Field Theory and Exact Stochastic Eqs. for interacting particle systems

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Introduction

- Usual way to derive the field theory for interacting particle systems : DOI-PELITI formalism

- Problem / Issue : it seems problematic to relate the field theory to exact physical stochastic equations

$$\partial_t \rho = \cdots\cdots + \eta \leftarrow \text{Imaginary Noise}$$

ρ is the density

- Results :
  - Exact physical stochastic equations can be derived in all cases (Newtonian, Stochastic, lattice, continuum)
  - Derivation of a dual version of the Doi-Peliti field theory directly from these eqs.
  - $\partial_t \rho(\vec{r}) = \cdots\cdots$ close eq.

    No need of coarse graining !

PLAN

- Brief recall of Doi - Peliti formalism

  Pb of the imaginary noise for $A+A \to 0$

- From Doi - Peliti to exact physical stochastic equations and back.

  (a) Interacting particles on a lattice

  $$H = \sum_{i<j} n_i \, V(i-j) \, n_j$$

  Doi - Peliti $\to$ Field Theory $\to$ Exact PSE

  continuum limit + fields transformation

$\to$ Dean Eqs :

$$\partial_t \rho(\vec{r}) = \vec{\nabla}\left(\rho(\vec{r})\vec{\nabla}\frac{\delta F}{\delta\rho(\vec{r})}\right) + \vec{\nabla}\left[\sqrt{\rho}\,\vec{\eta}\right] \quad (\text{Ito})$$

$$F = T\int d\vec{r}\,\rho(\vec{r})\ln\rho(\vec{r}) + \frac{1}{2}\int d\vec{r}\,d\vec{r}'\,\rho(\vec{r})\,V(\vec{r}-\vec{r}')\,\rho(\vec{r}')$$

$$\langle \eta^\alpha(\vec{x},t)\,\eta^\beta(\vec{y},t')\rangle = 2T\,\delta^{\alpha\beta}\,\delta(\vec{x}-\vec{y})\,\delta(t-t')$$

(b) Show in a simple example (diffusion) how to write Exact PSE $\to$ Field Theory

(c) Extensions and Conclusion

Doi- Peliti Formalism for Newtonian and <u>Stochastic</u> dynamics

- FoKKer - Planck operator (Liouville) $\longleftrightarrow$ Schrödinger in imaginary time

- Example : $A+A \to 0$

$$\boxed{\frac{dP(n,t)}{dt} = (n+1)(n+2)P(n+2,t) - n(n-1)P(n,t)}$$

$$|P\rangle = \sum_n P(n,t)(a^+)^n|0\rangle \qquad [a,a^+]=1$$

$$\partial_t |P\rangle = -H|P\rangle$$

For $A+A\to 0$ $\quad \partial_t|P\rangle = \left(a^2 - a^{+2}a^2\right)|P\rangle$

$$\left(\begin{array}{l} a(a^+)^n|0\rangle = \\ = n(a^+)^{n-1}|0\rangle \end{array}\right)$$

- Field Theory via coherent states

Rules to get the Field Theory via coherent states

$$H(a^+, a) \longrightarrow \int \mathcal{D}\phi \, \mathcal{D}\phi^* \, e^{S(\phi, \phi^*)}$$

$$S(\phi, \phi^*) = \phi(t_f) - \underline{\underline{\int_{t_i}^{t_f} dt \left[\phi^*(t)\, \partial_t \phi(t) + H_{n.o.}(\phi^*, \phi)\right]}} + f(\phi^*(t_i))$$

$$\langle O(n) \rangle_{dyn} \xrightarrow{n = a^+ a} \langle O(\phi^* \phi) \rangle_{FT}$$

Often $\phi \sim \rho$ because $\langle \phi^* \phi \rangle = \langle \rho \rangle = \langle \phi \rangle$

$A + A \to 0$ $\qquad \mathcal{L}$ $\qquad$ Imaginary Noise

$$S = \phi(t_f) - \int_{t_i}^{t_f} \phi^* \partial_t \phi \, dt + \int_{t_i}^{t_f} \phi^2 (1 - \phi^{*2}) \, dt + f(\phi^*(t_i))$$

Shift $\phi^* = 1 + \hat{\phi}$ $\longrightarrow$ Standard Field Theory à la Martin - Siggia - Rose

$$S = \underbrace{- \int_{t_i}^{t_f} \hat{\phi} \, \partial_t \phi \, dt + \int_{t_i}^{t_f} \phi^2 (-2 \hat{\phi} - \hat{\phi}^2) \, dt}_{- \int_{t_i}^{t_f} \hat{\phi} (\partial_t \phi + 2 \phi^2) dt - \int_{t_i}^{t_f} \phi^2 \hat{\phi}^2 dt} + \tilde{f}(\phi(t_i) \hat{\phi}(t_i))$$

$\Downarrow$ Martin - Siggia - Rose Action for

$$\partial_t \phi = -2 \phi^2 + \eta(t)$$

$$\langle \eta(t) \eta(t') \rangle = -2 \phi^2(t) \delta(t - t')$$

Imaginary Noise

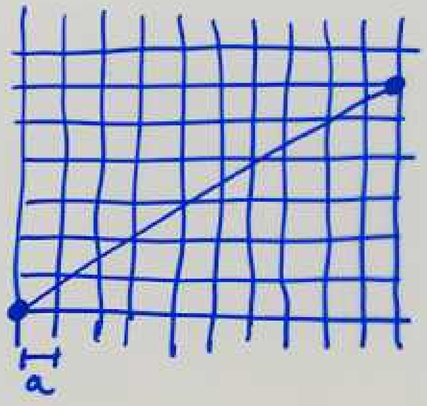
Interacting particles on a lattice

$$H = \frac{1}{2} \sum_{l,m} n_l \, V(l-m) \, n_m \qquad (V(0)=0)$$

Basic move between nearest neighbors $\quad \underset{l}{\bullet} \curvearrowright \underset{m}{\bullet} \quad n_l, n_m \to n_l - 1, n_m + 1$

$$P_{\Delta E} \to \frac{2T n_l}{1+e^{\beta \Delta E}} \xrightarrow{\text{Detailed Balance}} \prod_i \frac{1}{n_i!} \, e^{-\beta H(\{n_i\})}$$

Continuum limit



$$\left(\frac{1}{\rho}\right)^{\frac{1}{3}} \gg a$$

range of V $\gg a$

$$V(i-j) \longleftrightarrow \sum_i V(i) \sim O(1)$$

$$i \curvearrowright j \qquad\qquad n_i+1,\ n_j-1 \longrightarrow n_i,\ n_j$$

$$\Delta E = H(n_i, n_j) - H(n_i+1, n_j-1)$$

$$\Delta E = \sum_e n_e V(j-e) - \sum_e n_e V(i-e) + \dots \qquad \ll 1$$

$$P_{\Delta E} = \frac{2T(n_i+1)}{1+e^{\frac{\Delta E}{T}}} \simeq \left(T - \frac{\Delta E}{2}\right)(n_i+1)$$

$$\frac{dP}{dt} \simeq \sum_{\langle ij \rangle} \Big\{ \left(T - \frac{\Delta E}{2}\right)(n_i+1)\, P(n_i+1, n_j-1, \dots) - n_i \left(T - \frac{\Delta E}{2}\right) P(n_i, n_j, \dots)$$
$$+ \left(T + \frac{\Delta E}{2}\right)(n_j+1)\, P(n_i-1, n_j+1, \dots) - n_j \left(T + \frac{\Delta E}{2}\right) P(n_i, n_j, \dots) \Big\}$$

$$\partial_t |P\rangle \simeq \sum_{\langle ij \rangle} \Big\{ \left(T - \frac{\Delta E}{2}\right) a_j^+ a_i - \left(T - \frac{\Delta E}{2}\right) a_i^+ a_i$$
$$+ \left(T + \frac{\Delta E}{2}\right) a_i^+ a_j - \left(T + \frac{\Delta E}{2}\right) a_j^+ a_j \Big\} |P\rangle$$

$$\partial_t |P\rangle \simeq \Big[-T \sum_{\langle ij \rangle} (a_i^+ - a_j^+)(a_i - a_j)$$
$$- \sum_{\langle ij \rangle} \Big\{ \frac{\Delta E}{2} (a_j^+ - a_i^+) a_i - \frac{\Delta E}{2} (a_i^+ - a_j^+) a_j \Big\} \Big] |P\rangle$$

$$\boxed{\begin{aligned} \partial_t |P\rangle = & -T \int dx\ \vec{\nabla} a^+(x) \cdot \vec{\nabla} a(x) \\ & - \int dx \left(\vec{\nabla} a^+(x)\right) a(x) \left[\int dx'\, a^+(x')\, a(x')\, \vec{\nabla} V(x'-x) \right] \end{aligned}}$$

Field Theory

$$S = \int dx\,dt \left(-\phi^* \partial_t \phi\right) \qquad - \int dx\, \phi(x, t_f)$$

$$+ \int dx\,dt\ \phi^*\left[T \Delta \phi + \vec{\nabla}\cdot\left[\left(\int dx'\, \phi^*(x')\phi(x')\vec{\nabla}V(x'-x)\right)\phi(x)\right]\right]$$

"Interpretation in term of a stochastic equation"

After the shift $\phi^* = 1 + \hat{\phi}$ and the MSR procedure

$$\partial_t \phi = T\Delta\phi + \vec{\nabla}\left[\phi(x)\int dx'\, \phi(x')\, \vec{\nabla} V(x'-x)\right] + \eta$$

$$\langle \eta(x,t)\, \eta(x',t')\rangle =$$

$$= \delta(t-t')\left[\vec{\nabla}\phi(x)\vec{\nabla}V(x'-x)\phi(x') + \Delta V(x'-x)\phi(x')\phi(x)\right]$$

Depending on V it may be negative

→ Imaginary Noise. Pb!

→ Is there no $\boxed{\text{PHYSICAL}}$ equation on the density field?

Cole - Hopf - like transformation

- The density operator $\hat{\rho} = a^+ a$

NEW $\rightarrow$ FIELDS $\begin{cases} \rho = \phi^* \phi \\ \hat{\rho} = \ln \phi^* \end{cases}$

$\boxed{\text{the Jacobian is one}}$

- $-\phi^* \partial_t \phi \text{ "=" } \frac{(\partial_t \phi^*)}{\phi^*} \phi^* \phi = (\partial_t \hat{\rho}) \rho \text{ "=" } -\hat{\rho}\, \partial_t \rho$

- $\phi^* \Delta \phi \text{ "=" } -\frac{(\nabla \phi^*)}{\phi^*} (\phi^* \nabla \phi) = -\frac{\nabla \phi^*}{\phi^*} \left[\nabla(\phi \phi^*) - (\nabla \phi^*) \phi \right]$

$$= -(\nabla \ln \phi^*) \nabla(\phi \phi^*) + \left(\frac{\nabla \phi^*}{\phi^*} \right)^2 \phi \phi^*$$

$$= -(\nabla \hat{\rho})(\nabla \rho) + (\nabla \hat{\rho})^2 \rho$$

$$= \hat{\rho}\, \Delta \rho + (\nabla \hat{\rho})^2 \rho$$

- $\phi^* \vec{\nabla} \cdot \left[\phi(x) \int \phi^*(x') \phi(x') \vec{\nabla} V(x'-x) \right] =$

$$-\frac{(\vec{\nabla} \phi^*(x))}{\phi^*(x)} \phi(x) \phi^*(x) \int dx' \phi^*(x') \phi(x') \vec{\nabla} V(x'-x)$$

$$= -\vec{\nabla} \hat{\rho}(x)\ \rho(x) \int dx'\ \rho(x')\ \vec{\nabla} V(x'-x)$$

$$= \hat{\rho}(x)\ \vec{\nabla} \cdot \left[\rho(x) \int dx'\ \rho(x')\ \vec{\nabla} V(x'-x) \right]$$

$$S = -\hat{\rho}\left[\partial_t \rho - T\Delta\rho - \vec{\nabla}\cdot\left\{\rho(x)\int dx'\,\rho(x')\,\vec{\nabla}V(x'-x)\right\}\right]$$

$$\mp T\left(\vec{\nabla}\hat{\rho}\right)^2 \rho$$

$$\Downarrow$$

via MSR procedure

$$\Downarrow$$

$$\partial_t \rho = \underbrace{T\Delta\rho + \vec{\nabla}\cdot\left[\rho(x)\int dx'\,\rho(x')\,\vec{\nabla}V(x'-x)\right]}_{\vec{\nabla}\left(\rho\,\vec{\nabla}\frac{\delta F}{\delta\rho}\right)} + \vec{\nabla}\left[\sqrt{\rho}\,\vec{\eta}\right]$$

$$F = T\int dx\,\rho(x)\ln\rho(x) + \frac{1}{2}\int dx\,dy\,\rho(x)\rho(y)V(x-y)$$

$$\langle \eta^\alpha(x,t)\,\eta^\beta(y,t')\rangle = 2T\,\delta(x'-y)\,\delta(t-t')\,\delta_{\alpha\beta}$$

That's the Dean Equation!

- Is this result general?
- How can one avoid this painful procedure?

A simple example : DIFFUSION

$$\rho_i(t+dt)-\rho_i(t) = \boxed{d\rho_i(t) = dJ_i} \qquad \forall t$$
$$\rho_j(t+dt)-\rho_j(t) = \boxed{d\rho_j(t) = dJ_j}$$

- Stochastic equations with noise : (JUMP PROCESS)

Probability $\quad D\rho_i\, dt \qquad dJ_i = -1 \quad dJ_j = +1$

Probability $\quad 1 - D\rho_i\, dt \qquad dJ_j = dJ_i = 0$

- Field Theory à la Martin - Siggia - Rose

$$\overline{\int \prod d\rho \prod \delta\left(d\rho(t) - dJ_t\right)}^{J} = \int \prod d\rho\, d\hat\rho\; \overline{e^{-\sum \hat\rho\left(d\rho - dJ_t\right)}}^{J}$$

$$\overline{e^{\hat\rho_i dJ_i + \hat\rho_j dJ_j}}^{J} \simeq \left(1 - \rho_i D\, dt\right) + \rho_i D\, dt\; e^{-\hat\rho_i + \hat\rho_j}$$
$$\simeq \exp\left(D\rho_i\, dt\left(e^{-\hat\rho_i + \hat\rho_j} - 1\right)\right)$$

$$-\sum \hat{\rho}\, d\rho \longrightarrow \boxed{-\int \hat{\rho}\, \partial_t \rho \; dt}$$

$$D\rho_i\, dt\left(e^{-\hat{\rho}_i+\hat{\rho}_j}-1\right)+ D\rho_j\, dt\left(e^{-\hat{\rho}_j+\hat{\rho}_i}-1\right) \longrightarrow \boxed{D\int\left\{\rho_i\left(e^{-\hat{\rho}_i+\hat{\rho}_j}-1\right)+\rho_j\left(e^{-\hat{\rho}_j+\hat{\rho}_i}-1\right)\right\}dt}$$

Remark 1) Continuum limit leads to ~~the~~ same S for interacting particle with $V=0$

$$S = \int -\hat{\rho}\,\partial_t\rho + D\hat{\rho}\,\Delta\rho + D(\nabla\hat{\rho})^2\rho$$

Remark 2) Mapping to Doi-Peliti Field Theory via Cole-Hopf-like transformation

$$\hat{\rho} = \ln \phi^* \qquad \rho = \phi^*\phi$$

$$\hat{\rho}\,\partial_t\rho \rightarrow \phi^*\partial_t\phi$$

$$D\rho_i\left(e^{-\hat{\rho}_i+\hat{\rho}_j}-1\right)+D\rho_j\left(e^{-\hat{\rho}_j+\hat{\rho}_i}-1\right)=$$

$$D\left[\phi_i^*\phi_i\left(\frac{\phi_j^*}{\phi_i^*}-1\right)+\phi_j^*\phi_j\left(\frac{\phi_i^*}{\phi_j^*}-1\right)\right]=$$

$$= D\left[\phi_i\phi_j^* - \phi_i^*\phi_i + \phi_j\phi_i^* - \phi_j\phi_j^*\right]$$

$$= -D\left[(\phi_i^*-\phi_j^*)(\phi_i-\phi_j)\right]$$

Remark 3) The mapping can be performed using only operators

$$a^{+} = e^{\rho^{+}} \qquad a = e^{-\rho^{+}} \rho$$ is a canonical transformation

$$[\rho, \rho^{+}] = 1 \quad \ldots\ldots$$

Remark 4) The pb of imaginary noise was due to the incorrect identification of ϕ with the density field.

Newtonian Case

$$\rho(\vec{r},\vec{p}) = \sum_i \delta(\vec{r}-\vec{r}_i)\,\delta(\vec{p}-\vec{p}_i) \qquad (\nabla V(0)=0)$$

$$\partial_t \rho = \sum_i -\dot{\vec{r}}_i\,\vec{\nabla}_r\,\delta(\vec{r}-\vec{r}_i)\,\delta(p-\vec{p}_i) + \sum_i -\dot{\vec{p}}_i\,\vec{\nabla}_p\,\delta(\vec{r}-\vec{r}_i)\,\delta(p-\vec{p}_i)$$

$$\partial_t \rho = \sum_i -\frac{\vec{P}_i}{m}\,\nabla_{\vec{r}}\,\delta(\vec{r}-\vec{r}_i)\,\delta(\vec{p}-\vec{p}_i) + \sum_i \Big(\sum_j \vec{\nabla} V(\vec{r}_i-\vec{r}_j)\Big)\,\nabla_{\vec{p}}\,\delta(\vec{r}-\vec{r}_i)\,\delta(\vec{p}-\vec{p}_i)$$

$$\partial_t \rho(\vec{r},\vec{p}) = -\frac{\vec{p}}{m}\cdot\vec{\nabla}_r\,\rho(\vec{r},\vec{p}) + \int dr_1\,dp_1\,\rho(\vec{r}_1,\vec{p}_1)\,\vec{\nabla}_r V(\vec{r}-\vec{r}_1)\cdot\vec{\nabla}_{\vec{p}}\,\rho(\vec{r},\vec{p})$$

Deterministic eq. with stochastic initial condition

As previously : Martin - Siggia - Rose Field Theory

$$S = \int -\hat{\rho}\left[\partial_t \rho + \frac{\vec{p}}{m}\,\vec{\nabla}_r\,\rho(\vec{r},\vec{p}) + \dots\right]$$

Cole - Hopf - like transformation
$$\begin{cases} \rho(\vec{r},\vec{p}) = \phi^*(\vec{r},\vec{p})\,\phi(\vec{r},\vec{p}) \\ \hat{\rho}(\vec{r},\vec{p}) = \ln \phi^*(\vec{r}\,\vec{p}) \end{cases}$$

$\hookrightarrow$ Field Theory derived by Doi

Conclusion and Perspectives

1) In all cases (Newtonian, Stochastic, lattice, continuum) one can derive exact physical stochastic equations

2) The field Theory derived from these eqs is a dual version (on physical fields) of the Doi-Peliti field Theory

3) Application of this duality to off-equilibrium phase transitions (driven or glassy cases)
cf. Lefèvre talk.

4) Numerical Simulations

5) Microscopic $\qquad\qquad\qquad\qquad$ Macroscopic

$$\partial_t \rho = \cdots \xrightarrow[\text{via RG}]{\text{Coarse-graining}} \partial_t \rho = \cdots$$

Exact physical stochastic eqs $\qquad\qquad$ Macroscopic hydrodynamics eqs