Some Applications of Scalar Curvature Deformation in General Relativity

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TYPES OF DEFORMATION

- CONFORMAL

- LOCALIZED
  - LOHKAMP
  - (LOCAL) FISCHER-MARSDEN

APPLICATIONS

- CONSTRUCT SOLUTIONS OF CONSTRAINT $R(g) = 0$ WHICH ARE IDENTICALLY SCHWARZSCHILD OUTSIDE A COMPACT SET.
  (FULL CONSTRAINTS: C-Schoen; Chruściel-Delay)
- CONSTRUCT SUCH DATA SO THAT EINSTEIN EVOLUTION IS ASYMPTOTICALLY SIMPLE (PENROSE COMPACTIFICATION)
  (Chruściel-Delay; C.)
- CONSTRUCT MULTI-HORIZON DATA SETS, INCLUDING EXAMPLES WITH TRIVIAL TOPOLOGY.
  (Chruściel-Delay, Chruściel-Mazzeo; P. Miao; C.)
- LOCALIZED GLUING CONSTRUCTIONS.
  (Isenberg-Mazzeo-Pollack; Chruściel-Delay, Chruściel-Isenberg-Pollack)
  C.

LOCAL FISCHER-MARSDEN: $\Omega \Subset (M,g)$ smooth.

Suppose L_g^* has trivial Kernel on Ω.

Ω not static

adjoint of $L_g = R_g'$,

$L_g^*(f) = -(\Delta_g f)g + \mathrm{Hess} f - f\,\mathrm{Ric}(g)$.

There exists $\varepsilon > 0$ s.t. for $S \in C_c^\infty(\Omega)$, $\|S\|_{C^{0,\alpha}} < \varepsilon$,

there is a smooth $h \in C_c^\infty(\bar{\Omega})$ s.t. $\|h\|_{C^{2,\alpha}} \leq C\|S\|_{C^{0,\alpha}}$ and

$$\boxed{R(g+h) = R(g) + S}$$

STATIC VERSION (FLAT CASE): $\Omega \Subset \mathbb{R}^3$ smooth

There is a nbhd. U of flat metric in $C^{4,\alpha}(\bar{\Omega})$ and an $\varepsilon > 0$ s.t. for smooth $g \in U$ and $S \in C_c^\infty(\Omega)$, $\|S\|_{C^{0,\alpha}} < \varepsilon$, there is a smooth $h \in C_c^\infty(\bar{\Omega})$ with $\|h\|_{C^{2,\alpha}} \leq C\|S\|_{C^{0,\alpha}}$ and

$$\boxed{R(g+h) - (R(g) + S) \in \zeta K}$$

where $K = \ker L_\delta^* = \mathrm{sp}\{1, x^1, x^2, x^3\}$, ζ = fixed bump fn $\in C_c^\infty(\Omega)$.

APPLICATION: C. '99/'00

(M,g) AF, $R(g)=0$, $N \subset M$ is AF end
$$\begin{array}{c} N \\ \cup\!\!\!| \\ E_r \cong \{|x| \geq r\} \end{array}$$

Suppose $g = \left(1+\frac{m_0}{2|x|}\right)^4 \delta + O(|x|^{-2})$ (dense by Schoen-Yau)

For all $\varepsilon>0$, there is an $R>0$ and a smooth $\bar{g}$ with $\boxed{R(\bar{g})=0}$, $\bar{g} = \begin{cases} g & \text{on } M \setminus E_R \\ g_S & E_{2R} \end{cases}$

an appropriate Schwarzschild

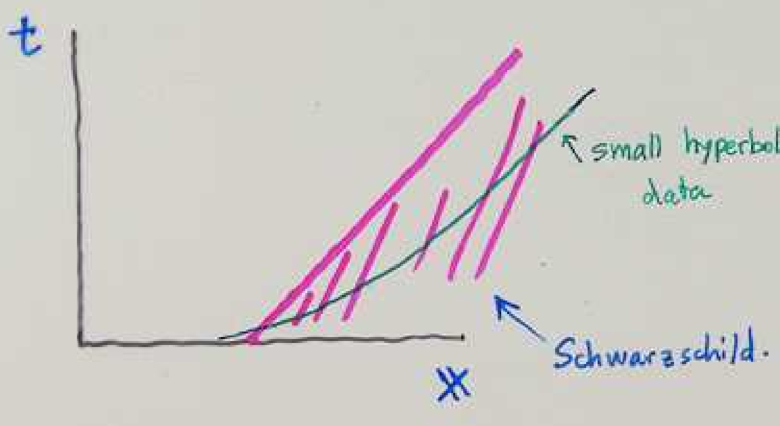
IDEA OF PROOF: PATCH g to $g_S = \left(1+\frac{m}{2|x-c|}\right)^4 \delta$ in $A_R = B_{2R} \setminus B_R$. TO KILL SCALAR CURVATURE in A_R, USE LOCAL DEFORMATION TO HANDLE PROJECTION TRANSVERSE TO $\mathcal{S}K$, USE PARAMETERS m, c TO HANDLE PROJECTION TO $\mathcal{S}K$.

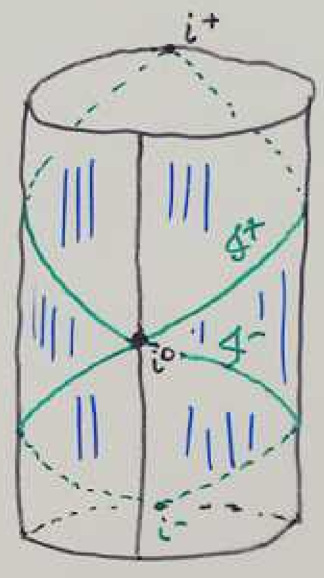
MOTIVATION TO MAKE SUCH A CONSTRUCTION :

- INSPIRED BY CUTLER-WALD '89 : CONSTRUCTION OF SOLUTION TO EINSTEIN-MAXWELL CONSTRAINTS $R(g) = 2|B|^2$, $\mathrm{div}_g B = 0$, S.T. B COMPACTLY SUPPORTED AND g IDENTICAL TO SCHWARZSCHILD OUTSIDE BALL CONTAINING SUPPORT OF B.

- SCALING B to ϵB, PRODUCE SMALL DATA FOR EINSTEIN EVOLUTION, SCHWARZSCHILD OUTSIDE FIXED BALL, TENDING TO MINKOWSKI DATA.

- APPLY H. FRIEDRICH STABILITY THEOREM





FIRST EXISTENCE RESULT : CHRUŚCIEL – DELAY

- APPLY CONSTRUCTION TO FAMILIES OF SMALL INITIAL DATA WITH **PARITY SYMMETRY** ($x \mapsto -x$).

  THE PATCHING AND COMPACT PERTURBATION CAN BE DONE PRESERVING THE SYMMETRY. HENCE,

$$\boxed{\int_{A_R} x^i R(g)\, dx = 0 \qquad i = 1,2,3}$$

- FOR THE PROJECTION ONTO CONSTANT FUNCTIONS, WE USE $R(g) \approx \sum_{i,j} g_{ij,ij} - g_{ii,jj}$ (R large) :

$$\int_{A_R} R(g)\, dx \approx \int_{\partial A_R} (g_{ij,i} - g_{ii,j})\nu^j\, d\xi$$

$$\approx 16\pi\,(m - m_0)$$

ESTIMATE UNIFORM AS DATA TENDS TO FLAT

FOR LARGE ENOUGH R (A SINGLE R WORKS FOR WHOLE FAMILY)

THE QUEST FOR MORE DATA

g AF

- Suppose $R(g)=0$, with $g = u^4\delta$ outside compact set:

$$\Delta u = 0 \quad \text{outside compact set}$$

$$u \to 1 \quad \text{at } \infty$$

center. of. mass

$$u(x) = 1 + \frac{m}{2|x|} - \frac{m\vec{c}\cdot\vec{x}}{2|x|^3} + O(|x|^{-3}).$$

- In above construction (without imposing parity)

$$\boxed{\alpha_0 \int_{A_R} x \cdot R(g)\, dx \approx mc - m_0 c_0 .}$$

if $m_0 \downarrow 0$, may need to take $R \uparrow$ to force error terms to be small **enough** (for given range of c)

- **"Enough"**: use fixed-point/degree methods to guarantee choice of (m,c) near (m_0,c_0) to make $R(g)=0$.

- **Idea**: arrange asymptotics so mass dominates

Theorem (C., '05-'06) (LINEARIZATION-STABILITY-TYPE THM.)

Let $h \in C_c^\infty$ (symmetric (0,2)) satisfy $L(h) = 0$

$L = L_\delta = R'_\delta$

For $|\varepsilon|$ small, consider $g_\varepsilon = u_\varepsilon^4 (\delta + \varepsilon h)$ AF, $R(g_\varepsilon) = 0$. There is $R_0 > 0$ s.t. for all ε small enough, there is $\bar{g}_\varepsilon$, $R(\bar{g}_\varepsilon) = 0$, $\bar{g}_\varepsilon = g_\varepsilon$ in B_{R_0}, $\bar{g}_\varepsilon$ is Schwarzschild in E_{2R_0}, and s.t.

this data evolves to a maximal Ricci-flat spacetime (with $(\mathbb{R}^3, \bar{g}_\varepsilon)$ as a totally geodesic Cauchy surface) which admits a conformal compactification.

Moreover, $\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \bar{g}_\varepsilon = h$.

REMARKS: $L(h) = -\Delta(\mathrm{tr}\, h) + \mathrm{div}(\mathrm{div}(h))$.

- TT-tensors belong to $\ker(L)$.
  - Easy to construct examples with compact support
  - General formulas for TT-tensors (at flat metric) are available (Beig; Dain-Friedrich)

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weighted function space

Prop: Suppose $h_0 \in X_K$, with $L(h_0) = 0$.

Given $\varepsilon > 0$, there is $R_0 > 0$, and h_1 supported in B_{2R_0} s.t. $L(h_1) = 0$ and $\| h_0 - h_1 \|_{X_K} < \varepsilon$.

Idea of proof: Cut-off h_0 to ψh_0. Use variational method to solve $L(h) - f \in SK$, where $f = -L(\psi h_0)$.

"small" "small"

Projection shows $L(h) = f$, i.e. $L(h + \psi h_0) = 0$.

Basic ideas of construction of small data :

- Arrange asymptotics so mass dominates.

① Lower bound on mass for small data (Bartnik)

$$\boxed{m(g) \geq \int_{\mathbb{R}^3} \left(R(g) + \frac{1}{8}|\partial g|^2\right) dx}$$

② Estimates on conformal metric :

$\gamma_h = \delta + h$, $Lh = 0$, h cpt support.

Let $g_h = u_h^4 \gamma_h$, $\Delta_{\gamma_h} u_h = \frac{1}{8} \underbrace{R(\delta+h)}_{\leq C\|h\|^2_{X_K}} u_h$

Write $u_h = 1 + w_h$, explicit formula for w_h using fundamental solution for Δ:

$$u_h(x) = 1 + \frac{m_h}{2|x|} + |x|^{-2}\, \mathcal{O}(\|h\|^2).$$

- $\|h\|^2_{X_K} \gtrsim m_h \gtrsim \|\partial h\|^2_{L^2} - \|h\|^4_{X_K}$

- Given a fixed h ($Lh = 0$, compact support), scale to εh, obtain
  - $m_\varepsilon \gtrsim \varepsilon^2$
  - $|x|^2 \left| w_\varepsilon - \frac{m_\varepsilon}{2|x|} \right| \lesssim \varepsilon^2$

- The projection to ξK is like

$$(m - m_0,\ mC - m_0 C_0)$$

(plus error terms in R^{-1})

scale $\sim \varepsilon^2 \lesssim m_\varepsilon$

- Can solve gluing problem at some large R_0, independent of $|\varepsilon|$ small.

- $\frac{\partial \bar{g}_\varepsilon}{\partial \varepsilon}\Big|_0 = h$ (WLOG, $B_{R_0} \supseteq \mathrm{supp}(h)$)

Pf: Let $v = \frac{\partial u_\varepsilon}{\partial \varepsilon}\Big|_0$. $\frac{\partial \bar{g}_\varepsilon}{\partial \varepsilon} = h + 4v\delta$.

$$\Delta_{g_\varepsilon} u_\varepsilon - \frac{1}{8} \underset{\sim \varepsilon^2}{R(g_\varepsilon)} u_\varepsilon = 0 \implies \begin{array}{l} \Delta v = 0 \\ v \to 0 \text{ at } \infty. \end{array} \quad ///$$

LOCALIZED DEFORMATION

LOHKAMP: (M^n, g) $(n \geq 3)$ smooth Riemannian. $U \subset M$ open. $f \in C^\infty(M)$, $f = R(g)$ on $M \setminus U$, $\boxed{f < R(g)}$ on U. Given $\varepsilon > 0$, there is a smooth g_ε which agrees with g outside U_ε (ε-nbhd of U) satisfies $\boxed{\|g_\varepsilon - g\|_{C^0(M)} < \varepsilon}$ and

$$\boxed{f - \varepsilon \leq R(g_\varepsilon) \leq f}.$$

"Scalar curvature hammocks"

REMARK: WE CANNOT IN GENERAL EXPECT TO BE ABLE TO PUSH SCALAR CURVATURE UP: E.G. $(\mathbb{R}^3, \delta)$ DOES **NOT** ADMIT A COMPACTLY SUPPORTED PERTURBATION h WITH $R(\delta + h) \geq 0$ (NOT $\equiv 0$), BY

POSITIVE MASS THEOREM / SCHOEN-YAU π_1^3- THEOREM

Application: Scalar-flat data on $\mathbb{R}^3$ with multiple minimal spheres

- * Combination of constructions by Chruściel-Delay, and P. Miao

$\swarrow M$

Thm. (Chruściel-Delay) There exist metrics on $\mathbb{R}^3 \setminus \{0 = x_0, \pm x_1, \dots, \pm x_k\}$ of zero scalar curvature of the following form: there are radii r, r_i, masses m, m_i, s.t. in $M \cap B(\pm x_i, r_i)$ the metric is Schwarzschild of mass m_i centered at $\pm x_i$; outside $B(0,r)$, metric is Schw. of mass m, centered at 0. The masses and radii are chosen so the metric contains minimal spheres $|x \mp x_i| = \frac{m_i}{2}$ $\quad (r_i > \frac{m_i}{2})$.

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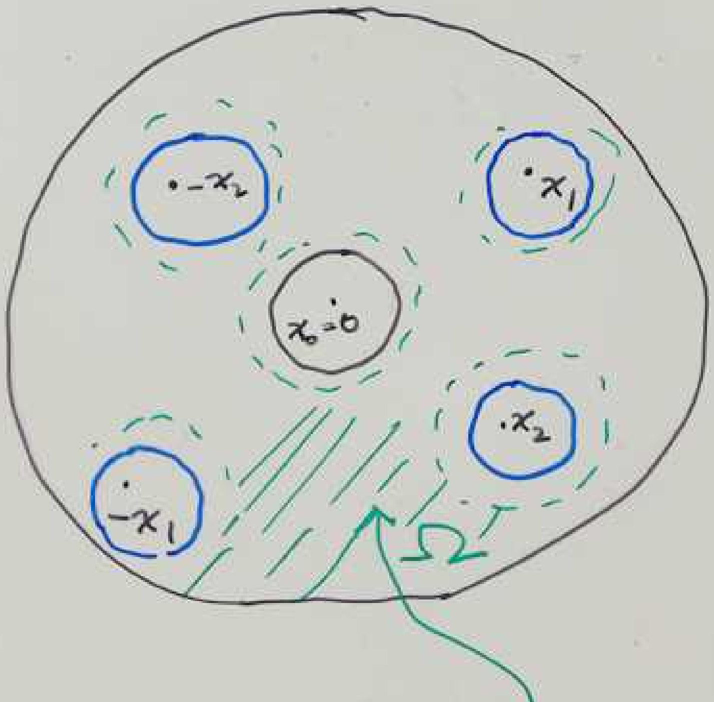
*: Existence of scalar-flat AF metric on $\mathbb{R}^3$ with a horizon due to Beig-O'Murchadha.

- Also more recent work of Yu Yan.



Parity-Symmetric

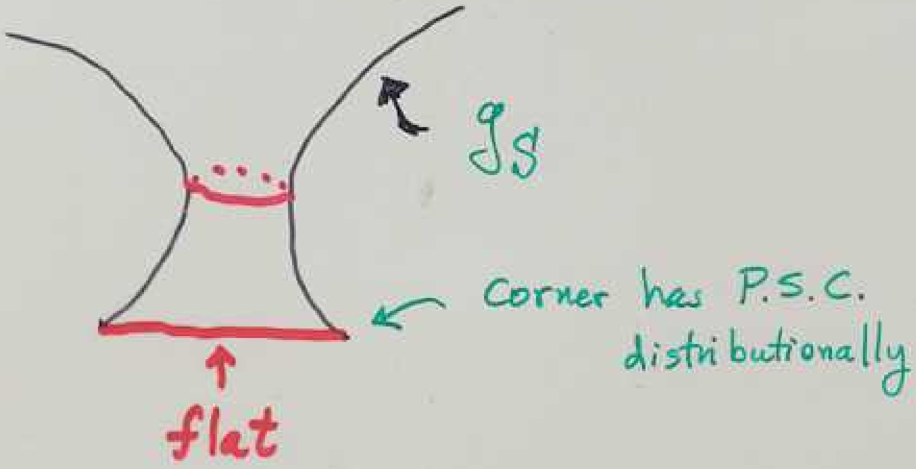
- Key idea: Make metric near flat by using <u>small</u> <u>masses</u>.

- Try to do local gluing: by parity symmetry, need only project onto constant functions:

$$\int_\Omega R(g+h)\, dx = 16\pi \left(m - m_0 - \sum_{i=1}^{K} 2m_i\right) + O(\delta^2)$$

$0 < m, m_i < \delta$

- Adjust topology by P. Miao's construction:



- Mollify conformal factor:

- Now apply Lohkamp
    - Push scalar curvature down into range $(-\varepsilon, 0]$

      not too far!
    - Local deformation: metric controlled in C^0.
- Finally, use a conformal deformation to move metric to $R(u_\varepsilon^4 g_\varepsilon) = 0$
- Good estimates on u_ε $\rightsquigarrow$ horizon persists! (Barrier)