On the relation between length and time scales in glassy systems

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Outline

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1. Motivation
   - What is this talk about
   - A few examples of glassy systems
2. Definitions and a general inequality
3. Sketch of the proof
4. The exact relation: a glimpse from mean field
5. Open problems

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Jean-Philippe Bouchaud (Jan 10, 2006)

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Near a glass transition:

No sign of criticality in static 2-point functions (e.g. $\langle \rho(x)\rho(y)\rangle$).

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Example (1): Antiferromagnetic Potts model on sparse random graphs

$G = (V, E)$: graph with degree k.
Configuration: $x = \{x_i;\ i \in V\}$, $x_i \in \{1, \dots, q\}$

$$H(x) = \sum_{(i,j)\in E} \mathbb{I}(x_i = x_j)\,.$$

Heath-bath dynamics at temperature T.

G is a uniformly random graph, and $k > k_*(q)$

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Ideal glass transition at $T_{\rm d} > 0$.

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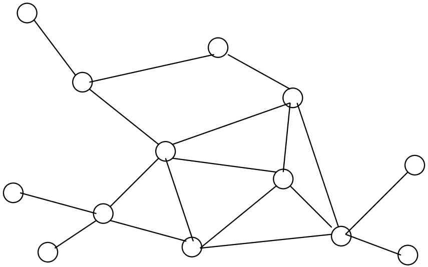

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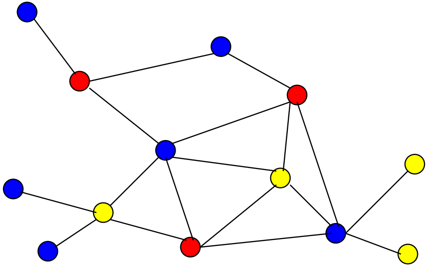


Example (2): Lattice glass (Biroli-Mézard)

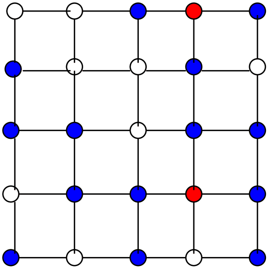
$G = 1, \dots, L^d$: d dimensional lattice
$n = \{n_i : i \in G\}$, $n_i \in \{0, 1\}$.

$$H(n) = -\mu \sum_i n_i + \beta \sum_i n_i \, \mathbb{I}\Big(\sum_{|j-i|=1} n_j \geq m \Big) .$$

Sluggish dynamics at large μ, β.

No signature in the two point susceptibility.

$$m = 3\,.$$



$$H(n) = -16\mu + 2\beta\,.$$

General definitions (statics)

Configuration : $x = \{x_i : \ i = 1, \dots, N\} \in \mathcal{X}^N$, ($\mathcal{X}$ finite set).

Energy function : $E(x) = \sum_{a=1}^{M} E_a(x_{i_1(a)}, \dots, x_{i_k(a)})$.

Gibbs distribution: $\mu(x) \propto \exp\{-E(x)\}$.

Graph representation: $E(x) = E_a(x_1, x_2, x_4) + E_b(x_1, x_2) + \dots$

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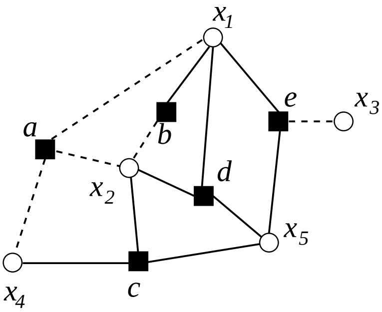

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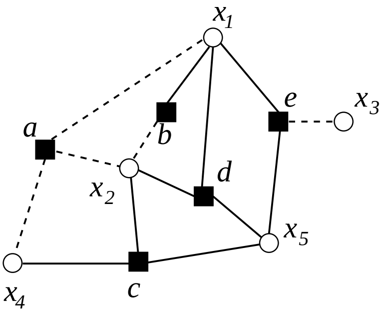

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General definitions (dynamics)

Initial configurations $x(0) \sim \mu$.

The spin x_i tries to flip in the interval dt with probability dt.

x_i changes to x_i' with probability $\kappa_i^x(x_i')$ depending only on the neighbors of i (locality).

Aperiodic, irreducible + detailed balance

$$\mu(x)\kappa_i^x(x_i') = \mu(x)\kappa_i^x(x_i').$$

Hypotheses

1. Degree $\leq k < \infty$.

2. For each i there exist a permitted ('empty') state x_i^* s.t. $\mu(x_i^*|x_{\sim i}) \geq \mu_* > 0$.

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$$C_i(t) \equiv \max_{f:|f(x)|\le 1} \Big[\langle f(x_i(0)) f(x_i(t)) \rangle - \langle f(x_i(0)) \rangle \langle f(x_i(t)) \rangle \Big] \, .$$

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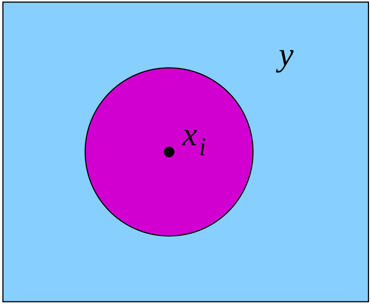
Length scale: an alternative definition.

$\langle f(x_i) \rangle_{i,r}^{y}$ conditional expectation on Ball(i, r) with b.c. y.



$$G'_i(r) = \max_{f:|f(x)|\le 1} \left| \langle f(x_i) \langle f(x'_i) \rangle^{x}_{i,r} \rangle - \langle f(x_i) \rangle^2 \right| .$$

$$\tau'_i(\varepsilon) \equiv \inf\{ r : \dots \}$$

Bouchaud-Biroli:

`Pick an equilibrium reference configuration ...`

$$G_i'(r) = \max_{f:|f(x)|\leq 1} \left| \langle f(x_i) \langle f(x_i') \rangle_{i,r}^{x} \rangle - \langle f(x_i) \rangle^2 \right| .$$

$$\tau_i'(\varepsilon) \equiv \inf\{ r : \ldots \}$$

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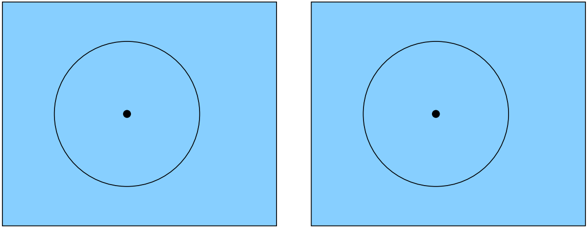
Proposition

$$C_1 \ell_i(\varepsilon') \le \tau_i(\varepsilon) \le \exp\left\{ C_2 |Ball(i, \ell_i(\varepsilon''))| \right\} .$$

where $\varepsilon' = c_1 \varepsilon^{1/2}$*, and* $\varepsilon'' = c_2 \varepsilon^2$*.*

Sketch of the proof (lower bound)

(via coupling and disagreement percolation, Häggstrom, Sinclair, Peres, etc. . .)
Take two copies of the system, initialize with a thermalized configuration and run them in parallel:



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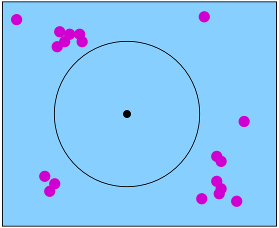

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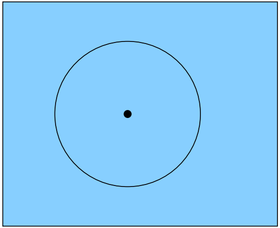

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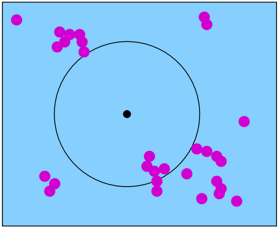

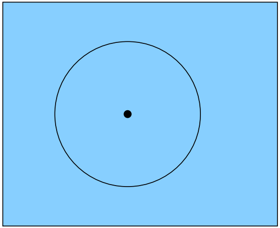

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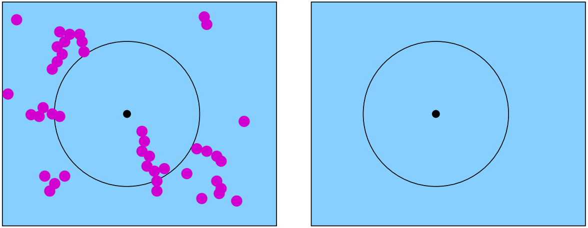

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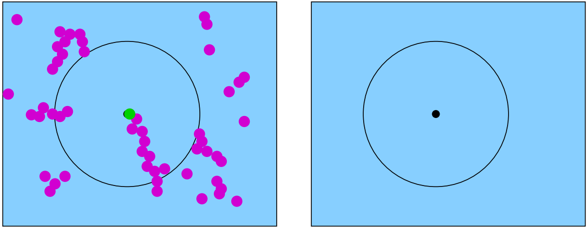

Usual dynamics. Never flips in $G \backslash \mathrm{Ball}(i, r)$.

Take $f = f(x_i)$ s.t. $\langle f \rangle = 0$,
and $r = \ell_i(\varepsilon)/2$:

$$\varepsilon \le \langle f \langle f \rangle_{i,r} \rangle = \lim_{t \to \infty} \langle f(0) f(t) \rangle_{(2)} \le \langle f(0) f(\tau) \rangle_{(2)} \approx \langle f(0) f(\tau) \rangle_{(2)} \, .$$

If $\tau = \delta \cdot r$.

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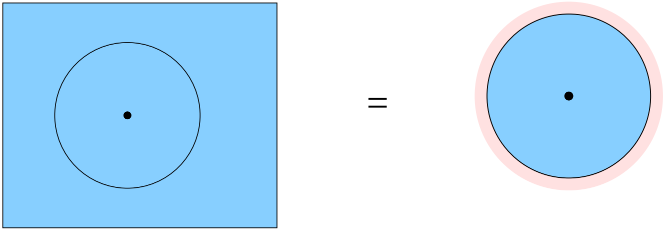
Sketch of the proof (upper bound)



$$\langle f(x_i(0))f(x_i(\tau))\rangle = \Big\langle \langle f(x_i'(0))f(x_i'(\tau))\rangle_{i,r}^{\{x(t)\}} \Big\rangle$$

Take $r = \ell$ and $t = \exp\{C|\text{Ball}(i,\ell)|\}$.

$$\begin{aligned} \langle f(x_i(0)) f(x_i(\tau)) \rangle &= \Big\langle \langle f(x_i'(0)) f(x_i'(\tau)) \rangle_{i,r}^{\{x(t)\}} \Big\rangle \\ &\approx \Big\langle \langle f(x_i(0)) \rangle_{i,r}^{\{x(t)\}} \langle f(x_i'(\tau)) \rangle_{i,r}^{\{x(t)\}} \Big\rangle \\ &= \langle f(x_i) F(x_{\sim i,r}) \rangle \le \varepsilon \end{aligned}$$

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The exact relation: a glimpse from mean field

Two models on random sparse graphs:

p-spin:

$$H(\sigma) = -\beta \sum_{(i_1 \dots i_p) \in G} J_{i_1 \dots i_p} \sigma_{i_1} \cdots \sigma_{i_p} \, . \tag{1}$$

FA model

$$H(n) = -\beta \sum_{i \in G} n_i \tag{2}$$

modify definitions for this!

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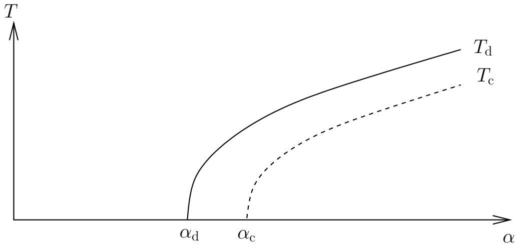
Schematic phase diagram

$1\,\frac{1}{2}$ order phase transition at $\alpha_{\rm d}(T)$.



$$\ell_i \sim (\alpha_{\rm d} - \alpha)^{-1/2}\,.$$

[AM/Semerjian, AM/Mézard]

Numerically

$$\tau_i \sim (\alpha_{\rm d} - \alpha)^{-\gamma}$$

$\gamma > 1$. (Activation energy $\ll$ volume)

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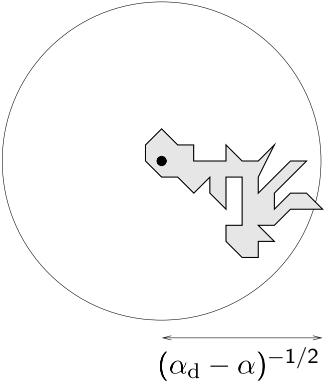
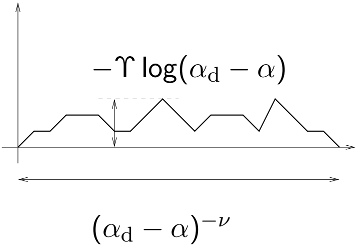
Solution of the puzzle





MCT-like exponents ν and Υ

Open problems

- Relation between ℓ and 4-point functions (partial results).
- Dynamic scaling in mean field: $\tau \sim \ell^z$.
- Geometry of excitations in finite d.