Pinning of random directed polymers: smoothing of the transition and some path properties

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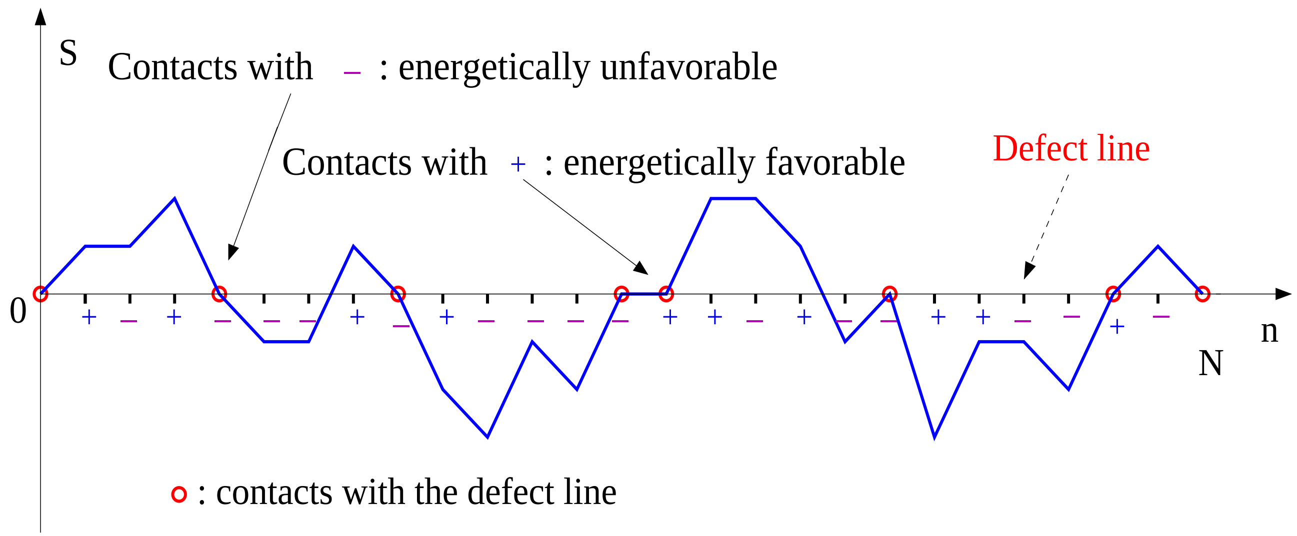
Outline of the talk

- Pinning of directed polymers on a defect line
- The mathematical model
- Applications: (1+1)–interface wetting, DNA denaturation...
- Depinning transition: known results
- Smoothing of the transition
- Estimates on path delocalization
- Open problems

Random pinning of directed polymers at a defect line



Does the polymer stick to the line, or does it wander away?

Some references (for this or related models):

Forgacs, Luck, Nieuwenhuizen, Orland 1986

Derrida, Hakim, Vannimenius 1992

Alexander, Sidoravicius 2005

The mathematical model

$\{S_n\}_{n=0,1\ldots}$ homogeneous Markov chain, state space $\mathcal{S}$, law $\mathbf{P}$.

$S \equiv 0 \in \mathcal{S}$: defect line. Initial condition: $S_0 = 0$ (on the line).

First return to zero: assume that, for some $1 \leq \alpha < \infty$,

$$K(n) \equiv \mathbf{P}(\min\{k > 0 \text{ such that } S_k = 0\} = n) \sim n^{-\alpha}, \quad n \to \infty.$$

Notice: if $\alpha < 2$ the return times are not integrable

Introduce the pinning measures:

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{\exp\left(\sum_{n=1}^{N}(\beta\omega_n - h)1_{S_n=0}\right)1_{S_N=0}}{Z_{N,\omega}}$$

where: $N \in \mathbb{N}$, $\beta, h \geq 0$, $\omega = \{\omega_n\}_{n=1,2\ldots} \in \mathbb{R}^N$ (charges)

Random model:

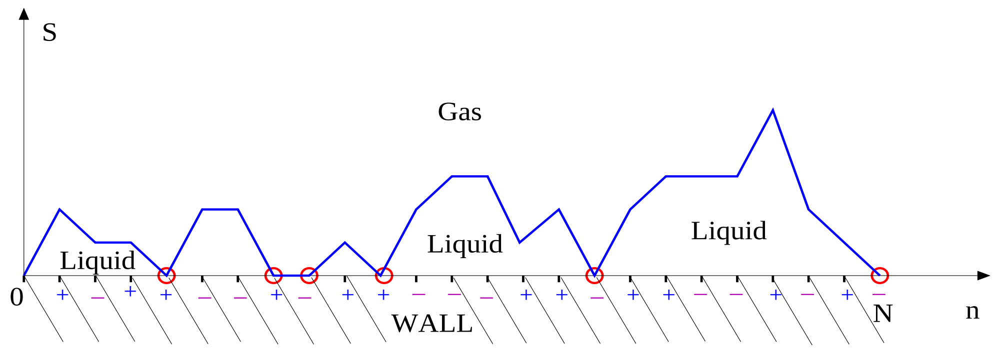
$\{\omega_n\}_{n=1,2\ldots}$: i.i.d. random variables with law $\mathbb{P}$ and properties:

- symmetry: $\omega_n \sim -\omega_n$
- normalization: $\mathbb{E}\,\omega_n^2 = 1$
- finite exponential moments: $\mathbb{E}\,e^{t\omega_n} < \infty$ for $t \in \mathbb{R}$
- some "smoothness" conditions (not detailed here)

Examples to keep in mind:
ω_n Gaussian $\mathcal{N}(0,1)$, or $\omega_n = \pm 1$ symmetric

Applications of the model

- $(1+1)$–dimensional wetting. $\mathcal{S} = \mathbb{Z}^+$. S: random walk, with bounded i.i.d. increments, conditioned to $S_n \geq 0$. Here, $\alpha = 3/2$



- pinning of $(1+d)$–dimensional directed polymers. Here, $\mathcal{S} = \mathbb{Z}^d$ and $\alpha = 3/2$ if $d = 1$, $\alpha = d/2$ if $d \geq 2$

- Poland–Scheraga model of DNA denaturation (separation of the two DNA strands at high temperature).
Here, $\mathcal{S} = \mathbb{Z}$ and $\alpha \simeq 2.11$ (Kafri, Mukamel, Peliti 2000)

Free energy and depinning transition

Simple facts: $\mathbb{E}\,\log Z_{N,\omega}$ is super–additive in N, and

$$\lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega} = f(\beta,h) \qquad \mathbb{P}(\,\mathrm{d}\omega) - a.s.$$

Preliminary remark: $f(\beta,h) \ge 0$. Indeed,

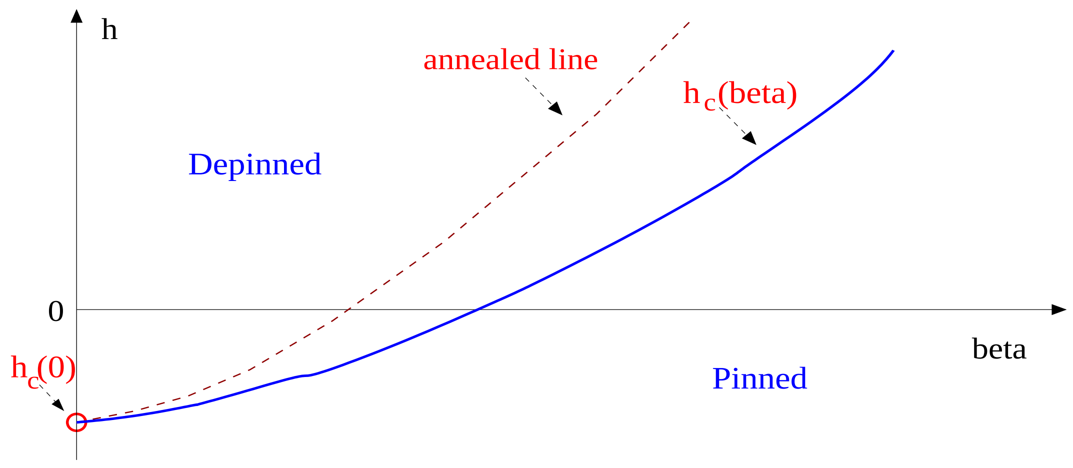
$$\frac{1}{N}\mathbb{E}\,\log Z_{N,\omega} \ge \frac{1}{N}\mathbb{E}\,\log \mathbf{E}\left\{e^{\sum_{n=1}^{N}(\beta\omega_n - h)1_{S_n=0}};\, S_n \neq 0\ \forall n < N, S_N = 0\right\}$$

$$= \frac{1}{N}(\beta\mathbb{E}\,\omega_1 - h) + \frac{1}{N}\log \mathbf{P}(\text{first return to 0 occurs at } N) \longrightarrow 0.$$

This suggests to define:

- the pinned (localized) region $\mathcal{P} = \{(\beta,h) : f(\beta,h) > 0\}$
- the depinned (delocalized) region $\mathcal{D} = \{(\beta,h) : f(\beta,h) = 0\}$

Depinning transition: known results I (free energy)



The two regions are separated by a critical line $h_c(\beta)$:

$\mathcal{P} = \{(\beta, h) : h < h_c(\beta)\}$, $\mathcal{D} = \{(\beta, h) : h \geq h_c(\beta)\}$

Properties (Alexander, Sidoravicius 2005):

- $h_c(\beta)$ continuous, increasing, convex, and $|h_c(\beta)| < \infty$.
- $h_c(0) \leq 0$ and $h_c(0) < 0$ iff the Markov chain is transient.

The order parameter

The order paramter is the contact fraction:

$$\ell_N = \frac{L_N}{N} \equiv \frac{1}{N}|\{1 \le n \le N : S_n = 0\}|$$

related to the free energy by

$$\partial_h f(\beta, h) = \lim_{N\to\infty} \mathbb{E}\,\mathbf{E}_{N,\omega}(\ell_N).$$

$\mathbf{E}_{N,\omega}(\cdot)$: finite–volume, disorder–dependent Gibbs measure (with parameters β, h)

Smoothness of the transition

What happens at the critical point $h_c(\beta)$?

Non–random case $\beta = 0$

Transition is either of first or of higher order:

- If $\sum_{n\geq 1} n\, K(n) < +\infty$: first order (e.g., $\alpha > 2$)

$$f(0,h) \sim c(h_c(0) - h), \quad h \to h_c(0)^-$$

- If $\sum_{n\geq 1} n\, K(n) = +\infty$: higher order
(Non–integrable return times, e.g. $1 \leq \alpha < 2$)

$$f(0,h) \sim c(h_c(0) - h)^{1/(\alpha-1)}, \quad h \to h_c(0)^-$$

Random case $\beta > 0$

Theorem 1 [G. Giacomin, F.T., Commun. Math. Phys. to appear]
For $0 < \beta < \infty$ there exists $c(\beta) < \infty$ such that, for $1 \leq \alpha < \infty$, and $h < h_c(\beta)$,

$$0 < f(\beta, h) < \alpha\, c(\beta)(h_c(\beta) - h)^2.$$

I.e., the transition is always continuous (at least of second order)

Remark: $c(\beta)$ can be large for β small: $c(\beta) \sim c/\beta^2$.

Moreover, finite–size scaling results (F.T., preprint), of the type: if $|h - h_c(\beta)| \sim N^{-1/3}$, then $\ell_N \leq N^{-1/3}$ with probability ~ 1 (probably not optimal).

Discussion of the smoothing result

• This solves somes controversies in the literature: e.g. for the Poland–Scheraga model ($\alpha \simeq 2.11 > 2$), some numerics indicated first–order transition for $\beta > 0$ (e.g., Garel, Monthus 2005) while others suggested higher order (e.g., Coluzzi 2005)

• When $\alpha > 3/2$, the disorder really smooths the transition

• The rounding mechanism is very different from that of the 2–d Random Field Ising Model (Aizenman–Wehr 1989)!
For the 2d–RFIM: Imry–Ma argument (effect of boundary conditions suppressed by random field fluctuations).
In our case, boundary conditions play no role

Depinning transition: known results II (paths)

Is the polymer really pinned (or depinned) in $\mathcal{P}$ (or $\mathcal{D}$)?

• For $h < h_c(\beta)$, long excursions of S from the line are exponentially suppressed:

$$\mathbb{E}\,\mathbf{P}_{N,\omega}(S_k \neq 0, S_{k+1} \neq 0, \cdots, S_{k+s} \neq 0) \leq c_1 e^{-c_2 f(\beta,h)|s|}, \quad \forall k \in \mathbb{N}.$$

More refined (and $\mathbb{P}(\,\mathrm{d}\omega)$–a.s.) statements: see Biskup, den Hollander 1999, Albeverio, Zhou 1996, Sinai 1993 in a related model (copolymer).

Easier fact (from convexity of $f(\beta, .)$):

$$0 < \lim_{N\to\infty} \mathbf{E}_{N,\omega}(\ell_N) < 1, \quad \mathbb{P}(\,\mathrm{d}\omega) - a.s.$$

That is, finite fraction of time on the defect line

• For $h > h_c(\beta)$, one has $\ell_N = (1/N)|\{n : S_n = 0\}| \to 0$ (zero fraction of time on the defect line)

This just follows from $\partial_h f(\beta, h) = 0$ and

$$\lim_{N\to\infty} \mathbf{E}_{N,\omega}(\ell_N) = \partial_h f(\beta, h) = 0, \quad \mathbb{P}(\mathrm{d}\omega) - a.s.$$

However, this is unsatisfactory: one expects

$$L_N (= N\ell_N) = O(1) \quad \text{(in some sense)}.$$

Estimates on path delocalization

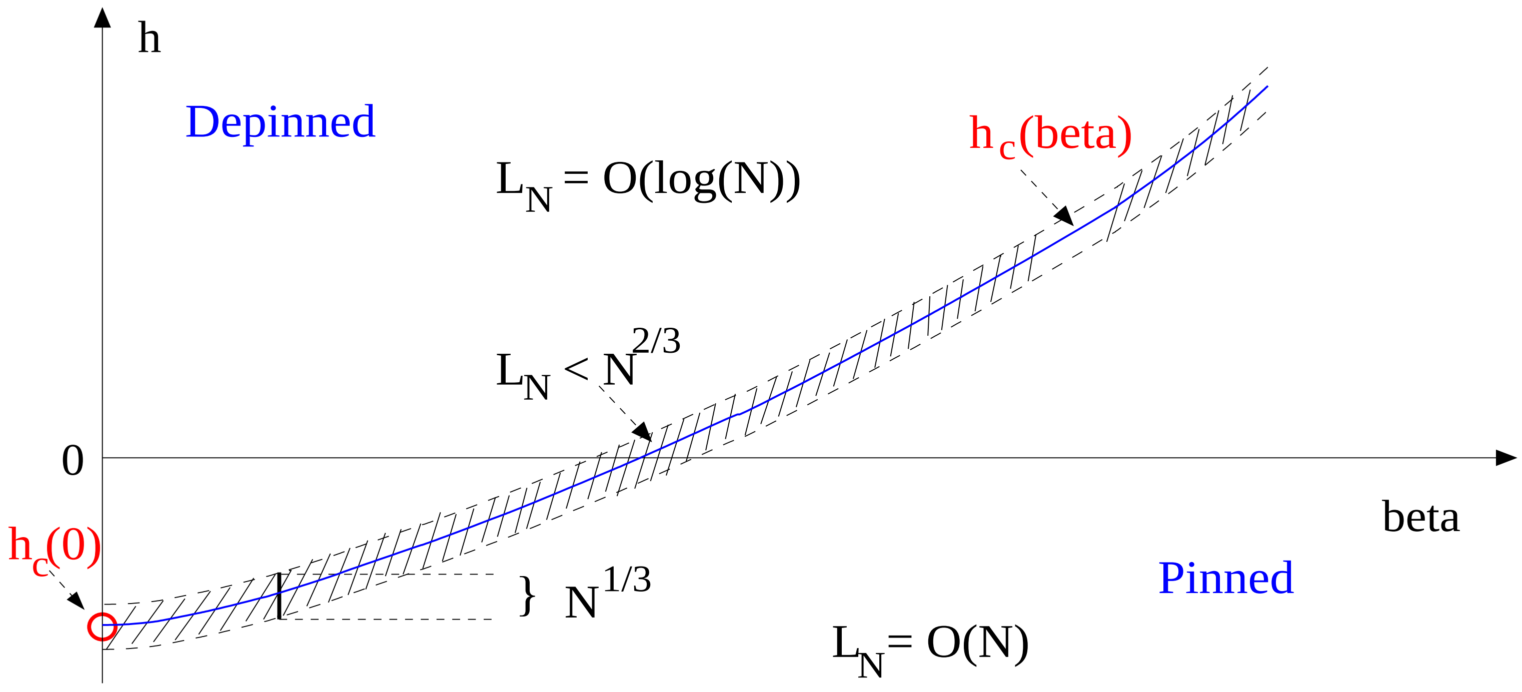
Theorem 2 [G. Giacomin, F.T., Probab. Theory Rel. Fields 2005]

Let $h > h_c(\beta)$ (depinned region).
Recall $L_N = |\{1 \leq n \leq N : S_n = 0\}|$ (number of contacts with the line). Then, $\exists c_1, c_2 > 0$ such that for any N,

$$\mathbb{E}\mathbf{P}_{N,\omega}(L_N \geq m) \leq e^{-c_1 m} \quad \forall m \geq c_2 \log N$$

Therefore, $L_N = o(N)$ improved to $L_N = O(\log N)$, “in average”.

To summarize known results



Open problems

• Is the transition of second order or higher?
Some literature claims that it is of infinite order for $\alpha \leq 3/2$, i.e., $f(\beta, h)$ would be infinitely differentiable in h at $h_c(\beta)$

• Where are the contacts with the line, in the depinned region?
Expected: only close to the endpoints
Only weak results on this issue.

• Does the critical line coincide with the annealed one?
Expected: for β small and $\alpha < 3/2$, yes. Otherwise, not.