Rigidity and positivity of the mass for asymptotically hyperbolic manifolds

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Based on joint work with Lars Andersson and Mingliang Cai

Introduction

- The classical positive mass theorem (Schoen-Yau, Witten) is a fundamental result about AF Riemannian manifolds.
- Developments in physics over the past few years have increased interest in the geometrical and physical properties of asymptotically hyperbolic (AH) Riemannian manifolds.
- Proofs of the positivity of mass in the AH setting have been given in
  - 3 space dimensions by Gibbons et al ('83)
  - $n \geq 3$ space dimensions by X. Wang ('01), and by Chruściel and Herzlich ('03)

  using spinor based arguments.
- Want to describe an approach to the proof in the AH case that does not require spin assumption, based on the general minimal surface methodology of Schoen and Yau.

Positive Scalar Curvature á la Schoen-Yau

A key observation in the Schoen-Yau approach to the study of compact manifolds of positive scalar curvature is the following.

Theorem (Schoen-Yau)

Suppose M^n is a manifold of positive scalar curvature, $S > 0$. If Σ is a compact stable minimal hypersurface in M then Σ carries a metric of positive scalar curvature.

(stable $\iff$ $A''(0) \geq 0 \quad \forall$ variations of Σ.)

This result enabled Schoen and Yau to inductively construct a large class of compact manifolds M^n, $3 \leq n \leq 7$, that do not carry metrics of positive scalar curvature.

Positive Scalar Curvature á la Schoen-Yau

In particular,

- the n-torus T^n, or more generally,
- $T^n \# N$ or more generally,
- M^n with nonzero degree map to T^n

do not carry metrics of positive scalar curvature.

In fact, if such a manifold carries a metric of nonnegative scalar curvature it must be flat (cf., also Gromov-Lawson).

These results solved a basic precursor problem to the positive mass problem, namely ...

Precursor Problem

Problem: Suppose that M^n has nonnegative scalar curvature, $S \geq 0$, and is isometric to $\mathbb{R}^n$ outside a compact set K. Show that M^n is *globally* isometric $\mathbb{R}^n$

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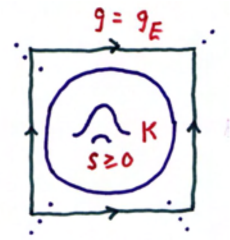

Solution: Put K inside a large box.

The positive mass theorem

Theorem (Schoen-Yau)

Suppose (M^n, g), $3 \leq n \leq 7$, *is an asymptotically flat manifold with nonnegative scalar curvature,* $S(g) \geq 0$. *Then* M *has mass* ≥ 0 *(and* $= 0$ *iff* M *is isometric to Euclidean space).*

Remarks:

- mass = ADM mass
- Physically, M corresponds to a maximal ($H = 0$) spacelike hypersurface in spacetime satisfying the Einstein equations (with $\Lambda = 0$). For then the Gauss equation and weak energy condition imply $S \geq 0$.

Lohkamp's version of the proof ('99)

Lohkamp's approach was to compactify the problem, i.e., make use of the precursor result.

Theorem

Suppose (M^n, g) is an asymptotically flat manifold with nonnegative scalar curvature, $S(g) \geq 0$ with negative mass $m < 0$. Then g can be deformed to a metric g_1 such that

- *$S(g_1) \geq 0$ (and > 0 at some points),*
- *Outside a compact set, (M, g_1) is isometric to $(\mathbb{R}^n, g_E)$.*

Positivity of mass - the AH case

Roughly, (M^{n+1}, g) is asymptotically hyperbolic provided M has an end diffeomorphic to $\mathbb{R}^{n+1} \setminus \bar{B}_R(0)$, such that on this end,

$$g = g_0 + h$$

where,

$$\begin{aligned} g_0 &= \text{hyperbolic metric}, \\ &= \frac{1}{1+r^2}dr^2 + r^2 d\Omega^2 \end{aligned}$$

and $h \to 0$ at a suitable rate as $r \to \infty$.

Ex. Schwarzschild-Anti-de Sitter space (spatial part).

$$g = \frac{1}{1 - \frac{2m}{r^{n-1}} + r^2}dr^2 + r^2 d\Omega^2$$

Satisifies $S = -n(n+1)$ and mass $= m$ (by all definitions)

Positivity of mass - the AH case

Theorem (Wang, Chruściel-Herzlich)

Suppose M^{n+1} is an asymptotically hyperbolic spin manifold with scalar curvature $S \geq -n(n+1)$. Then M has mass $m \geq 0$ (and $= 0$ iff M is isometric to standard hyperbolic space $\mathbb{H}^n$.

Remark: Physically, M corresponds to a maximal ($H = 0$) spacelike hypersurface in spacetime satisfying the Einstein equations with $\Lambda = -n(n+1)/2$. For then the Gauss equation and weak energy condition imply $S \geq -n(n+1)$.

Aim: Obtain a proof without spin assumption, using "minimal suface" methodology. But instead of area functional, work with the "brane action" as developed by Witten-Yau.

Taking hints from Lohkamp's approach ...

The rigidity problem

Problem: Suppose that (M^{n+1}, g) satisfies,

$$S(g) \geq -n(n+1)\,,$$

and is isometric to $\mathbb{H}^{n+1}$ outside a compact set K. Show that (M^{n+1}, g) is *globally* isometric to $\mathbb{H}^{n+1}$.

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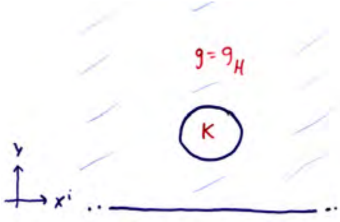
Our solution: Partially compactify. Consider the half space model:

Outside K,

$$g = \frac{1}{y^2}\left(dy^2 + \sum_{i=1}^{n}(dx^i)^2\right)$$



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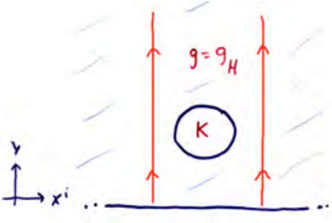
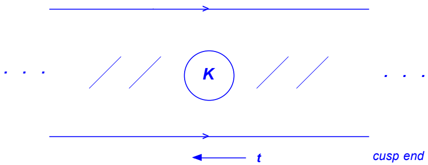


The rigidity problem, cont.



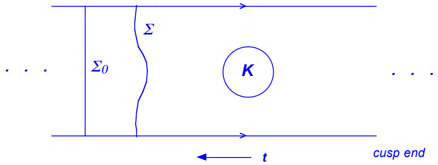
Make change of variable $t = -\ln y$. Then outside K, we have

$$M' = \mathbb{R} \times T^n\,, \qquad g' = dt^2 + e^{2t} h\,,$$

where h is a flat metric on T^n. I.e., (M', g') is a standard hyperbolic cusp, perturbed on a compact set.

Claim: (M', g') is globally isometric to the hyperbolic cusp $(\mathbb{R} \times T^n, dt^2 + e^{2t} h)$.

The rigidity problem, cont.



Fix a t-slice $\Sigma_0 = T^n$. Consider hypersurfaces $\Sigma \subset M'$ homologous to Σ_0. For $\Sigma \sim \Sigma_0$, consider the "brane action" $\mathfrak{L}$,

$$\mathfrak{L}(\Sigma) = A(\Sigma) - nV(\Sigma)\,,$$

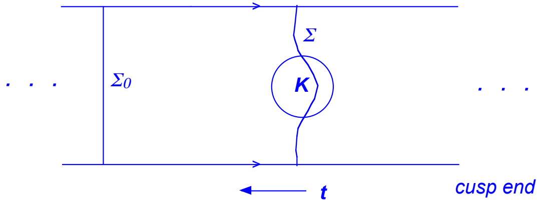
where

$$\begin{aligned} A(\Sigma) &= \text{area of } \Sigma \\ V(\Sigma) &= \text{volume enclosed by } \Sigma\,. \end{aligned}$$

Want to minimize $\mathfrak{L}$ in $[\Sigma_0]$. Consider a minimizing sequence ...

The rigidity problem, cont.

By GMT there exists a smooth minimizer; denote it by Σ:



Can construct nonzero degree map from Σ to $\Sigma_0 \approx T^n \Rightarrow \Sigma$ does not carry a metric of positive scalar curvature.

Under these circumstances can prove rigidity.

Local warped product splitting

Theorem

Assume (M^{n+1}, g) has scalar curvature S satisfying,

$$S \geq -n(n+1)\,.$$

Let Σ^n be a compact hypersurface in M, which does not carry a metric of positive scalar curvature. If Σ locally minimizes the action $\mathcal{L}$, then there is a neighborhood U of Σ that splits as a warped product,

$$U = (-\epsilon, \epsilon) \times \Sigma, \qquad g|_U = du^2 + e^{2u} h\,,$$

where h is Ricci flat.

Infinitesimal rigidity

- Consider normal variations $u \to \Sigma_u$ of $\Sigma = \Sigma_0$, with variation vector field $V = \frac{\partial}{\partial u}|_{u=0} = \phi N$, $\phi \in C^\infty(\Sigma)$.
- First variation of $\mathcal{L}$ implies Σ has mean curvature $H = n$.
- **Second Variation Formula.** Let $\ell(u) = \mathcal{L}(\Sigma_u)$. Then

$$\ell''(0) = \int_\Sigma \phi L(\phi) dA$$

where

$$\begin{aligned} L(\phi) &= -\triangle\phi + \frac{1}{2}(S_\Sigma - S - |A|^2 - H^2)\,\phi \\ &= -\triangle\phi + \frac{1}{2}(S_\Sigma - S_n - |A_0|^2)\,\phi \end{aligned}$$

$(S_n = S + n(n+1) \geq 0.)$

Infinitesimal rigidity, cont.

- Let λ_1 be first eigenvalue of L, and let ϕ be associated eigenfunction; can choose $\phi > 0$.
- By second variation ($\ell''(0) \geq 0$), $\lambda_1 \geq 0$.
- Let $\tilde{S}$ be the scalar curvature of Σ in the conformally rescaled metric $\tilde{h} = \phi^{\frac{2}{n-2}} h$. Then,

$$\begin{aligned} \tilde{S} &= \phi^{-\frac{n}{n-2}}(-2\triangle\phi + S_\Sigma\,\phi + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi}) \\ &= \phi^{-\frac{2}{n-2}}(2\lambda_1 + S_n + |A_0|^2 + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi^2}) \geq 0 \end{aligned}$$

$\Rightarrow \lambda_1 = 0$, $S = -n(n+1)$ along Σ, $A_0 = 0$, ($\iff$ $A = h$), $\phi = const.$ $\Rightarrow$ Σ is scalar, in fact, Ricci flat.

The deformation problem

Problem Let (M^{n+1}, g) be an asymptotically hyperbolic manifold with scalar curvature,

$$S(g) \geq -n(n+1)\,.$$

Show that if the mass $m < 0$, then g can be deformed near infinity to a metric g_1 such that (perhaps after a homothetic change),

- $S(g_1) \geq -n(n+1)$,
- Outside a compact set, (M, g_1) is isometric to $\mathbb{H}^n$.

- Transition from SS-ADS with $m < 0$:

$$g_1 = \frac{1}{1 + \frac{2|m|}{r} + r^2} dr^2 + r^2 d\Omega^2 \qquad (S = -6)$$

- to hyperbolic space with $K = -\frac{1}{a^2}$, $a < 1$:

$$g_2 = \frac{1}{1 + \frac{r^2}{a^2}} dr^2 + r^2 d\Omega^2 \qquad (S = -6/a^2)$$

so that globally,

$$S \geq -\frac{6}{a^2}$$

The basic example

Consider the metric,

$$g = \frac{1}{1 + \frac{f(r)}{r}} dr^2 + r^2 d\Omega^2$$

- where for SS-ADS,

$$f(r) = f_1(r) = r^3 + 2|m|\,,$$

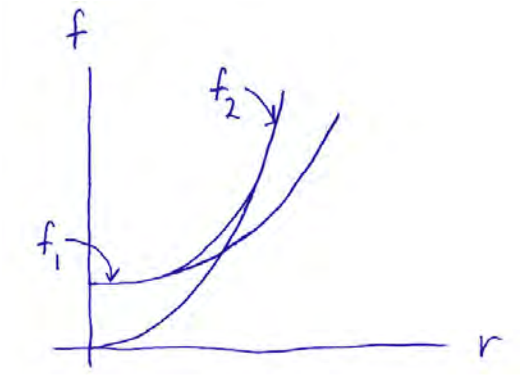
- and for hyperbolic space with $K = -\frac{1}{a^2}$, $a < 1$,

$$f(r) = f_2(r) = \frac{r^3}{a^2}$$

The metric g has scalar curvature,

$$S(g) = -\frac{2}{r^2} f'$$

The basic example



$$f_1(r) = r^3 + 2|m| \quad \text{and} \quad f_2(r) = \frac{r^3}{a^2}$$

Suitably smoothing the corner gives the desired transition.

Deformation result

Suppose (M^{n+1}, g) is AH in the sense of Wang. Then on the AH end, g has the expansion,

$$g = \frac{1}{r^2+1}dr^2 + r^2 d\Omega^2 + \frac{h}{r^{n-1}} + O(\frac{1}{r^n})$$

where h is a symmetric 2-tensor on S^n. Then,

$$\text{mass} \; = \int_{S^n} \mathrm{tr}\, h$$

Theorem

If the mass aspect $= \mathrm{tr}\, h < 0$, then g can be deformed on an arbitrarily small neighborhood of infinity to the hyperbolic metric, while preserving (after a homothetic rescaling) the scalar curvature inequality $S \geq -n(n+1)$.