On problems related to Bartnik's definition of quasi-local mass

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Outline of the talk:

- A brief review of Bartnik's quasi-local mass functional $m_B(\cdot)$ and its properties, like positivity, etc.

- Problems related (or motivated) by $m_B(\cdot)$
  - Horizons in an AF, time-symmetric, **vacuum** initial data which is topologically $\mathbb{R}^3$
  - Static metric extension conjecture
  - An inequality relating the ADM mass of (M^3, g) and the electrostatic capacity of ∂M.

Definition of $m_B(\cdot)$ (Time-symmetric case)

Consider

$$\mathcal{PM} = \{(\mathrm{M}^3, \mathrm{g}) \text{ is an AF manifold satisfying } R(g) \geq 0 \text{ and } (\mathrm{M}, \mathrm{g}) \text{ has no horizons}\}.$$

A horizon is a closed surface with zero mean curvature. It follows from Meeks-Simon-Yau that any $(M, g) \in \mathcal{PM}$ is topologically $\mathbb{R}^3$.

Let (Ω^3, g_Ω) be a bounded region, isometrically contained in some $(M, g) \in \mathcal{PM}$, Bartnik defines

$$m_B(\Omega, g_\Omega) = \inf\{m_{ADM}(M, g) \mid (\Omega, g_\Omega) \subset (M, g) \in \mathcal{PM}\}.$$

It follows from PMT that $m_B(\Omega, g_\Omega) \geq 0$.

Properties of $m_B(\cdot)$:

- $m_B(\cdot)$ is monotone: If $\Omega_1 \subset \Omega_2 \subset\subset (M,g) \in \mathcal{PM}$, then $m_B(\Omega_1) \le m_B(\Omega_2)$.

- $m_B(\cdot) > 0$ unless (Ω, g_Ω) is locally flat (proved by Huisken-Ilmanen)

- Asymptotic to ADM mass: $\lim_{n\to\infty} m_B(\Omega_n, g) = m_{ADM}(M,g)$ for any exhaustion $\{\Omega_n\}$ of $(M,g) \in \mathcal{PM}$ (proved by Huisken-Ilmanen).

- Suppose $(\Omega, g_\Omega) \subset (M,g)$ and $m_B(\Omega, g_\Omega) > m_{ADM}(g)$, then (M,g) must contain a horizon.

Problem 1: Existence of horizons

Theorem(Schoen-Yau 83)

$$Matter\ condensation \Rightarrow horizons\ form.$$

(A theorem for (M^3, g, K), where $K \neq 0$)

Question: What about *no matter*?

First step: Time-symmetric case. Necessarily, we assume (M, g) has *trivial topology*.

A general construction of initial data which is AF, $R \equiv 0$ and topologically $\mathbb{R}^3$:

Take a conformal class of metrics $[g]$ on S^3, assume

$$Y([g]) = \inf_{\bar{g} \in [g]} \{ \frac{\int R(\bar{g})\ d\bar{g}}{Vol(\bar{g})^{\frac{1}{3}}} \} > 0,$$

let G be a positive solution to

$$\triangle_g G - \frac{1}{8} R(g) G = 0 \text{ on } S^3 \setminus \{P\}, \quad \lim_{x \to P} G = \infty.$$

Then $(S^3 \setminus \{P\}, G^4 g)$ is such an initial data.

A basic fact: $(S^3 \setminus \{P\}, G^4 g)$, up to a constant scaling, depends only on $[g]$.

Question: What conditions on $[g]$ (or g) guarantee horizons to form in $(S^3 \setminus \{P\}, G^4 g)$?

Theorem(Beig – Ó Murchadha 91) Let $\{g_\epsilon\}$ be a sequence of metrics on S^3 such that

$$g_\epsilon \to g_0 \text{ in } C^2, \text{ where } R(g_0) \equiv 0,$$

then $(S^3 \setminus \{P\}, G_\epsilon^4 g_\epsilon)$ admits a horizon for ϵ sufficiently small.

Theorem(Yan 04) Let g be a metric on S^3 satisfying

$$Ric(g) \geq \mu, \; Vol(g) \geq V, \; Diameter(g) \leq D.$$

Let $r > \frac{3}{2}$. If

$$R(g) > 0 \text{ and } \int_{S^3} |R(g)|^r \, dg < \delta,$$

for some **small** $\delta = \delta(\mu, V, D, r) \leq 1$, then $(S^3 \setminus \{P\}, G^4 g)$ admits a horizon.

Proposition: There exists a metric g on S^3 with $Ric(g) > 0$ such that $(S^3 \setminus \{P\}, G^4 g)$ admits a horizon.

Theorem (M) Suppose $[g]$ is a conformal class of metrics on S^3 which admits a positive Ricci curvature metric. Consider

$$V^+ = \sup\{Vol(S^3, \bar{g}) \mid Ric(\bar{g}) \geq 2, \ \bar{g} \in [g]\}.$$

If

$$V^+ \geq \frac{1}{2} Vol(S^3, h),$$

where (S^3, h) is the standard unit sphere in the Euclidean space $\mathbb{R}^3$, then $(S^3 \setminus \{P\}, G^4 g)$ admits **no** horizons.

Problem 2: Static metric extension conjecture

Conjecture(Bartnik) Suppose $m_B(\Omega, g_\Omega)$ is defined. Then it is achieved by an asymptotically flat metric g on $\mathbb{R}^3 \setminus \Omega$ such that g is *static, matter free* and satisfies the boundary condition

$$g|_{T\partial\Omega} = g_\Omega|_{T\partial\Omega} \quad \text{and} \quad H(\partial\Omega, g) = H(\partial\Omega, g_\Omega).$$

Interior equation: suggested by Bartnik's calculation, verified by Corvino's work on scalar curvature deformation.

Boundary condition: suggested by the idenity

$$R = 2K - (|A|^2 + H^2) - 2D_\nu H.$$

Proposition Given an AF metric g on $\mathbb{R}^3 \setminus \Omega$ with $R(g) \geq 0$ such that

$$g|_{T\partial\Omega} = g_\Omega|_{T\partial\Omega} \quad \text{and} \quad H(\partial\Omega, g) \leq H(\partial\Omega, g_\Omega).$$

If $K_{\partial\Omega} > 0$, $H(\partial\Omega, g) > 0$ and $H(\partial\Omega, g)$ does **not** identically agree with $H(\partial\Omega, g_\Omega)$, then there exists another $\bar{g}$ on $\mathbb{R}^3 \setminus \Omega$ with $\mathbb{R}(\bar{g}) \geq 0$,

$$\bar{g}|_{T\partial\Omega} = g_\Omega|_{T\partial\Omega} \quad \text{and} \quad H(\partial\Omega, \bar{g}) \leq H(\partial\Omega, g_\Omega)$$

such that

$$m_{ADM}(\bar{g}) < m_{ADM}(g).$$

Theorem(M 03) Let σ be a metric on S^2 and h be a function on S^2. If σ and h is sufficiently close to the Euclidean data on S^2 and satisfies some symmetry condition, then there exists an AF metric g and a function u on $\mathbb{R}^3 \setminus S^2$ such that

$$\begin{cases} D_g^2 u - uRic(g) & = & 0 \\ \triangle_g u & = & 0 \end{cases}$$

and

$$g|_{TS^2} = \sigma \quad \text{and} \quad H(S^2, g) = h.$$

Problem 3: Relation between ADM mass and electrostatic capacity

Let (M^3, g) be an AF manifold with a compact inner boundary Σ^2. Define

$$Cap(\Sigma, g) = \inf \left\{ \int_M |\nabla \phi|^2 \, dg \mid \phi|_\Sigma = 0, \ \lim_{x \to \infty} \phi = 1 \right\}.$$

Theorem(Bray 99) If (M^3, g) is AF with $R(g) \geq 0$ and Σ consists of *minimal surfaces*, then

$$m_{ADM} \geq Cap(\Sigma, g)$$

and "$=$" holds if and only if (M^3, g) is the spatial Schwarzschild manifold outside its horizon.

Remark: This theorem may be thought as a generalization of the classic result of Bunting and Masood-ul-Alam, which is formulated in term of the *staticity* of (M^3, g).

Question: What can we say if Σ is not necessarily a minimal surface?

Theorem(Bray-M) Let (M^3, g) be AF and topologically is $\mathbb{R}^3$ minus a ball. If $R(g) \geq 0$ and $\int_\Sigma H^2 \, d\mu \leq 16\pi$, then

$$m_{ADM}(g) \geq Cap(\Sigma, g) \left(1 - \sqrt{\frac{1}{16\pi} \int_\Sigma H^2 \, d\mu}\right)$$

and " $=$ " holds if and only if (M^3, g) is isometric to

$$\left(S^2 \times [r_0, \infty), \; \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\sigma^2\right)$$

for some $m \geq 0$ and $r_0 \geq 2m$.

Theorem(Bray-M) Let (M^3, g) be AF and topologically is $\mathbb{R}^3$ minus a ball. If $R(g) \geq 0$, then

$$Cap(\Sigma, g) \leq \sqrt{\frac{|\Sigma|}{4\pi}} \left(\frac{1 + \sqrt{\frac{1}{16\pi} \int_\Sigma H^2 \ d\mu}}{2} \right)$$

and " $=$ " holds if and only if (M^3, g) is isometric to

$$\left(S^2 \times [r_0, \infty), \ \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\sigma^2 \right)$$

for some $m \geq 0$ and $r_0 \geq 2m$.