Positive Energy-Momentum Theorem for AdS-Asymptotically Hyperbolic Manifolds

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- General Relativity is a **geometric** theory of gravitation
- Hamiltonian description of GR $\Longrightarrow$ Notions of Energy and Momentum well-defined for isolated systems and are geometric **invariants**
- Energy-momentum is a **linear form** on some space of Killing vector fields
- Positive energy theorems $\Longrightarrow$ attents to understand the relation between the total mass of a space-time and the local energy density (stress-energy tensor T)

What is an isolated system ?

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Definition

A Riemannian manifold (M^n, g, k) *is said to be asymptotically flat if there exist a compact set* $K \subset M$*, a real number* $R > 0$ *and a homeomorphism* $\Phi : M \setminus K \longrightarrow \mathbb{R}^n \setminus B(0, R)$ *called a chart at infinity such that*

$$g_{ij} = b_{ij} + e_{ij}, \quad e_{ij} \xrightarrow{\infty} 0 \text{ and } k_{ij} \xrightarrow{\infty} 0,$$

where b *is the Euclidean metric.*

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The energy-momentum is then

$$p^\mu := (m, p^i) \in \mathbb{R}^{n,1},$$

where the mass part is

$$m = - \int_{S_\infty} \delta_b g + \mathrm{d}\mathrm{tr}_b g.$$

Positive Energy-Momentum Theorem

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Theorem

Let (M^n, g, k) be an asymptotically flat spin Riemannian manifold with L^1 constraints and such that $e_{ij} = g_{ij} - b_{ij} \in C^{2,\alpha}_{\tau}$, $k_{ij} \in C^{1,\alpha}_{\tau-1}$, $\tau < \frac{-(n-2)}{2}$. If the dominant energy condition is satisfied then

p^μ is causal future-directed.

If moreover p^μ is null then (M, g, k) is isometrically embeddable in Minkowski.

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- Schoen-Yau 1980 $3 \leq n \leq 7$
- Witten 1981 (and Parker-Taubes) $\forall n \geq 3$
- The rigidity case has been proved by Beig-Chruściel in dimension 3 (1995)

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Both contain hypersurfaces that are isometric to $\mathbb{H}^n$.
But the second fundamental form differs when $\mathbb{H}^n$ is seen as hypersurface of Minkowski or AdS.

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That explains the terminology **AdS-asymptotically hyperbolic**.
Rq. The dimension of the *slice* M is supposed to be $n = 3$ till the end of the exposal except if the contrary is specially mentionned.

More precisely

Definition

A Riemannian manifold (M^3, g, k) *is said to be AdS-asymptotically hyperbolic if there exist compact set* $K \subset M$*, a real number* $R > 0$ *and a homeomorphism* $\Phi : M \setminus K \longrightarrow \mathbb{R}^3 \setminus B(0, R)$ *called a chart at infinity such that*

$$\begin{cases} e := g - b = O(e^{-\tau r}), & \partial e = O(e^{-\tau r}), \quad \partial^2 e = O(e^{-\tau r}), \\ k = O(e^{-\tau r}), & \partial k = O(e^{-\tau r}), \end{cases}$$

for $\tau > 3/2$ *and where* ∂ *is taken with respect to the hyperbolic metric* $b = \mathrm{d}r^2 + (\sinh r)^2 g_{S^2}$ *with* g_{S^2} *the standard metric on the sphere* S^2*.*

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- **Hamiltonnian Description** of General Relativity $\Rightarrow$ Notions of Energy and Momentum
- The mass of an asymptotically hyperbolic manifold is not a real but a **mass functional**
- In fact the mass is a **linear form** on $\mathbb{R}^{n,1}$
- The difficulty is to find the notion of **positivity** suitable for the nature of the mass functional

Mass Formula (Zero Extrinsic Curvature)

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where $f \in N_b := \{f \in C^\infty(\mathbb{R}^n) | \mathrm{Hess}^b f = fb\}$, with N_b isometric to $\mathbb{R}^{n,1}$ via the application

$$\begin{array}{ccc} \mathbb{R}^{n,1} & \longrightarrow & N_b \\ y_k & \longmapsto & x_k := y_{k|\mathbb{H}^n} \end{array},$$

where $(y_k)_{k=0}^n$ are the standard coordinates of Minkowski.
Remark. N_b corresponds to normal Killing vectors.

Positive Mass Theorem

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Theorem

Let (M^n, g) an asymptotically hyperbolic spin Riemannian manifold with scalar curvature $\mathrm{Scal}^g \geq -n(n-1) = \mathrm{Scal}^b$. Then

$$M^\mu \in \mathbb{R}^{n,1} \text{ is causal future-directed.}$$

Furthermore if M^μ is null then $(M^n, g) \cong \mathbb{H}^n$.

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Furthermore if M^μ *is null then* $(M^n, g) \cong \mathbb{H}^n$.

- Proved by Chruściel-Herzlich (2003) $\forall n \geq 3$
- The geometric invariant is not M^μ but its Minkowskian norm

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where $(f, \alpha) \in N_b \oplus \mathfrak{Kill}(\mathbb{H}^3)$, with $\mathfrak{Kill}(\mathbb{H}^3)$ the space of Killing forms on $\mathbb{H}^3$and N_b defined as before.

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- Hamiltonnian description $\Rightarrow \mathcal{H} \in \mathfrak{Kill}\left(\mathrm{AdS}^{3,1}\right)^*$
- Clearly $\mathfrak{Kill}\left(\mathrm{AdS}^{3,1}\right) \cong \mathbb{R}^{3,1} \oplus \mathfrak{Kill}(\mathbb{H}^3)$
- We will consider $\mathfrak{Kill}(\mathbb{H}^3) \cong \mathfrak{so}(3,1) \cong \mathfrak{sl}(2, \mathbb{C})$

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Energy-Momentum of an AdS-asymptotically hyperbolic (M, g, k) is $\mathcal{H} = M \oplus \xi \in \mathfrak{M} \oplus \mathfrak{sl}(2, \mathbb{C})$ where $\mathfrak{M} = (\{A \in \mathrm{M}_2(\mathbb{C}) / A^* = A\}, -\det) \cong \mathbb{R}^{3,1}$.

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We define the Hermitian matrix

$$Q := \begin{pmatrix} \widehat{M} & \xi \\ \xi^* & M \end{pmatrix},$$

where $\widehat{M}$ is the transposed comatrix of M and $\xi^* =^t \overline{\xi}$.

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where $\widehat{M}$ is the transposed comatrix of M and $\xi^* =^t \overline{\xi}$.
Remark. $SL(2, \mathbb{C})$ naturally acts on Q.

Theorem

Let (M^3, g, k) be an orientable and complete AdS-asymptotically hyperbolic manifold satisfying the dominant energy condition. Then the Hermitian matrix Q is non-negative.

Suppose M has a boundary ∂M with induced metric $\breve{g}$ and second fundamental form $\breve{k}$ then define the vector field

$$\vec{k} := (-\mathrm{tr}\breve{k} + (n-1))e_0 + k(\nu)$$

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Theorem

Let (M^3, g, k) be an orientable and complete AdS-asymptotically hyperbolic manifold with compact boundary, satisfying the dominant energy condition and the boundary condition:
$\vec{k}$ is causal future-directed.
Then the Hermitian matrix Q is non-negative.

Comments

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- This gives an **upper bound** for the angular momentum of non singular AdS-asymptotically space-times
- The restriction on the parameters of the Kerr-AdS metrics is then a **necessary physical property**
- The orbit of Q under $\mathrm{SL}(2, \mathbb{C})$ depends upon **4** independant real parameters (Kerr-AdS solutions have only **2** real parameters)

Connections

We suppose $(M, g, k) \subset (N, \gamma)$ where $\gamma = -\mathrm{d}t^2 + g_t$ is a Lorentzian metric on $]-\epsilon, \epsilon[\times M$ for ϵ small enough.

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AdS metric is denoted by $\beta = -\mathrm{d}t^2 + b_t$.
$\nabla, \overline{\nabla}, D$ denote respectively the Levi-Civita connections of γ, g and β.Let us take a spinor field $\psi \in \Gamma(\Sigma)$ and a vector field $X \in \Gamma(TM)$, then

$$\begin{cases} \nabla_X \psi & = & \overline{\nabla}_X \psi - \frac{1}{2} k(X) \cdot e_0 \cdot \psi \\ \langle k(X), Y \rangle_\gamma & = & \langle \nabla_X Y, e_0 \rangle_\gamma \end{cases} .$$

In these formulae $\cdot$ denotes the Clifford action with respect to the metric γ, and $e_0 = \partial_t$. We will use different notations when we have to make the difference between the Clifford action with respect to the metric γ or β.

Definition.

The Killing equation on a spinor field $\tau \in \Gamma(\Sigma)$ *is*

$$\widehat{D}_X \tau := D_X \tau + \frac{i}{2} X \cdot_\beta \tau = 0 \quad \forall X \in \Gamma(TM),$$

where D *denotes the Levi-Civita connection of AdS along* M*. Such a* $\widehat{D}$*-parallel spinor field is called a* β*-imaginary Killing spinor and we denote* $\tau \in IKS(\Sigma)$*. In the same way, a* $\widehat{\nabla}$*-parallel spinor field (where* $\widehat{\nabla}_X := \nabla_X + \frac{i}{2} X \cdot_\gamma$*) is called a* γ*-imaginary Killing spinor.*

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$$\begin{array}{rcl} IKS(\Sigma) & \longrightarrow & N_b \oplus \mathfrak{so}(3,1) \\ \sigma & \longmapsto & V_\sigma \oplus \alpha_\sigma \end{array}$$

where $V_\sigma = \langle \sigma, \sigma \rangle$ and $\alpha_\sigma(X) = \langle X \cdot e_0 \cdot \sigma, \sigma \rangle$

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- The study of the boundary integrals of the integrated Bochner-Lichnerowicz formula and some **analytical properties** of the Dirac operator $\widehat{\mathfrak{D}}$ gives

$$\forall \sigma \in IKS(\Sigma) \quad \mathcal{H}(V_\sigma, \alpha_\sigma) \geq 0.$$

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- It comes out that this application is the quadratic form Q
- The non-negativity of each $\mathcal{H}(V_\sigma, \alpha_\sigma) \Longrightarrow Q$ is non-negative

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Theorem

Let (M^3, g, k) be an AdS-asymptotically hyperbolic manifold satisfying the assumptions of the positive energy-momentum theorem. If $trQ = 4M^0 = 0$ then (M^3, g, k) is isometrically embbedable in $\mathrm{AdS}^{3,1}$.

We would like to weaken the rigidity condition to Q is **degenerate**.

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Theorem. *Let us suppose that* (M, g, k) *satisfies the assumptions of the positive energy-momentum theorem and that the matrix* Q *is degenerate. Then there exists some* $\widehat{\nabla}$*-parallel spinor field* ξ *such that* $\left\langle \widehat{\mathfrak{R}}\xi, \xi \right\rangle = 0$ *and consequently* (M, g, k) *is isometrically embeddable in a stationary pp-wave space-time.*

If furthermore the constant function (ξ, ξ) *is non-zero then* (M, g, k) *admits a vacuum Cauchy development which is a solution of the Einstein equations (with the cosmological constant -3) carrying a Killing vector field.*

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- Conjecture: Q is degenerate $\Longrightarrow (M, g, k)$ is isometrically embeddable in AdS

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- Conjecture: Q is degenerate $\Longrightarrow (M, g, k)$ is isometrically embeddable in AdS
- It probably needs the geometry at infinity

THANKS