Slow dynamics in systems with long-range interactions

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PLAN

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  - Extensivity vs. additivity
  - Ensemble inequivalence: negative specific heat, temperature jumps
- Models
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  - Free electron laser
- Slow dynamics
  - Quasi-stationary states
  - Metastability
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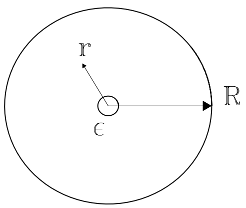
Long range interactions

- Energy of a particle at the center of a sphere of radius R where matter is homogeneously distributed



$$U = \int_{\varepsilon}^{R} 4\pi r^2 dr\ \rho\ \frac{1}{r^\alpha} = 4\pi\rho \int_{\varepsilon}^{R} r^{2-\alpha} dr \propto \left[r^{3-\alpha}\right]_{\varepsilon}^{R} \sim R^{3-\alpha}$$

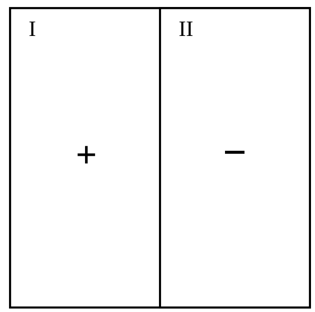
The contribution of the surface of the sphere can be neglected only if $\alpha > 3$.
Long range if $\alpha \leq 3$ ($\alpha \leq d$)

- Physical examples
  - Gravity $\alpha = 1, d = 3$, singularity at the origin
  - Coulomb $\alpha = 1, d = 3$, Debye screening
  - Dipolar $\alpha = 3, d = 3$, shape dependence
  - Onsager vortices $\alpha = 0, d = 2$
  - Mean-Field $\alpha = 0$, any d.

Extensive but not additive

$$H = -\frac{J}{2N}\sum_{i,j}\sigma_i\sigma_j$$

The Curie-Weiss Hamiltonian is EXTENSIVE $H \sim N$ but not ADDITIVE

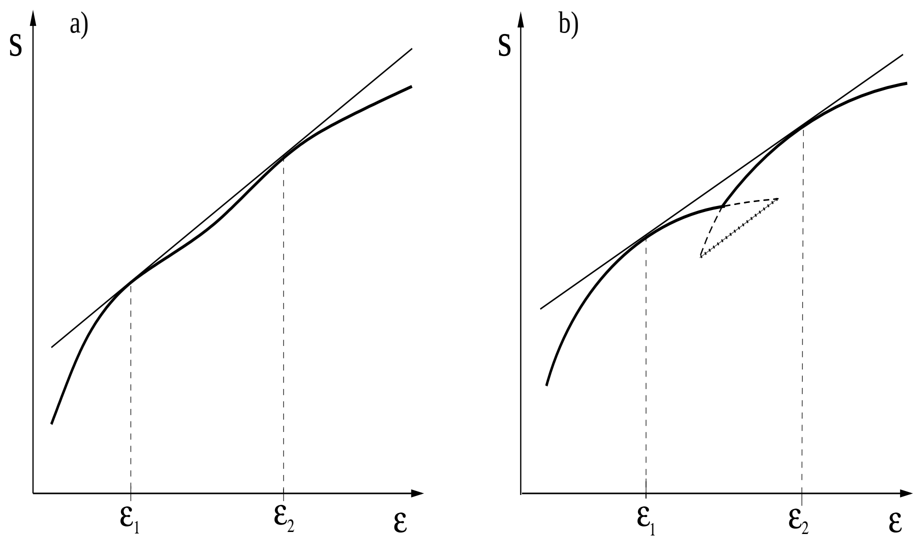


Zero magnetization state $M = \sum_i \sigma_i = 0$
$E_{I+II} = 0$, $E_I = E_{II} = -J/8N$
Hence

$$E_{I+II} \neq E_I + E_{II}$$

Ensemble inequivalence

Convex intruders

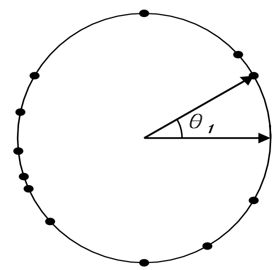


- Negative heat capacity (negative susceptibility, etc.)
- Temperature jumps

XY models

Simple mean-field models with Hamiltonian dynamics

$$H_{XY} = \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{J}{2N} (\sum_{i=1}^{N} \vec{s}_i)^2 - \frac{K}{4N^3} \left[(\sum_{i=1}^{N} \vec{s}_i)^2 \right]^2 , \quad \vec{s}_i = (\cos\theta_i, \sin\theta_i)$$

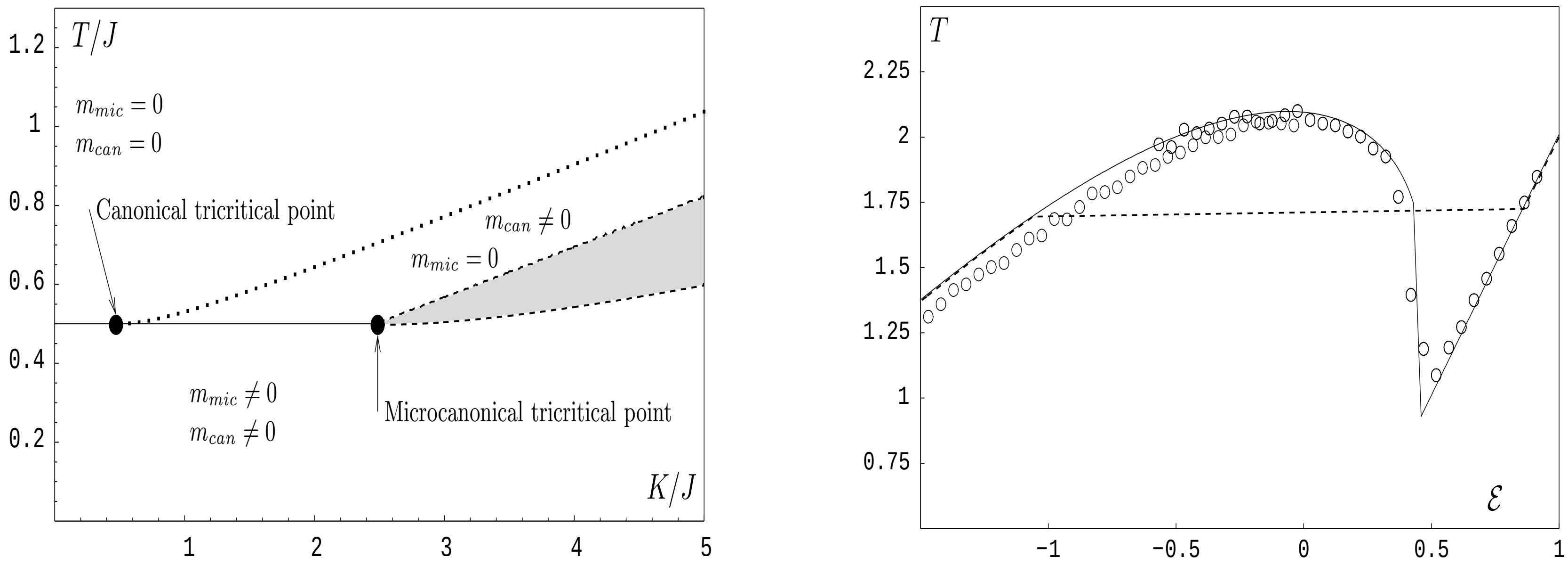


Simplifications of

- Gravitational and charged sheet models
- Wave-particle interactions

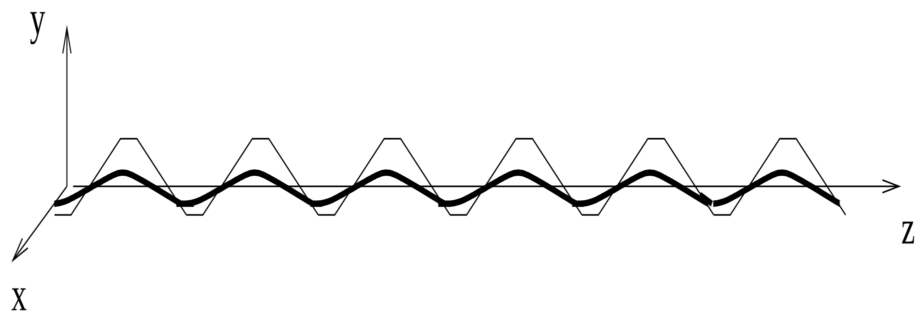
with D. Mukamel (Weizmann) and P. De Buyl (ULB, Bruxelles)

Phase diagram and caloric curves



- At $K/J = 0$ (HMF model), second order phase transition at $T/J = 0.5$. Ensembles are equivalent.
- For $K/J > 1/2$ ensembles are inequivalent. Negative specific heat for $1/2 < K \leq 5/2$; Temperature jumps for $K > 5/2$.
- Right figure shows the caloric curve for $K/J = 10$. The points are results of a molecular dynamics simulation with $N = 100$

Free Electron Laser

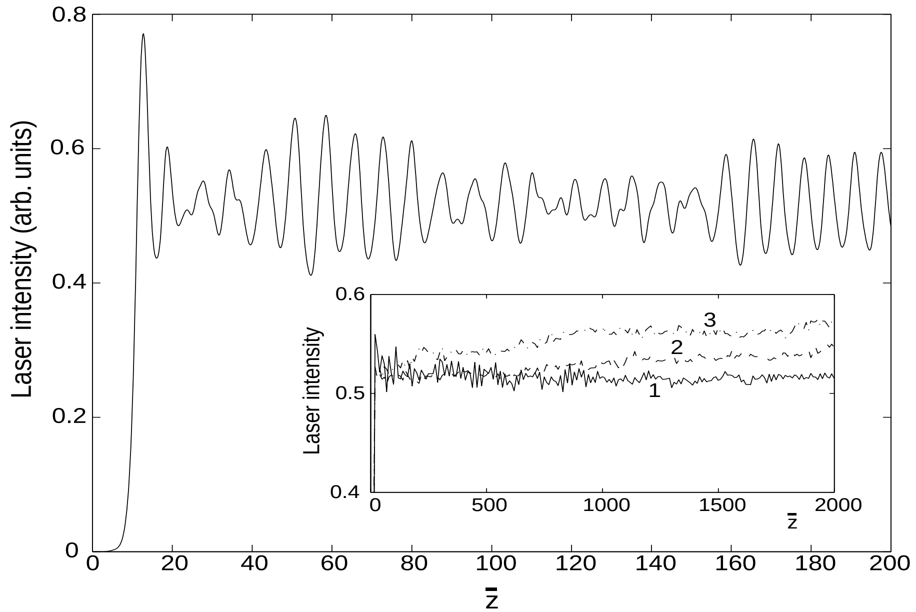


Colson-Bonifacio model

$$\begin{aligned} \frac{d\theta_j}{dz} &= p_j \\ \frac{dp_j}{dz} &= -\mathbf{A}e^{i\theta_j} - \mathbf{A}^* e^{-i\theta_j} \\ \frac{d\mathbf{A}}{dz} &= i\delta\mathbf{A} + \frac{1}{N}\sum_j e^{-i\theta_j} \end{aligned}$$

with A. Antoniazzi (Florence), J. Barré (Nice), T. Dauxois (ENS-Lyon), D. Fanelli (Florence and Stockholm), G. De Ninno (Sincrotrone Trieste)

Quasi-stationary states



$N = 5000$ (curve 1), $N = 400$ (curve 2), $N = 100$ (curve 3)

On a first stage the system converges to a quasi-stationary state. Later it relaxes to Boltzmann-Gibbs equilibrium on a time $O(N)$. The quasi-stationary state is a Vlasov equilibrium, sufficiently well described by Lynden-Bell's Fermi-like distribution arising from "violent relaxation" (constrained maximum entropy principle).

Vlasov equation

In the $N \to \infty$ limit, the single particle distribution function $f(\theta, p, t)$ obeys a Vlasov equation.

$$
\begin{aligned}
\frac{\partial f}{\partial z} &= -p\frac{\partial f}{\partial \theta} + 2(A_x \cos\theta - A_y \sin\theta)\frac{\partial f}{\partial p} \quad , \\
\frac{\partial A_x}{\partial z} &= -\delta A_y + \frac{1}{2\pi}\int f \cos\theta \, d\theta dp \quad , \\
\frac{\partial A_y}{\partial \bar{z}} &= \delta A_x - \frac{1}{2\pi}\int f \sin\theta \, d\theta dp \ .
\end{aligned}
$$

with $\mathbf{A} = A_x + iA_y = \sqrt{I}\exp(-i\varphi)$

Vlasov equilibria

Coarse grained entropy maximization (Lynden-Bell 1968, Chavanis, 1996)

$$s(\bar{f}) = -\int dp d\theta \left(\frac{\bar{f}}{f_0} \ln \frac{\bar{f}}{f_0} + \left(1 - \frac{\bar{f}}{f_0}\right) \ln \left(1 - \frac{\bar{f}}{f_0}\right)\right).$$

$$S(\varepsilon, \sigma) = \max_{\bar{f}, A_x, A_y} [s(\bar{f}) | H(\bar{f}, A_x, A_y) = N\varepsilon; \int d\theta dp \bar{f} = 1; P(\bar{f}, A_x, A_y) = \sigma].$$

Non Gaussian momentum distribution

$$\bar{f} = f_0 \frac{e^{-\beta(p^2/2 + 2A \sin\theta) - \lambda p - \mu}}{1 + e^{-\beta(p^2/2 + 2A \sin\theta) - \lambda p - \mu}} \, .$$

Non-equilibrium field amplitude

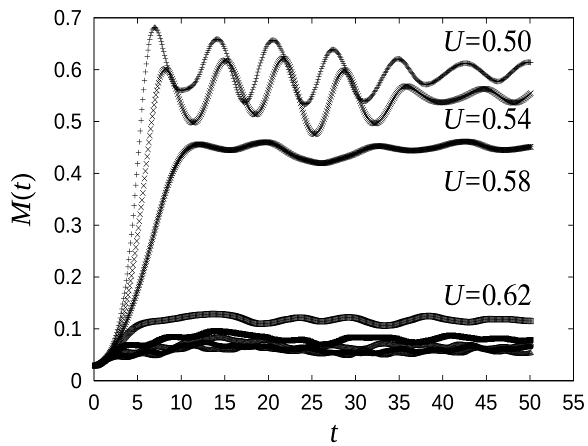
$$A = \sqrt{A_x^2 + A_y^2} = \frac{\beta}{\beta\delta - \lambda} \int dp d\theta \, \sin\theta \, \bar{f}(\theta, p).$$

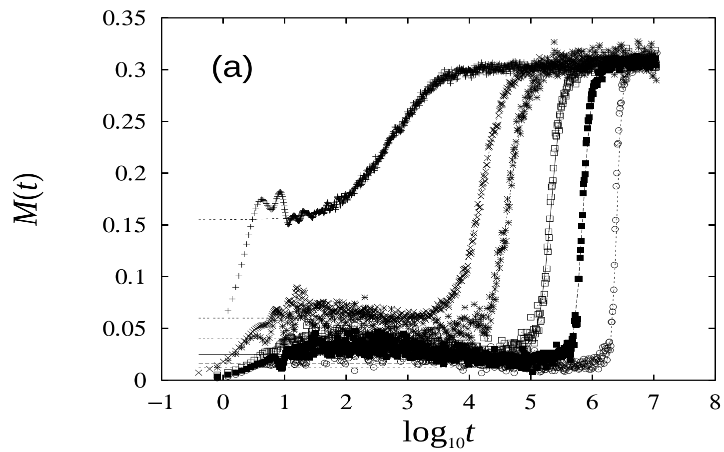
Dynamics of HMF

For the HMF model (XY model with $(J = 1, K = 0)$) there is a second order phase transition at $\epsilon_c = E_c/N = 0.25$ ($T_c = 0.5$).

In the Vlasov equation context, the phase transition is signalled by the instability of the homogeneous state with Gaussian momentum distribution. For other initial states, this Vlasov instability can appear at energy densities ϵ_{inst} that lie below ϵ_c. For $\epsilon_{inst} < \epsilon < \epsilon_c$ the finite N system can remain trapped in quasi-stationary states whose life-time increases with a power of N.

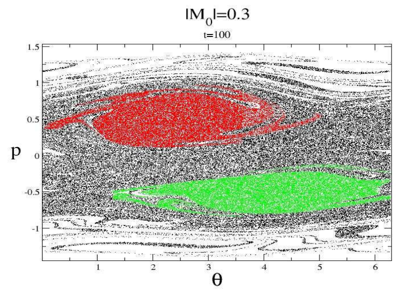
For the so called “water bag" states, where the particles are homogeneously distributed on the circle and momenta have a symmetric square distribution, the Vlasov instability energy density is $\epsilon_c = 1/12$ and the life-time of metastable states increases as $t \sim N^{\alpha}$, $\alpha \sim 1.7$.

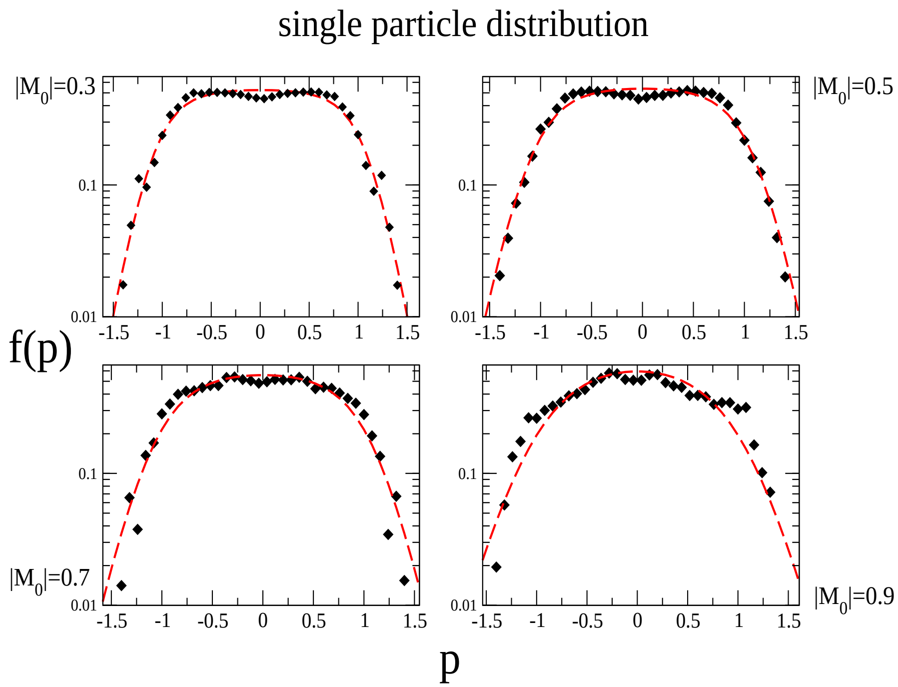




HMF macroparticle

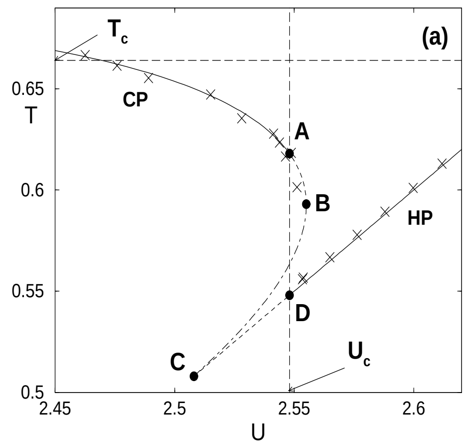
Refinements of maximum entropy methods should include the macroparticle concept (Tennyson, 1994)

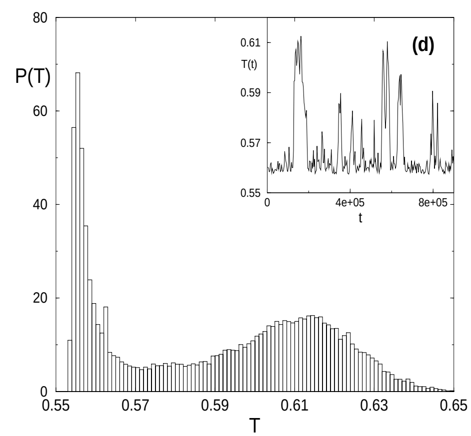


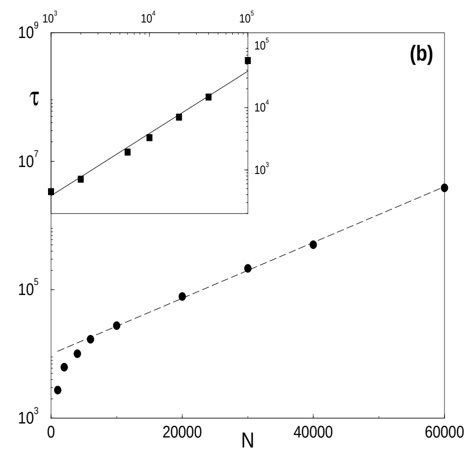


Metastability

At a microcanonical first order transition, temperature has a bimodal distribution.
Once prepared in a local entropy maximum the system relaxes to the global entropy maximum on a time that increases with $\exp(N\Delta s)$, where Δs is the entropy density barrier size.



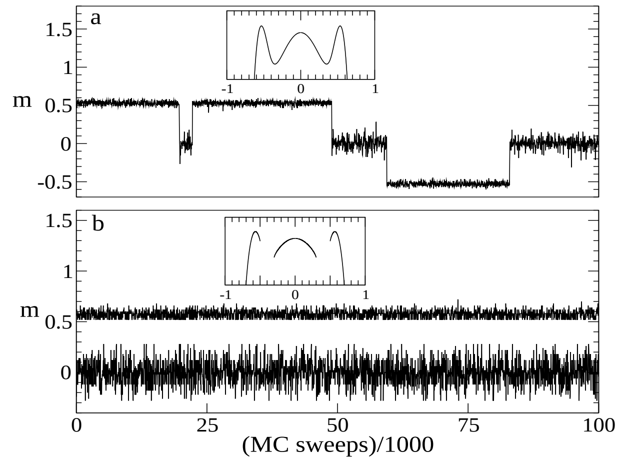




Broken ergodicity

Ising model with short and long-range interactions on a ring

$$H = -\frac{K}{2}\sum_{i=1}^{N}(S_iS_{i+1}-1) - \frac{J}{2N}\left(\sum_{i=1}^{N}S_i\right)^2 ,$$



with D. Mukamel and N. Schreiber (Weizmann)

Conclusions

- Microcanonical and canonical ensemble disagree for long range interactions at canonical first order transitions.
- Negative specific heat and temperature jumps are typical signatures of *ensemble inequivalence*.
- Collective phenomena for wave-particle interactions (FEL) are the result of constrained maximum entropy principles (Vlasov equilibria).
- Quasi-stationary states appear whose life-time increases with system size.
- Metastable states have a life-time which increases exponentially with system size.
- Due to non-additivity, broken ergodicity is a generic feature of systems with long-range interactions.

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