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# Towards a spectral approach to ageing

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# Plan

1. Ageing.
2. The REM-like trap model.
3. Sinai's random walk.

**Setting:** Markov processes, $X_t$, on some state-space $\mathcal{S}$ with generator $\mathcal{L}$. In this talk I will always consider the setting of a reversible Markov chain with reversible measure $\mathbb{Q}$, so $\mathcal{L}$ is self-adjoint.
**General question:** Relation between long-time properties of the dynamics and spectral properties of $\mathcal{L}$.

**Dynamic types:**
Rapid mixing: Process converges to equilibrium exponentially fast.
Spectral signature: Spectral gap between eigenvalue $0$ and rest of spectrum.

Metastability: State space decomposes into "quasi-invariant" sub-spaces; multiple time-scales: rapid mixing within quasi-invariant subspace at "short" times, transitions between quasi-invariant subspace at "large" time-scale.
Spectral signature: finite set of "exponentially small" followed by spectral gap. Small eigenvalues correspond to inverse exit times from metastable states.

Ageing: Systems are neither mixing nor metastable, but show slow (power-law) transients towards equilibrium.
Spectral signature: Subject of this talk.

Consider a (stochastic) process $X_t$. Define a correlation function

$$f(t_w, t) = C(X_{t_w}, X_{t_w+t})$$

General definition: $X_t$ ages, if $f(t_w, t)$, for $t_w, t$ large, depends on $t_w$. More restrictive: $f(t_w, t)$ is a function of $t/t_w$. [sometimes of $t^\theta/t_w$ (sub/super-ageing)].

This depends on the particular choice of correlation function. A more ”intrinsic” characterisation would maybe nicer.

For a system to age, at time $t_w$, it should be (typically) found in a states with typical “reaction time” $T(t_w)$.

A tentatative scenario (clearly not sufficiently general)

For $A \subset \mathcal{S}$, let $\bar{\lambda}(A)$ denote the smallest eigenvalue of the generator $\mathcal{L}$ with Dirichlet conditions on $A^c$.
Note: If $0 \in A$, then

$$\mathbb{P}_0\left[X_t \in A\right] \sim e^{-t\bar{\lambda}(A)}$$

Find an increasing family of sets, $0 \in \mathcal{S}_i \subset \mathcal{S}$, such that, for all $i$,

- $\mathcal{S}_i \subset \mathcal{S}_{i+1}$
- $\bar{\lambda}_i \equiv \bar{\lambda}(\mathcal{S}_i) > \bar{\lambda}(\mathcal{S}_{i+1}) \equiv \bar{\lambda}_{i+1}$
- $\mathbb{Q}(\mathcal{S}_{i-1}|\mathcal{S}_i) \sim 0$.

Then, roughly, at times $\bar{\lambda}_i^{-1} < t < \bar{\lambda}_{i+1}^{-1}$, $X_t$ should be distibuted as $\mathbb{Q}(\cdot|\mathcal{S}_i)$, and hence localised within $\mathcal{S}_i \backslash \mathcal{S}_{i-1}$.
$\mathcal{S}_i \backslash \mathcal{S}_{i-1}$ will in many examples decompose into smaller subsets, $\mathcal{S}_{i,k}$, with $\bar{\lambda}(\mathcal{S}_{ik})$ comparable to $\bar{\lambda}(\mathcal{S}_i)$.

Can one translate such a picuture directly into statements about the spectrum of $\mathcal{L}$?

## References:


- **General and trap models:**
  R. Mélin and P. Butaud, Glauber dynamics and ageing, J. de Physique. I, (1997).
- **Spherical SK-model:** Ageing related to spectral properties of random matrix, Ben Arous, Dembo, Guionnet, Prob. Theor. Rel. Fields. 2000
- **Sinai's random walk:** D.S. Fisher, P. Le Doussal, C. Monthus, Random walkers in 1-D random environments: exact renormalization group analysis, Phys. Rev. E (1999).
  C. Monthus and P. Le Doussal, Localization of thermal packets and metastable states in Sinai model, Phys. Rev. E (2002).
  P. Le Doussal and C. Monthus, Exact solutions for the statistics of extrema of some random 1D landscapes, Application to the equilibrium and the dynamics of the toy model, Physica A (2003)
  C. Monthus and P. Le Doussal, Energy dynamics in the Sinai model. Physica A (2004).
  C. Montus, Ma-Dasgupta renormalization studies of various disordered systems, 2004

**Trap models:**

$\mathcal{G} = (\mathcal{S}, \mathcal{E})$ finite graph.

$\underline{E} \equiv \{E_i,\ i \in \mathcal{S}\}$ iid r.v.

$Y(t)$ continuous–time random walk on $\mathcal{G}$ with $\underline{E}$–dependent transition rates, $c_{i,j}$

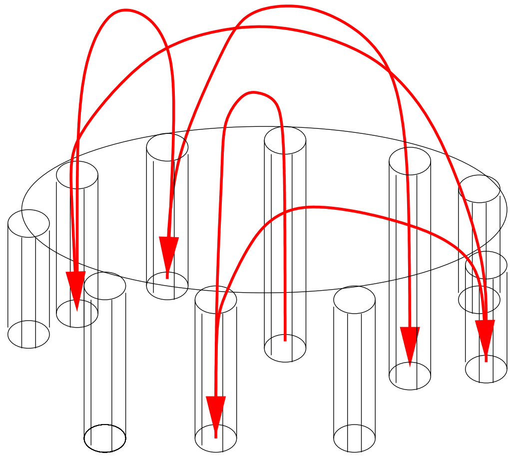
**REM-like trap model:**

$\mathcal{S} \equiv \{1, \ldots, N\}$, $\mathcal{G}$ complete graph.

$c_{i,j} = x_i \equiv e^{-E_i}/N$, for $i \neq j$

$E_i$ iid exp. with parameter $0 < \alpha < 1$.

$$\mathcal{L}_N \equiv \begin{pmatrix} \frac{(N-1)x_1}{N} & -\frac{x_1}{N} & \cdots & -\frac{x_1}{N} \\ -\frac{x_2}{N} & \frac{(N-1)x_2}{N} & \cdots & -\frac{x_2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{x_N}{N} & -\frac{x_N}{N} & \cdots & \frac{(N-1)x_N}{N} \end{pmatrix}$$



**Correlation function:**

$$\Pi_N(t, t_w) \equiv \mathcal{P}_N\left( Y_N(s) = Y_N(t_w),\ \forall s \in [t_w, t_w + t]\right).$$

**Ageing:** (Bouchaud-Dean '92) For almost all $\underline{E}$, and for all $\theta > 0$,

$$\lim_{t_w \uparrow \infty} \lim_{N \uparrow \infty} \Pi_N(\theta t_w, t_w) = \frac{\sin(\pi\alpha)}{\pi} \int_{\frac{\theta}{1+\theta}}^{1} u^{-\alpha}(1-u)^{\alpha-1} du$$

which is a somewhat generic behaviour for many ageing systems.

## Spectrum

**Proposition 1.** ▷ *Eigenvalues,* $0 = \lambda_1 < \lambda_2 \cdots < \lambda_N$, *of* $\mathcal{L}_N$ *are the zeros of*

$$\phi(\lambda) \equiv \sum_{j=1}^{N} \frac{\lambda}{x_j - \lambda}$$

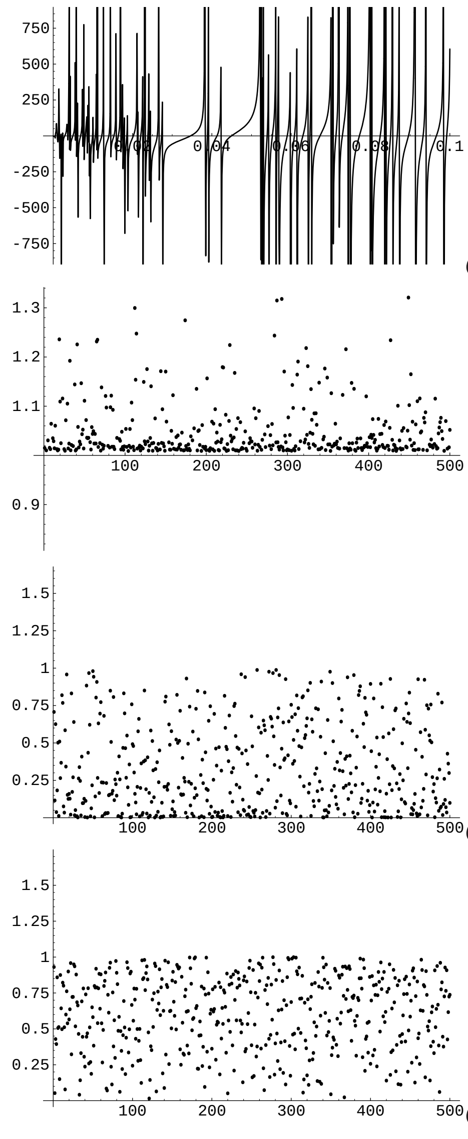
▷ *For all* $i$, $x_i < \lambda_{i+1} < x_{i+1}$

▷ *The* $j$*-th eigenvector,* $\psi^{(j)}$, *has components*

$$\psi_j^{(i)} \equiv \frac{x_j}{x_j - \lambda_i}$$

**Corollary.** Spectral distribution

$\sigma_N \equiv N^{-1} \sum_{j=1}^{N} \delta_{\lambda_j} \to \alpha x^{\alpha-1} dx$ on $\mathbb{R}_+$, a.s.




$\alpha = 0.5$

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$\alpha = 1.5$

**Proposition 2.** ▷ *Eigenvalues,* $0 = \lambda_1 < \lambda_2 \cdots < \lambda_N$*, of* $\mathcal{L}_N$ *are the zeros of*

$$\phi(\lambda) \equiv \sum_{j=1}^{N} \frac{\lambda}{x_j - \lambda}$$

▷ *For all* $i$*,* $x_i < \lambda_{i+1} < x_{i+1}$

▷ *The* $j$*-th eigenvector,* $\psi^{(j)}$*, has components*

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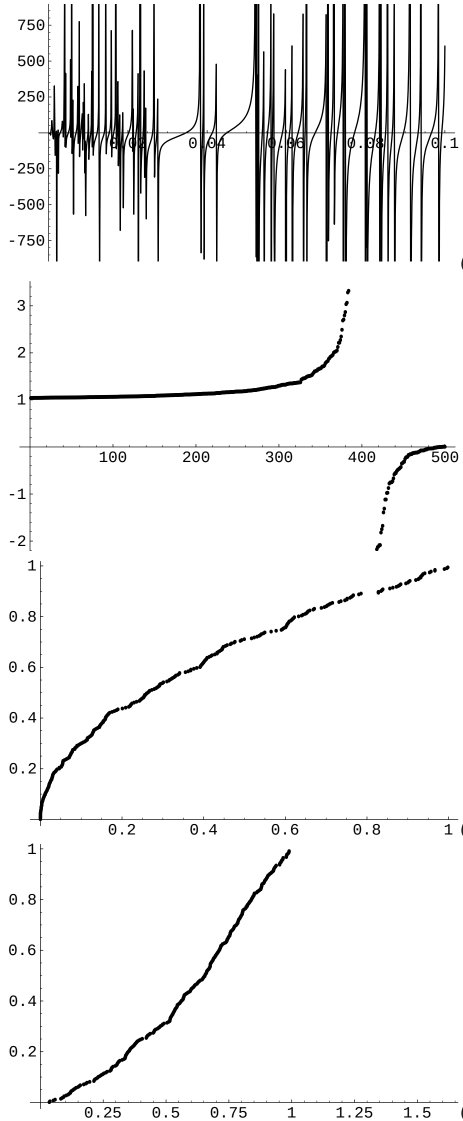


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WIAS

Consider the process as a process on the “waiting times”, i.e. set $x_N(t) \equiv x_{Y_N(t)}$. For any bounded function $h$, we can represent

$$\mathbb{E}_N\left(h(x_N(t))\right) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{\gamma_k e^{-\lambda_k t}}{x_j - \lambda_k} h(x_j)$$

where $\gamma_k \equiv \|\phi^k\|_2^2 = \sum_{j=1}^{N} \frac{x_j}{(x_j - \lambda_k)^2}$.
Key technical result:

$$\mathbb{E}_N\left(h(tx_N(t))\right) \to \frac{1}{2\pi i}\int_\gamma \frac{e^{-t\lambda}}{\lambda} \frac{\int_0^1 \frac{h(xt)}{\lambda - x} x^{\alpha-1} dx}{\int_0^1 \frac{1}{\lambda - x} x^{\alpha-1} dx} d\lambda, \quad \text{a.s.}$$

That is, the random variable $tx_N(t)$ or $t/\tau_N(t)$ converges to a random variable whose distribution is explicitly computed. All ageing results derive easily from here.

$X_n$ discrete time random walk on $\mathbb{Z}$, transition probabilities

$$\mathsf{P}(\,X_{n+1} = x+1 \,|\, X_n = x\,) = \omega_x \qquad \mathsf{P}(\,X_{n+1} = x-1 \,|\, X_n = x\,) = 1-\omega_x\,.$$

$0 < a < \omega_x < b < 1$ iid random variables s.t. $\mathbb{E}\left(\ln\left(\frac{\omega_x}{1-\omega_x}\right)\right) = 0$.

Well-know facts:

*Localisation:* [Sinai ’82]

There is random process, $\mathfrak{m}^{(n)}(\omega)$, depending only on the $\omega_x$, s.t.

$$\frac{X_n}{\ln^2 n} - \mathfrak{m}^{(n)} \to 0 \qquad \text{in } \mathcal{P}_0\text{–probability},$$

*Ageing:* [Dembo, Guionnet, Zeitouni ’01]

$$\lim_{n\uparrow\infty} P_0\left(X_n \sim X_{n^h}\right) = h^{-2}\left(\frac{5}{3} + \frac{2}{3}e^{-(h-1)}\right)$$

## Results

As is well-known, Sinai's walk is a random walk in a potential, i.e. invariant measure is

$$\mu(x) = \exp\left(-V(x)\right)$$

with

$$V(x) = \begin{cases} \sum_{i=1}^{x} \ln \frac{1-\omega_i}{\omega_i}, & \text{if } x \geq 1\,, \\ 0, & \text{if } x = 0\,, \\ -\sum_{i=x+1}^{0} \ln \frac{1-\omega_i}{\omega_i}, & \text{if } x \leq -1\,. \end{cases} \tag{1}$$

Rescaling: We will consider the process confined to a box $[-N, N]$. it is convenient to rescale to a unit box $[-1, 1]$ with spacing $1/N$. The invariant measure is then

$$\mu(x) = \exp\left(-\sqrt{N} V^{(N)}(x)\right)$$

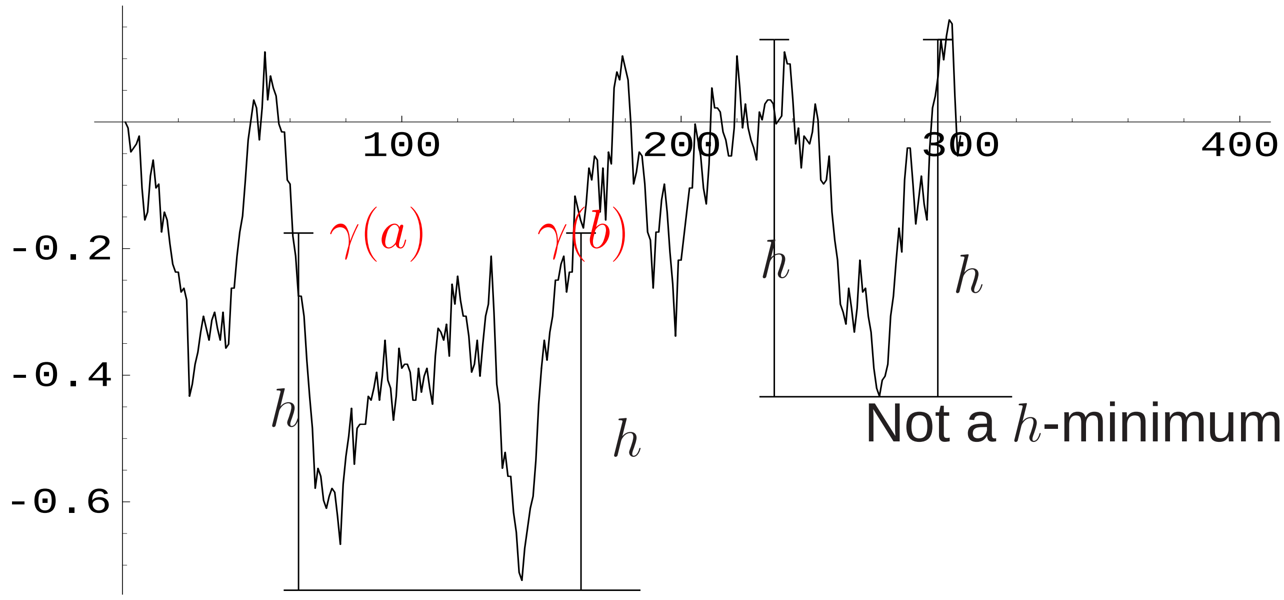
where, for $N$ large, $V^{(N)}$ can be strongly approximated by a Brownian motion, in the sense that

$$P^{(N)}\left(\sup_{x\in[-1,1]} \left|V^{(N)}(x) - B_x\right| > \frac{C_1 \ln N}{\sqrt{N}}\right) < \frac{C_2}{N^{C_3}}\,. \tag{2}$$

[Komlós-Major-Tusnády]

If $g$ is any continuous function on $[-1, 1]$, we call $x \in [-1, 1]$ an $h$**–minimum** (for $\gamma$), if there exist $a, b \in [-1, 1]$ with

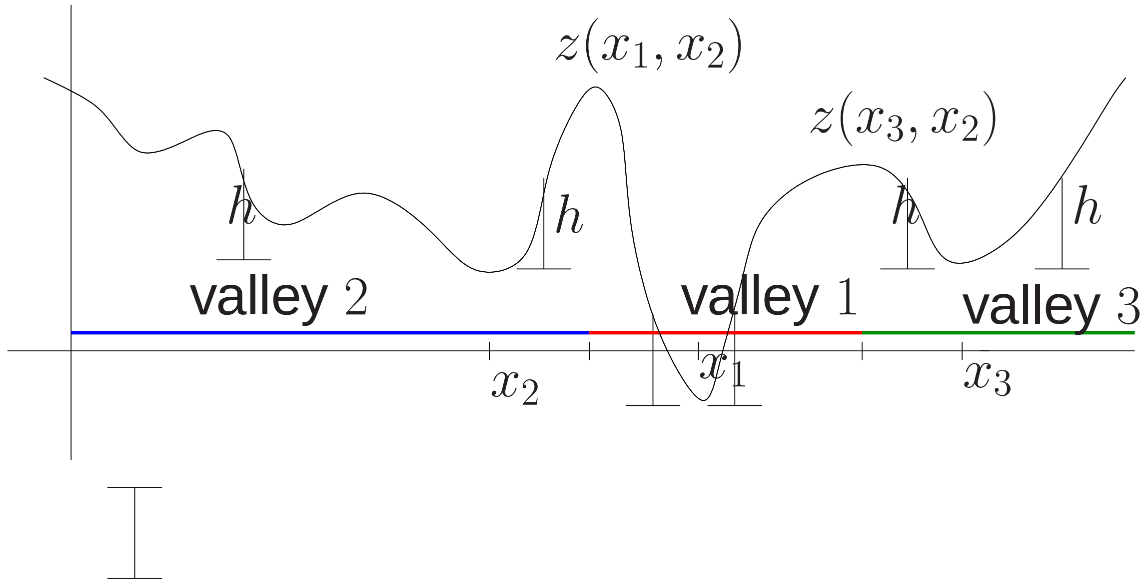
$$a < x < b, \quad \gamma(a) \geq \gamma(x) + h, \quad \gamma(b) \geq \gamma(x) + h, \quad \text{and } \gamma(x) = \min_{[a,b]} \gamma.$$




The absolute maxima between two consecutive $h$-minima, $x, y$, are called $h$-maxima, or saddle-points, $z^*(x, y)$.

Consider now $V^{(N)}$ on $[-1,1]$. For fixed $h$, we call $M_h^- \equiv \{x_1,\dots,x_n\}$ the set of $h$-minima.

The intervals between the consecutive $h$-maxima are called $h$-valleys.




Order $M_h^-$ in such a way that, if

$$S_{h,k} \equiv \{-1, 1, x_1, \dots, x_k\}, \quad 0 \le k \le n$$

then, for some $\delta > 0$,

$$V^{(N)}(z(x_k, S_{h,k-1})) - V^{(N)}(x_k) \ge \max_{n \ge j > k} \Big( v^{(N)}(z(x_j, S_{h,j-1})) - V^{(N)}(x_j) \Big) + \delta$$

***Theorem 3.*** *Given $h, \delta > 0$, with probability tending to one the following holds: $1 \leq q = |M_h^-|$, and if $\lambda_N^*$ denotes the principal eigenvalue of the operator $\mathcal{L}^{(N)}\left(I_N \setminus M_h^-(V_N)\right)$, then*

$$\sigma\left(\mathcal{L}_N\right) \cap [0, \lambda_N^*) = \left\{\lambda_1^{(N)} < \lambda_2^{(N)} < \cdots < \lambda_q^{(N)}\right\}$$

*and*

$$\begin{aligned} \lambda_k^{(N)} &\leq c(\kappa) N^2 e^{-\delta\sqrt{N}} \lambda_{k+1}^{(N)}, \qquad \forall k = 1, \ldots, q-1 \\ \lambda_q^{(N)} &\leq c(\kappa) e^{-\delta\sqrt{N}} \lambda_N^* \\ \lambda_N^* &\geq N^{-2} e^{-h\sqrt{N}} \end{aligned}$$

*Moreover, for $1 \leq k \leq q$,*

$$\begin{aligned} c(\kappa) N^{-2} \exp\left\{\sqrt{N}\left[V_N\left(z^*(x_k, S_{h,k-1})\right) - V_N(x_k)\right]\right\} \\ \leq \lambda_k^{(N)} \\ \leq c'(\kappa) \exp\left\{\sqrt{N}\left[V_N\left(z^*(x_k, S_{h,k-1})\right) - V_N(x_k)\right]\right\} \end{aligned}$$

*[in agreemant with results of LeDoussal and Monthus using Ma-Dasgupta RG]*

# Eigenfunctions

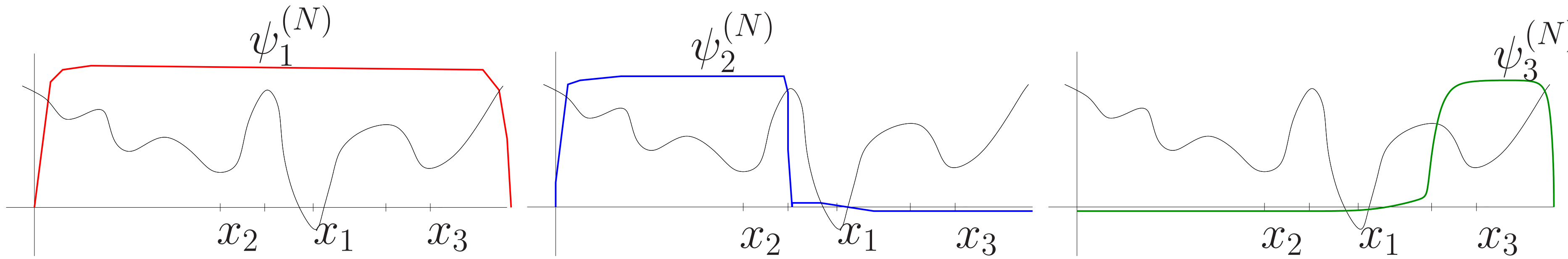
***Theorem 4.*** *For each* $1 \leq k \leq q$*, the simple eigenvalue* $\lambda_k^{(N)}$ *has eigenvector* $\psi_k^{(N)}$ *that satisfies*

$$\left\| \psi_k^{(N)} - \frac{h_{x_k, S_{k-1}}}{\|h_{x_k, S_{k-1}}\|_2} \right\|_2 \leq e^{-\frac{\delta}{8}\sqrt{N}}$$

where

$$h_{x_k, S_{k-1}}(x) \equiv \mathcal{P}_x\left(\tau_{x_k} < \tau_{S_{k-1}}\right)$$

x



## Consequences: Localisation

Define the $\ln n$ valley covering the origin can be written as $(a^{(n)}, m^{(n)}, b^{(n)})$.
Set $m^{(n)}(\omega) \equiv \mathfrak{m}^{(n)}(\omega) \ln^2 n$.
$A_n \equiv (a^{(n)}, b^{(n)}) \cap \mathcal{Z}, \qquad D_n \equiv \left((\mathfrak{m}^{(n)} - 2\delta_n) \ln^2 n, (\mathfrak{m}^{(n)} + 2\delta_n) \ln^2 n\right) \cap A_n$, where
$\delta_n \gg \sqrt{\frac{\ln \ln n}{\ln n}}$.

$$
\begin{aligned}
\mathsf{P}_0^\omega(X_n \in D_n) \geq\ & \mathsf{P}_0^\omega(X_n \in D_n\,, X_k \in A_n\ \forall 0 \leq k \leq n) \\
= & \frac{1}{\mu(0)} (1_0, (1 - \mathcal{L}(A_n))^n 1_{D_n}) \\
= & \sum_{j=1}^{|A_n|} \left(1 - \lambda_j^{(n)}\right)^n \left(\psi_j^{(n)}, 1_{D_n}\right) \psi_j^{(n)}(0)
\end{aligned}
$$

But: by the choice of $A_n$, all terms in the sum but the first vanish, and the first tends to one. This gives a refinement of Sinai's theorem.

Why choose $A_n$?
Obviously, if we choose a smaller interval, we get a lower bound that is too small! But we could choose a bigger one, $A \supset A_n$. Then we have still the same expression with $A_n$ replaced by $A$, and the sum may involve even smaller eigenvalues.
Consider the particular case where $n$ and $n' > n$ the smallest value such that $\mathfrak{m}^{(n')} \neq \mathfrak{m}^{(n)}$. Let $A = A_{n'}$. Assume that there are just two eigenvalues of $\mathcal{L}(A_{n'})$ smaller than $n^{-1}$. Then

$$\mathsf{P}_0^\omega\left(X_n \in D_{n'}\right) \geq \sum_{j=1}^{2} \left(\psi_j^{(n')}, 1_{D_{n'}}\right) \psi_j^{(n')}(0) \sim 0$$

but also

$$\left(\psi_1^{(n')}, 1_{D_{n'}}\right) \psi_1^{(n')}(0) \sim 1$$

and thus

$$\left(\psi_2^{(n')}, 1_{D_{n'}}\right) \psi_2^{(n')}(0) \sim -1$$

Now focus on times that are of the order of an inverse eigenvalue, $t/\lambda_2^{(n')}$. Then

$$\mathsf{P}_0^\omega\left(X_{t/\lambda_2^{(n')}} \in D_{n'}\right) \sim \left[\left(\psi_1^{(N)}, 1_{D_{n'}}\right)\psi_1^{(n')}(0) + e^{-t}\left(\psi_2^{(n')}, 1_{D_{n'}}\right)\psi_2^{(N)}(0)\right]$$
$$\sim 1 - e^{-t}$$

Since there is an infinity of values $n'$ that allow for this construction, there is an infinite of increasing random time-scales on which the process is exiting exponentially, a metastable state towards a more stable one.