# ASYMPTOTICALLY EXACT SCALING FOR NONEQUILIBRIUM STATIC & DYNAMIC CRITICAL BEHAVIOUR

Robin Stinchcombe, Oxford
stinch@thphys.ox.ac.uk

## Outline

Introduction

model, operator algebra

Scaling

- static : Method I
  - Method II
- dynamic

Conclusion

# SCALING

Direct approach to static/dynamic "critical" behaviour (typically where a characteristic length $\xi$ diverges), e.g.

near static equil$^{\underline{m}}$ trans$^{\underline{n}}$
" steady state nonequil$^{\underline{m}}$ trans$^{\underline{n}}$ *
in coarsening, etc.

* "simplest" example: ASEP in $d=1$ (driven lattice gas)

Will discuss asymptotically exact static & dynamic scaling for ASEP, etc.

→ REMINDERS about model

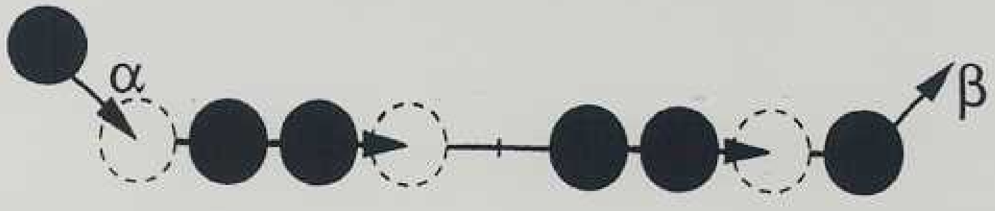
# SCALING APPROACH

Uses majority rule blocking of operators $D, E$.

Reduces using operator algebra

Resulting R.G. transformations exact at large scale factor $b$.

# (FULLY) ASYMMETRIC EXCLUSION PROCESS.



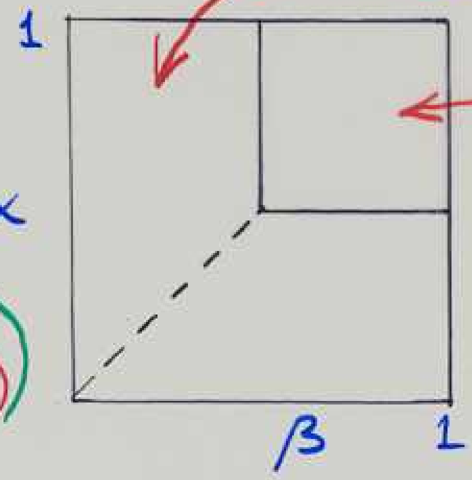

Mean field 

$J_{l\,l+1} = \rho_l\,(1-\rho_{l+1})$

steady state : $J_{l\,l+1} = \text{const.},\ J$

solved by $\rho_l = \frac{1}{2} + \dots \tan \xi^{-1}(l - \dots)$ $\quad J > J_c = \frac{1}{4}$

$\tanh \dots (\dots)$ $\quad J < J_c$




transition & $J_c$, $\rho_c$ correct

static exponents incorrect

(from exact static sol$^{\underline{n}}$

(Derrida et al: operator alg.))

mean field dynamics

$$\frac{\partial \rho}{\partial t} = J_{l-1\,l} - J_{l\,l+1}$$

kinks ...

disagrees with known $z = 3/2$

(Gwa & Spohn: Bethe ansatz)

Essler & de Gier.

# OPERATOR ALGEBRA APPROACH

(Derrida, Evans, Hakim, Pasquier)

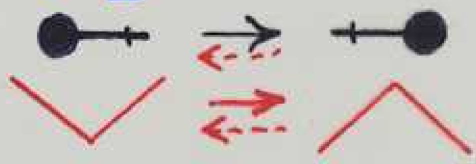
Configuration: string of operators

particle $D$
vacancy $E$

normalisation, shift $C \equiv D + E$

current (ASEP) $\Lambda = DE$

(Meakin et al
$\rightarrow$ KPZ)



steady state algebra

$$\boxed{\Lambda = C}$$

Dynamic generalisation* (Stinchcombe, Schütz)

$$\dot{D} = [C^{-1}, \Lambda] \quad \text{equ. of motion}$$

$$D C^{-1} \Lambda = \Lambda C^{-1} D \quad \text{constraint (quantiz}^{\text{n}}\text{)}$$

contains st. st. algebra as a special case

* generalises for any 1d lattice gas (non conserving cases, etc, etc)

# MAJORITY RULE FOR BLOCKING

Example: **$b=3$**:

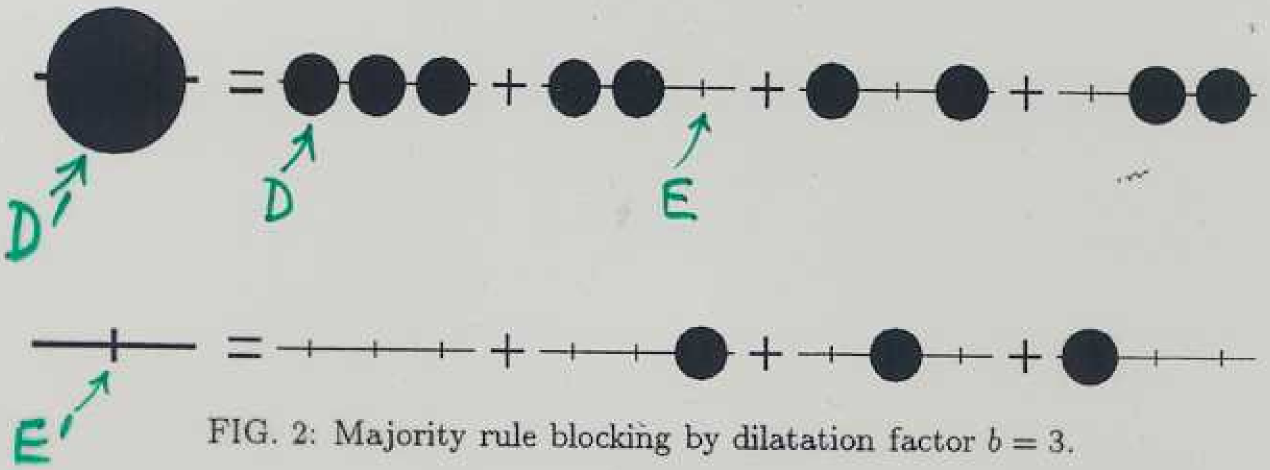



FIG. 2: Majority rule blocking by dilatation factor $b = 3$.

## For general dilatation $b$ (odd):

$$D'(b) + E'(b) = C'(b) = C^b = (D+E)^b$$

$$\Lambda'(b) = D'(b)\, E'(b)$$

# STEADY STATE SCALING

METHOD I, for cyclic b.c.'s (homogeneous...)

Average density, current, in size L:

$$\rho = Z_L^{-1} \,\mathrm{Tr}\, D C^{L-1} \qquad Z_L \equiv \mathrm{Tr}\, C^L$$
$$J = Z_L^{-1} \,\mathrm{Tr}\, \Lambda C^{L-2}$$

$\Rightarrow$ blocking

$$\rho' = Z_L^{-1} \,\mathrm{Tr}\, D' C^{L-b}$$
$$J' = Z_L^{-1} \,\mathrm{Tr}\, \Lambda' C^{L-2b}$$

Example $b=3$:

$$D'(3) = DDD + DDE + DED + EDD$$
$$= DC + CD + CDC - CC$$

$$\Lambda'(3) = 3C^4 + 3C[D,C]C + 4C^3 + [D,C^2] + [D,C]$$

Used (i) st. st. algebra

Now use (ii) cyclic invariance within Trace

(iii) $\mathrm{Tr}\, C^{L-(n+1)} / \mathrm{Tr}\, C^{L-n} = J$ in $(L-n)$-site system

(iv) large $L$

$$\Rightarrow \quad \boxed{\begin{aligned} \rho'(3) &= \rho(1+2J) - J \\ J'(3) &= 3J^2 + 4J^3 \end{aligned}}$$

(Scaling transformation for dilatation $b=3$)

# FLOW DIAGRAM ($b=3$)

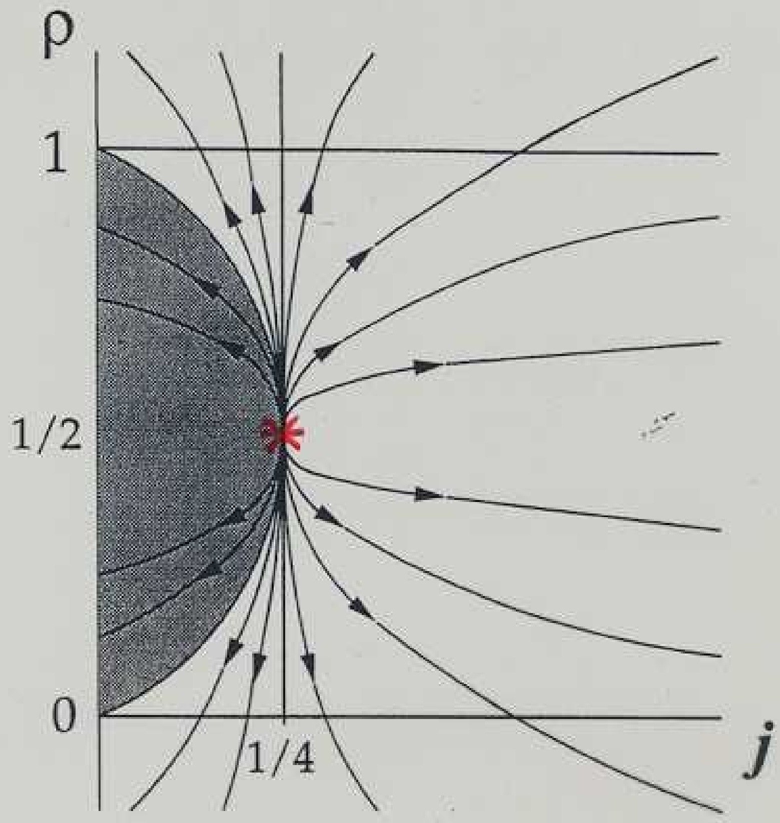



FIG. 3: Flow diagram of steady state scaling from the most primitive ($b = 3$) blocking. The critical fixed point is at $\rho^* = \frac{1}{2}$, $j^* = \frac{1}{4}$, and $j = 0$ is a line of fixed points.

– for SCALING

$$\boxed{\begin{aligned} \rho'(3) &= \rho(1+2j) - j \\ j'(3) &= 3j^2 + 4j^3 \end{aligned}}$$

* fixed point $\rho^* = \frac{1}{2}$, $j^* = \frac{1}{4}$ exact

eigenvalues $\lambda_j = \lambda_\rho^{\,2}$ exact relation

correct scaling scenario

# GENERALISATION TO ARBITRARY b

$$\boxed{\begin{aligned} J'(b) &= G_b(J) = J^b\, T_b(\mu_1/\mu_2)\, T_b(\mu_2/\mu_1) \\ \rho'(b) &= F_b(\rho, J) = \rho\, f_b(J) + g_b(J) \end{aligned}}$$

specific $f^n$'s of $J$ via $\mu_1 \mu_2$ $T_b$'s

where $T_b(y) = \sum_{r=\frac{1}{2}(b+1)}^{b} {}_bC_r\, y^r$ ,

$\mu_1 + \mu_2 = 1$ , $\mu_1 \mu_2 = J$ .

Gives

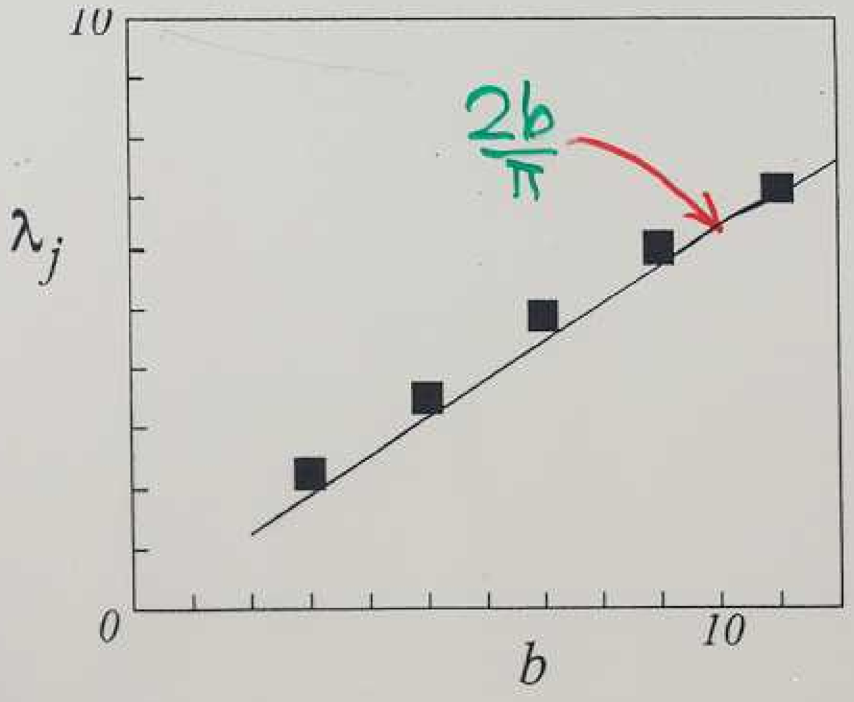
<u>f.p.</u> $\rho^* = \frac{1}{2}$ , $J^* = \frac{1}{4}$

<u>eigenvalues</u> $\lambda_J(b) = \lambda_\rho(b)^2$

exact

$$\lambda_\rho(b) = 2^{1-b} \sum_{r=\frac{1}{2}(b+1)}^{b} {}_bC_r\,(2r-b) \xrightarrow[b\ \text{large}]{} \underline{\underline{b^{1/2}\sqrt{2/\pi}}}$$

$$\Downarrow$$




$$\boxed{\begin{aligned} \xi &\sim (J - J^*)^{-1} \\ \alpha &\equiv (\rho - \rho^*) \\ &\sim (J - J^*)^{1/2} \end{aligned}}$$

exact

# METHOD II

(i) Applies at level of fields

(ii) Uses only $D + E = C$ , $\Lambda = DE$

Gives

$$\Lambda'(b) = C^{2b}\, \mathcal{G}_b(\Lambda C^{-2})$$

$$D'(b) = C^{b}\, \mathcal{F}_b(D C^{-1}, \Lambda C^{-2})$$

such that, for slow spatial variations,

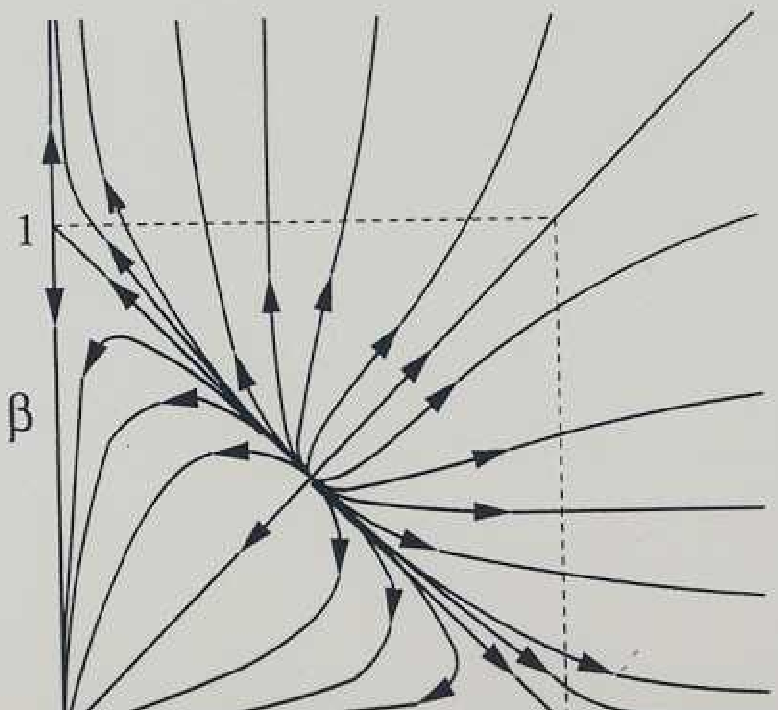
$$\mathcal{G}_b \to G_b$$

$$\mathcal{F}_b \to F_b$$

So recover previous scalings of $\rho, J$.

Allows generalisation to dynamic scaling, non-homogeneous situations

Flow, in terms of injection and ejection rates $\alpha$, $\beta$, for the $b=3$ steady state scaling:




Flow in terms of the injection and ejection rates $\alpha$ and $\beta$, for the $b=3$ steady state scaling

# DYNAMIC SCALING

Uses (i) Majority rule blocking

(ii) Method II

(iii) operator equation of motion in form

$$\partial D/\partial t = \partial \Lambda / \partial l$$

(iv) constraint equation automatic^ly satisfied by neglecting gradient terms beyond leading order.

Using (i), (ii), (iii), scaled version of (iii) becomes

$$\frac{\partial t}{\partial t'}\Big[\underbrace{C^{b-1} f_b(\Lambda C^{-2})}_{\substack{\text{operator equivalent}\\ \text{of } \lambda_\rho(b)}} + \underbrace{\ldots}_{\substack{\text{negligible in}\\ \text{critical regime}\\ \text{(factors } C^{-2D})}}\Big] = \frac{\partial l}{\partial l'}\Big(\underbrace{\partial C^{2b} G_b(\Lambda C^{-2})/\partial \Lambda}_{\substack{\text{operator}\\ \text{equiv. of}\\ \lambda_J(b)}}\Big)$$

At f.p.

$$\underbrace{\frac{\partial t}{\partial t'}}_{\ldots b^{z}} \underbrace{\lambda_\rho(b)}_{\ldots b^{1/2}} = \underbrace{\frac{\partial l}{\partial l'}}_{b} \underbrace{\lambda_J(b)}_{\ldots b} \quad \text{(b large)}$$

dynamic exponent $z = \frac{3}{2}$ (cf Bethe ansatz)

(i) Consistent with Galilean invariance)

(ii) Gives eg profile

$$\sigma = (l + \ldots)^{-1/2} \phi\left(\frac{t + \ldots}{(l + \ldots)^{3/2}}\right)$$

# CONCLUDING REMARKS

## SUMMARY

(i) Two procedures for static scaling:
Method I, for homogeneous situations, uses steady state algebra
Method II unrestricted, consistent with Method I

(ii) Can carry out both for any dilatation b

(iii) Give for any b exact relations

(a) $(\rho_c, J_c) = (1/2, 1/4)$

(b) $\lambda_\rho^2 = \lambda_J$ and consequence $\sigma \propto (J_c - J)^{1/2}$

(c) $b^z = b\lambda_J / \lambda_\rho$, via generalisation of Method II to dynamic scaling

(iv) Give exact individual static and dynamic exponents in limit of large b.

## FURTHER APPLICATIONS

(a) More on little-understood ASEP dynamics

(b) Since don't need 'reduced' steady state algebra in Method II, nor a representation of the general dynamic one, can in principle apply method to other models

Closest generalisation is to PASEP

Other possibilities include:
non-conserving, two-species, quasi 1d models.