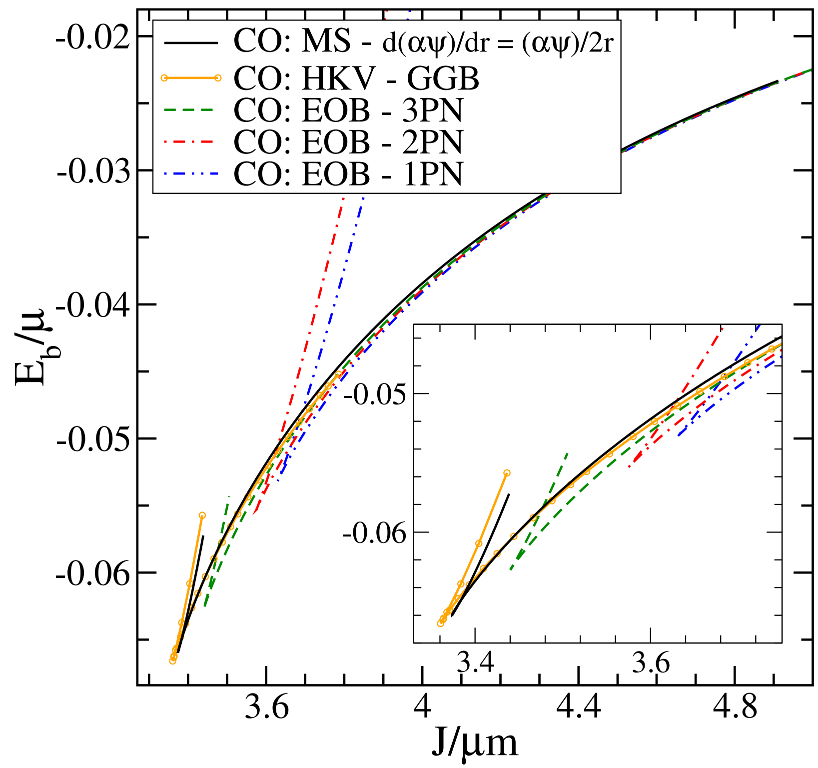
Sequences of quasi-circular orbits (corotating, $\tau = 0$)

Three different lapse boundary conditions



No difference – solution robust

Compare to GGB and post-Newtonian results

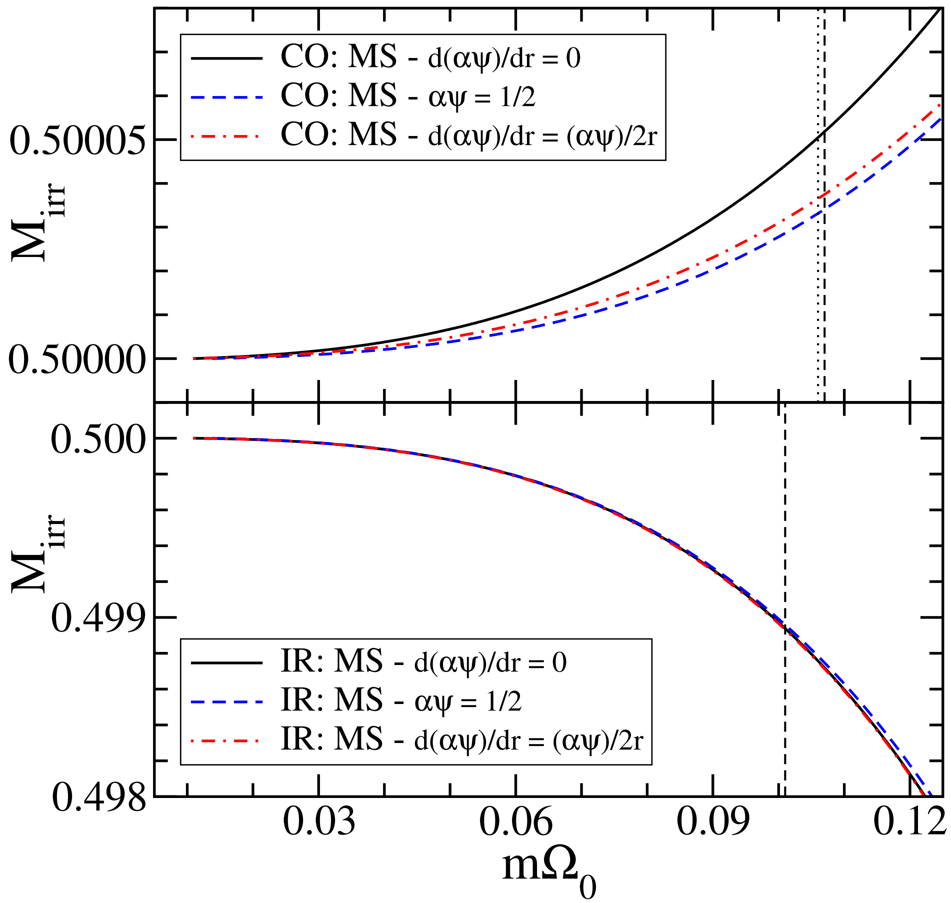


Excellent agreement

Testing the 2nd law

Normalize sequences by $\quad dE_{\mathrm{ADM}} = \Omega_0\, dJ_{\mathrm{ADM}}$

Irreducible mass along these sequences



⇐ corotating sequences
(three different lapse BC's)
$M_{\rm irr}$ slightly increasing during inspiral

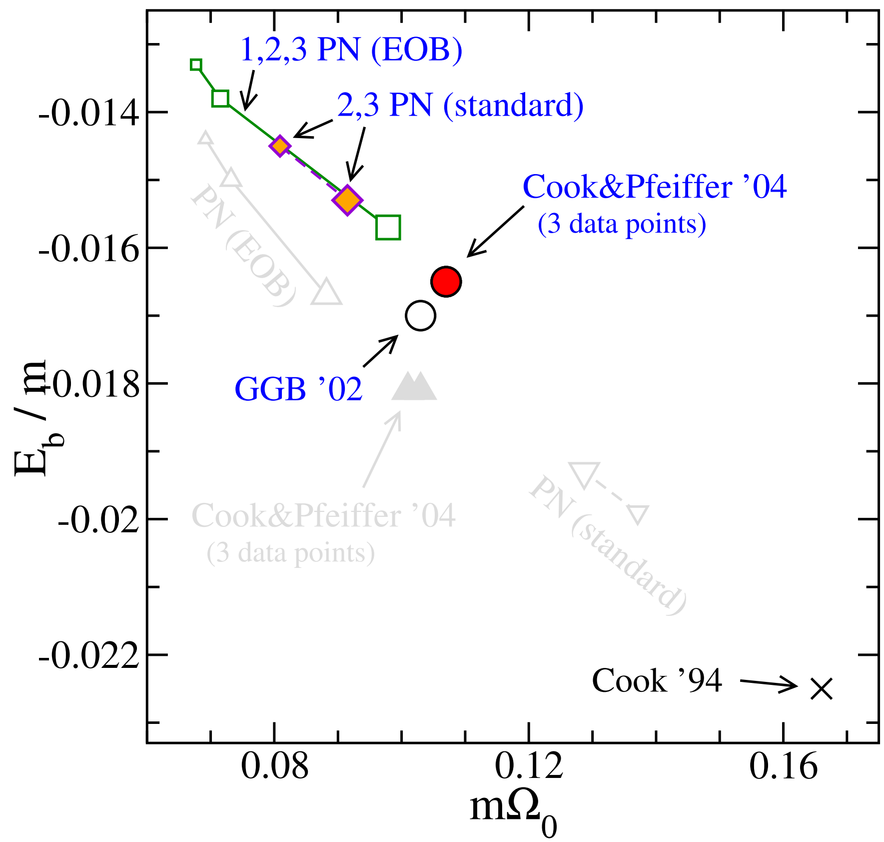
⇐ irrotational sequences
(three different lapse BC's)
$M_{\rm irr}$ decreasing during inspiral
– Definition of Spin insufficient (Cook)

ISCO location

Caution: ISCO is not a sharp, well-defined concept! Anyway...

Color: Corotating BH's

Grey: Irrotational BH's



- Excellent agreement between NR and PN
- Superior to Bowen-York w/ effective potential (cf. Cook 1994)

(Non-)uniqueness

Conformal thin sandwich equations are quite complicated.
Should we worry about finding solutions?

- **Existence?**
  - ⋆ CTS: Many mathematical results
  - ⋆ XCTS: **terra incognita**

- **Uniqueness?**
  - ⋆ CTS: (Almost) always unique
  - ⋆ XCTS: **terra incognita**

Some results on standard CTS equations

Free data based on "Teukolsky wave"
ingoing, $M=0$, odd parity, centered at $r=20$
(HP, Kidder, Scheel, Shoemaker 2005)

Mathematics:

1. asymptotically flat
2. no inner boundaries
3. maximal slice $\tau = 0$

$\longrightarrow$ Yamabe constant $\mathcal{Y}[\tilde{g}_{ij}]$:

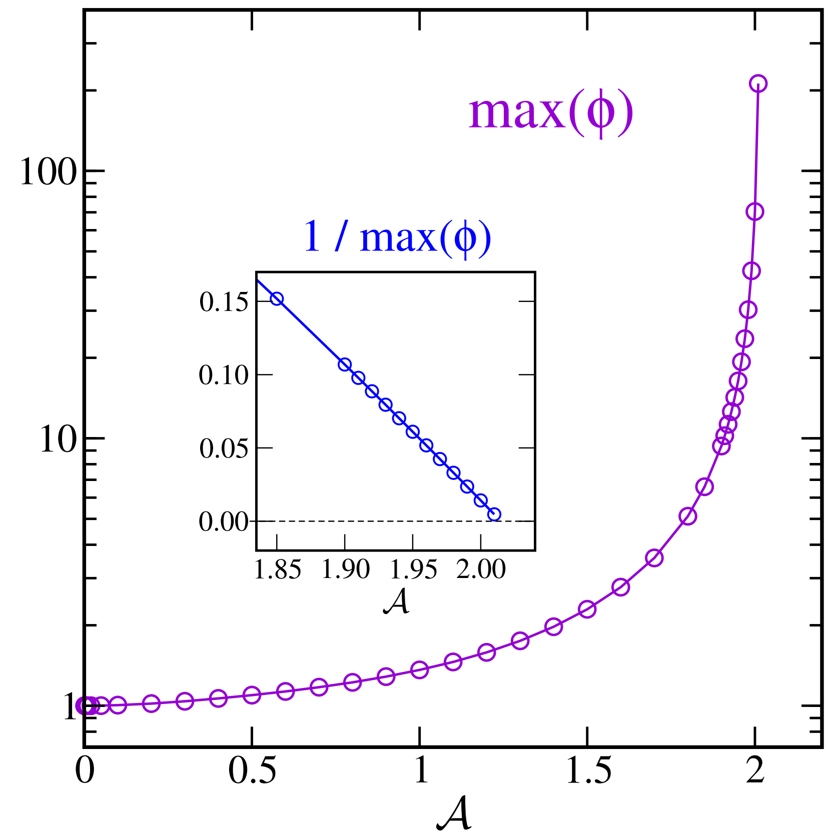
$\mathcal{Y}[\tilde{g}_{ij}] > 0 \Leftrightarrow$ existence & uniqueness

(Cantor 1977, Murray & Cantor 1981, Maxwell 2005)

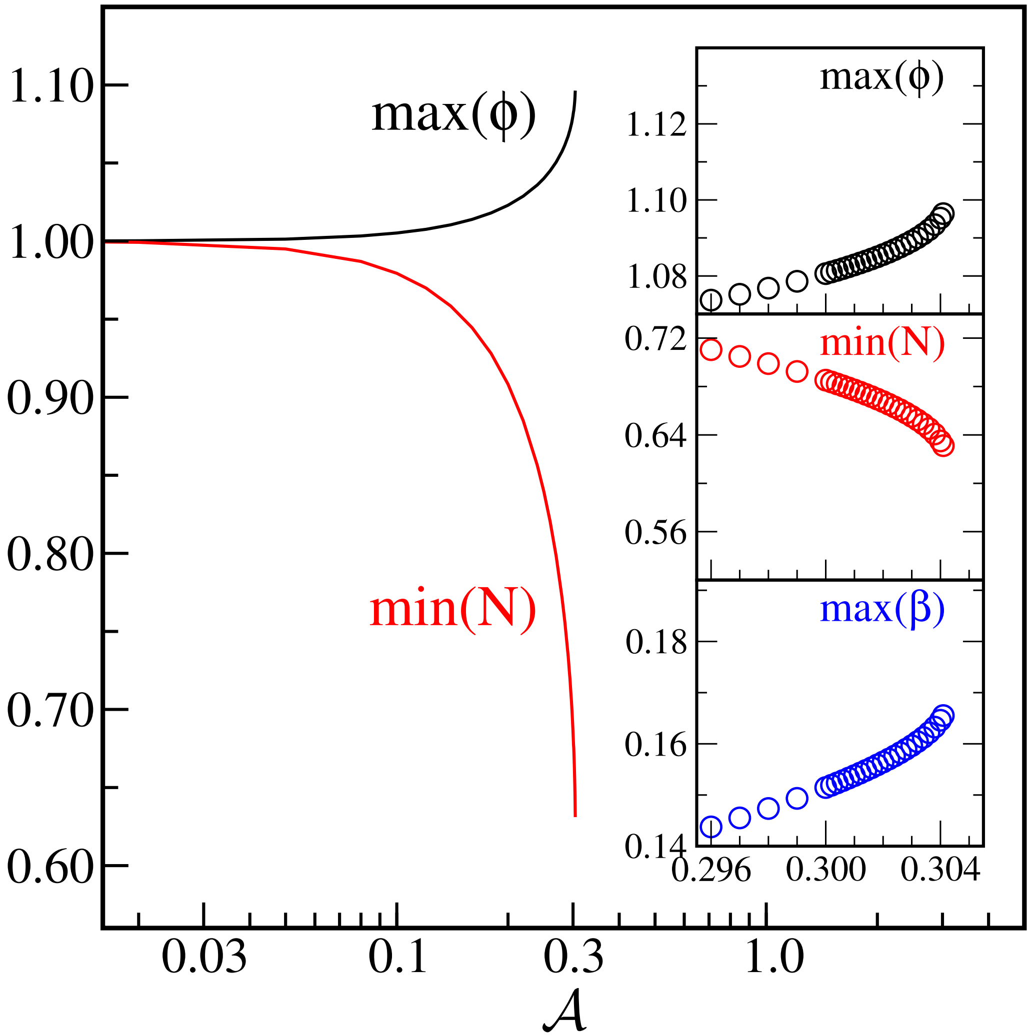
$$\tilde{g}_{ij} = \delta_{ij} + \mathcal{A}\, h_{ij}$$

$$\tilde{u}_{ij} = \quad \mathcal{A}\, \partial_t h_{ij}$$

$$\tau = 0, \quad \tilde{N} = 1$$



Extended system (HP & York, 2005)



$$\tilde{g}_{ij} = \delta_{ij} + \mathcal{A}\, h_{ij}$$

$$\tilde{u}_{ij} = \quad \mathcal{A}\, \partial_t h_{ij}$$

$$\tau = 0, \quad \partial_t \tau = 0$$

ϕ finite

$\phi \to 1.099$ as $\mathcal{A} \to \mathcal{A}_c$

Parabolic behavior

$\phi - \phi_c \propto (\mathcal{A}_c - \mathcal{A})^{1/2}$

Different $\mathcal{A}_c$

CTS $\mathcal{A}_c \approx 2.0$, XCTS $\mathcal{A}_c \approx 0.3$

Two branches

- We have found

$$\mathbf{u}_-(\mathcal{A}) = \mathbf{u}_c - \mathbf{v}_c\sqrt{\delta\mathcal{A}} \qquad\qquad \mathbf{u} = \Big(\phi, \beta^i, N\Big)$$

- *Second* branch??

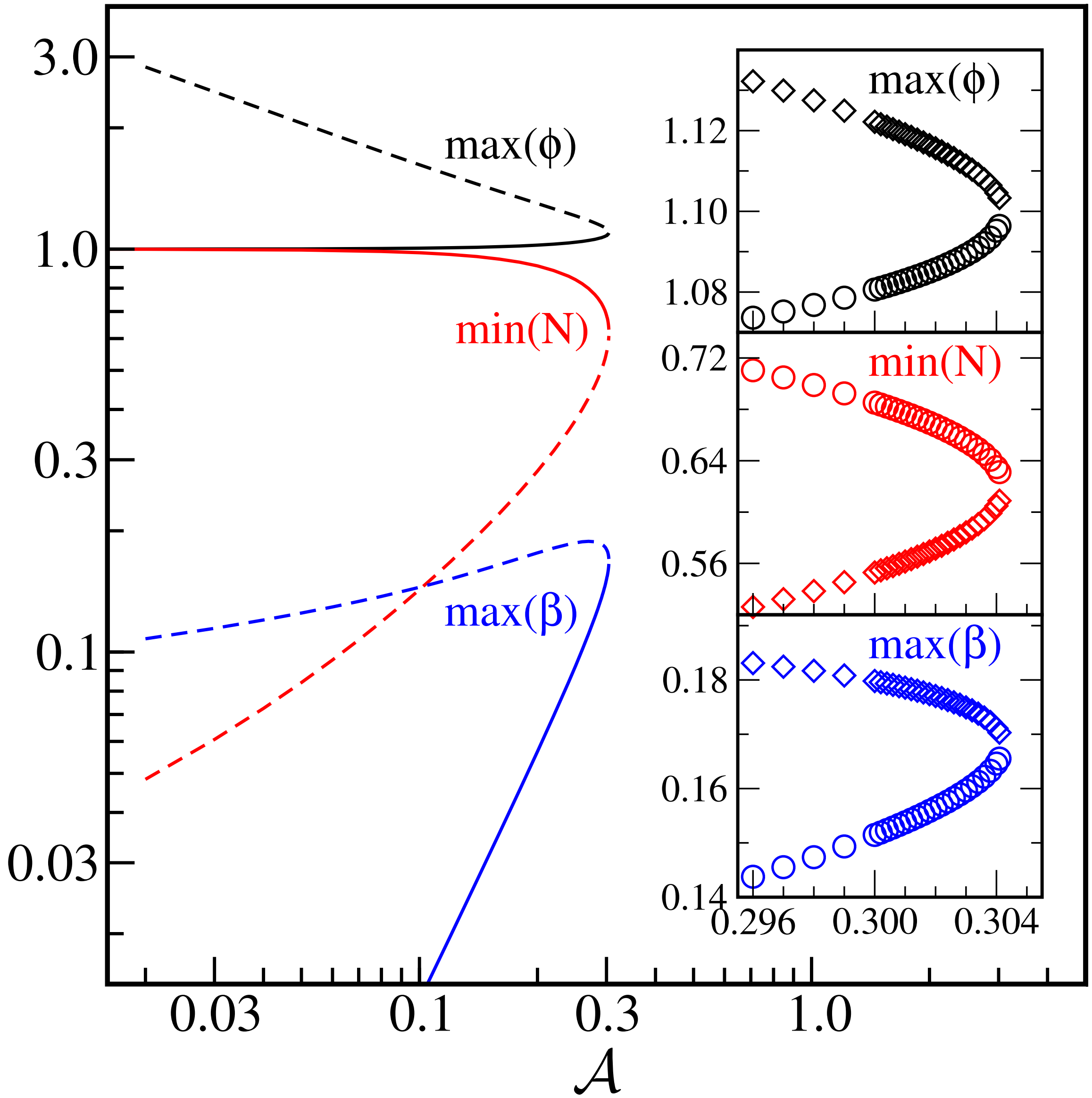
$$\mathbf{u}_+(\mathcal{A}) = \mathbf{u}_c + \mathbf{v}_c\sqrt{\delta\mathcal{A}}$$

- Problem: need good guess to converge to $\mathbf{u}_+$

$$\frac{\mathrm{d}\mathbf{u}_-(\mathcal{A})}{\mathrm{d}\mathcal{A}} = \frac{1}{2\sqrt{\delta\mathcal{A}}}\mathbf{v}_c \quad \Rightarrow \quad \mathbf{u}_+(\mathcal{A}) \approx \mathbf{u}_-(\mathcal{A}) + 4\,\delta\mathcal{A}\,\frac{\mathrm{d}\mathbf{u}_-(\mathcal{A})}{\mathrm{d}\mathcal{A}}$$

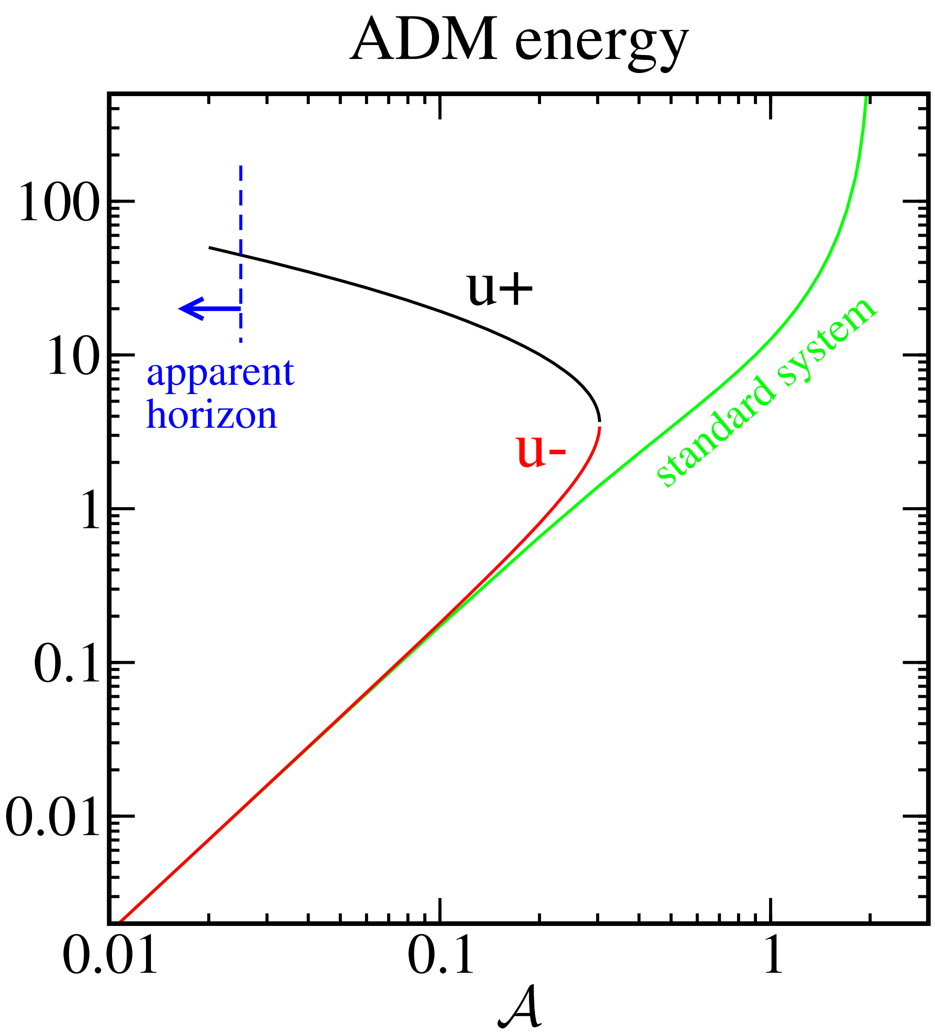
- **Finite-difference** numerical solutions $\mathbf{u}_-$ of five coupled nonlinear 3D elliptic equations!

Constructing the upper branch $\mathbf{u}_+$

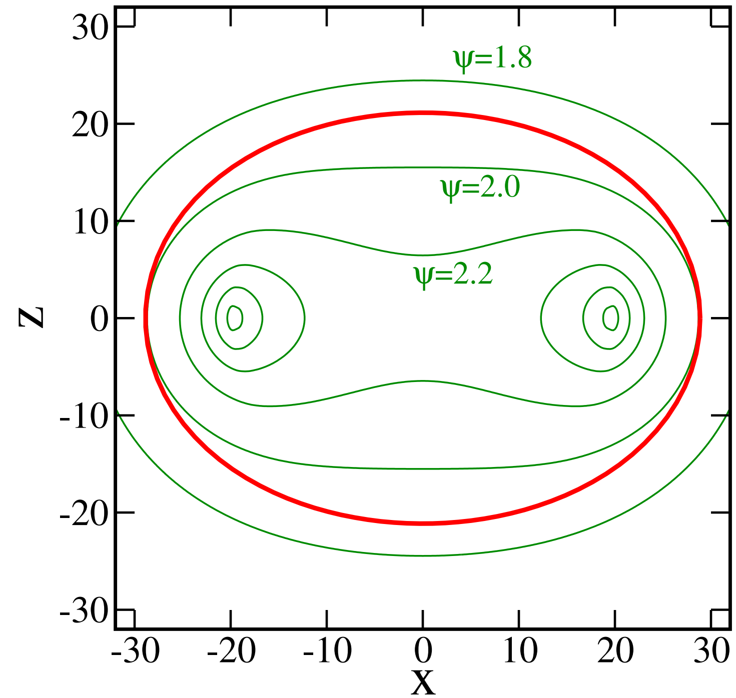


- Parabolic close to $\mathcal{A}_c$
- $\mathbf{u}_+$ and $\mathbf{u}_-$ meet at $\mathcal{A}_c$
- $\mathbf{u}_+$ deviates strongly from Minkowski at small $\mathcal{A}$
- **Two solutions for arbitrarily small $\mathcal{A}$!!**

Energy & Apparent horizon



Apparent horizons for $\mathcal{A} = 0.02$



A first lead by N O'Murchadha

The fifth equation is written in a deceiving form

$$\tilde{\nabla}^2(\tilde{N}\phi^7) - \tilde{N}\phi^7\left(\frac{1}{8}\tilde{R} + \frac{5}{12}\tau^2\phi^4 + \frac{7}{8}\tilde{A}_{ij}\tilde{A}^{ij}\right) - \phi^5\beta^k\partial_k\tau = 0$$

$\tilde{A}^{ij} = \frac{1}{2\tilde{N}}(\tilde{\mathbb{L}}\beta)^{ij}$, therefore there is a term

$$\tilde{\nabla}^2(\tilde{N}\phi^7) - \frac{7}{32}\left(\tilde{N}\phi^7\right)^{-1}\phi^{14}(\tilde{\mathbb{L}}\beta)_{ij}(\tilde{\mathbb{L}}\beta)^{ij} + \ldots = 0$$

with the wrong sign!

Summary

1. *The five conformal methods*
   - There are only 1.5:
   - EC/CTS (with weight-functions) supersedes the “old” methods
   - XCTS is an interesting and challenging variation

2. Astrophysical binary black hole initial data
   - XCTS equations preferred for physical reasons
   - Numerical solutions exist; mathematical results would be welcome

3. *Numerics*
   - Pseudospectral elliptic solver efficient, robust and flexible

4. *(Non-)uniqueness in the constraints*
   - XCTS has non-unique solutions
   - What happens at the critical point?
   - What determines existence of solutions? (not Yamabe!)