Nonconservation and Disorder in a Driven Diffusive System

1. Review of prototypical driven diffusive system
   Asymmetric Exclusion Process
   & nonequilibrium phase transitions

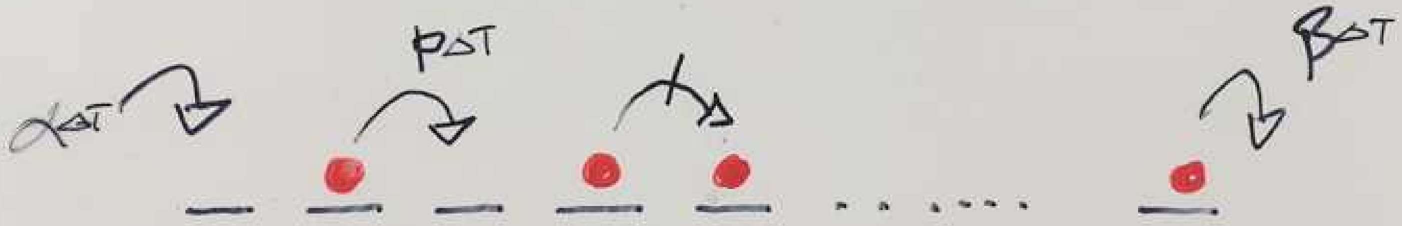
[2. Motivation from Molecular Motors]

3. New effects from nonconservation & disorder
   - Stretched exponential relaxation
   - Griffiths singularities ?
   - Localisation of Shocks

RA Blythe & MR Evans PRL 2002
MR Evans, R. Juhász, L Santen PRE 2003
MR Evans, T. Hanney, Y. Kafri PRE 2004

"Asymmetric Exclusion Process" 3b

- 1-D lattice of N sites with some particles on it



$i = 1$ .... $i = N$

$\tau_i = 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad \cdots\cdots \quad 1$

- Stochastic Dynamics

in ΔT each particle has probability $p \Delta T$ of attempting jump to the right

but single occupancy ; no overtaking

- Boundary Conditions

(i) periodic ⇒ conserved number of particles

(ii) open boundary conditions (KRUG 91)

Particle tries to enter at site 1 with prob. $\alpha \Delta T$

Particle at site N tries to leave with prob $\beta \Delta T$

ASEP

Periodic Boundary Conditions

- Steady State :

  All configurations (with correct particle number) equally probable

$$\# \bullet \, \_ = \# \_ \, \bullet$$

out   in

- Dynamics : dynamical exponent $z = 3/2$

$$\text{e.g. } \langle \tau_i \tau_j \rangle(t) - \rho^2 \sim \exp\left[- \text{const}\, t / N^{3/2} \right]$$

Bethe ansatz : Gwa & Spohn
Mallick & Golinelli (2004)

ASEP

Open Boundaries

Matrix Product solution
DEHP '93

$\underset{}{\bullet} \to D$

$\_ \to E$

e.g. $\text{Prob}[\_\ \bullet\ \_\ \bullet] = \dfrac{\langle\alpha|EDED|\beta\rangle}{Z_4}$

$$Z_N = \langle\alpha|C^N|\beta\rangle \qquad C = D+E$$

$$\boxed{\begin{aligned} DE &= D+E \\ \beta D|\beta\rangle &= |\beta\rangle \\ \alpha\langle\alpha|E &= \langle\alpha| \end{aligned}}$$

e.g. $J_N = \langle \underset{i}{\bullet}\ \underset{i+1}{\_} \rangle = \dfrac{\langle\alpha|C^{i-1}DEC^{N-i-1}|\beta\rangle}{Z_N}$

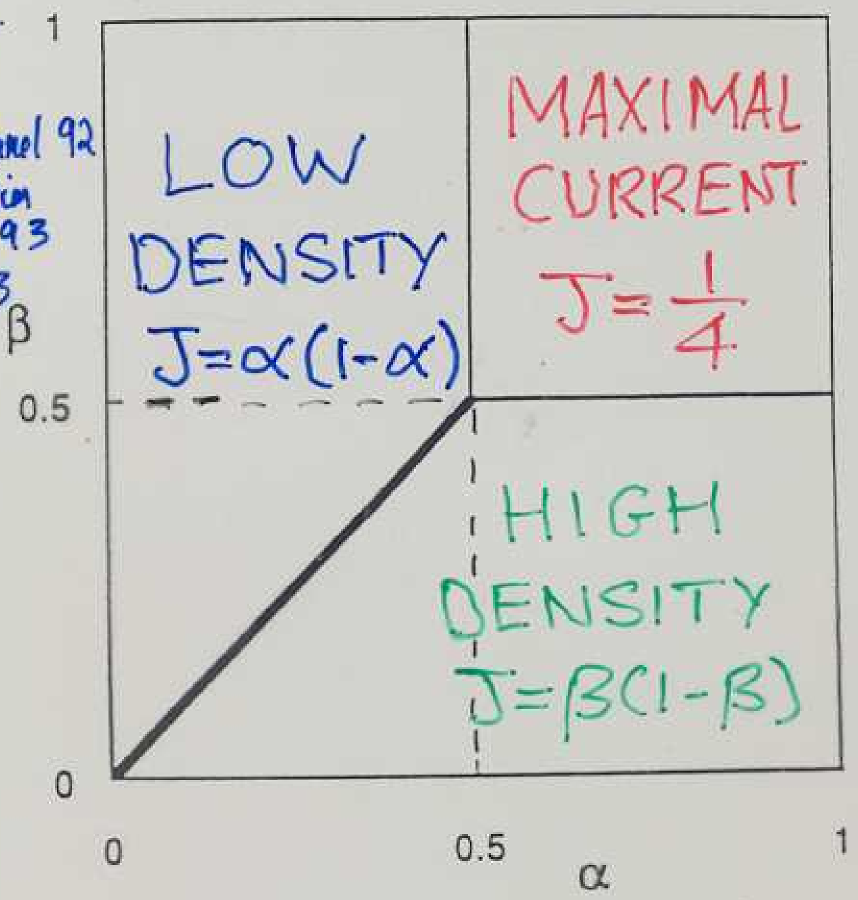
$$= \frac{Z_{N-1}}{Z_N}$$

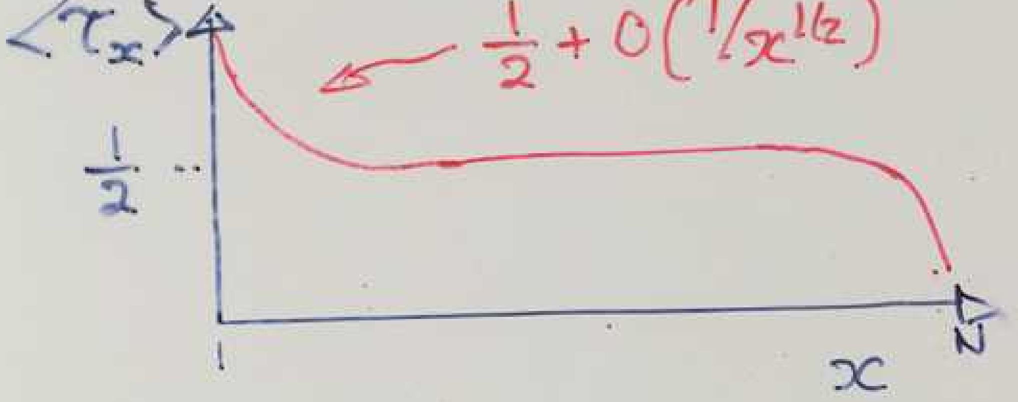
PHASE DIAGRAM

(LIMIT $N \to \infty$)

1 species Asymmetric Exclusion -

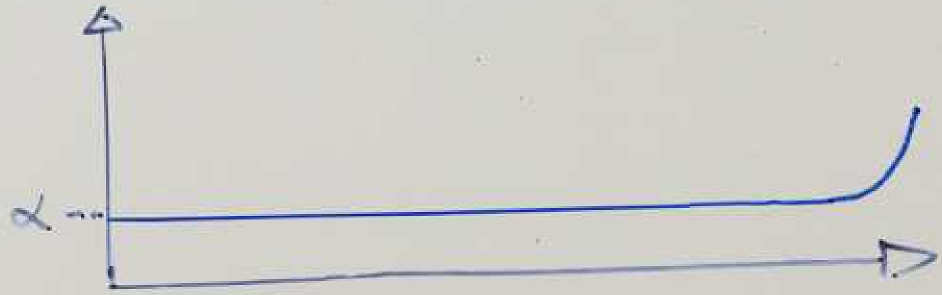
Krug '91
Derrida Domany Mukamel 92
Derrida Evans Hakim Pasquier 93
Schütz Domany 93





Power Law Decays

$z = 3/2$



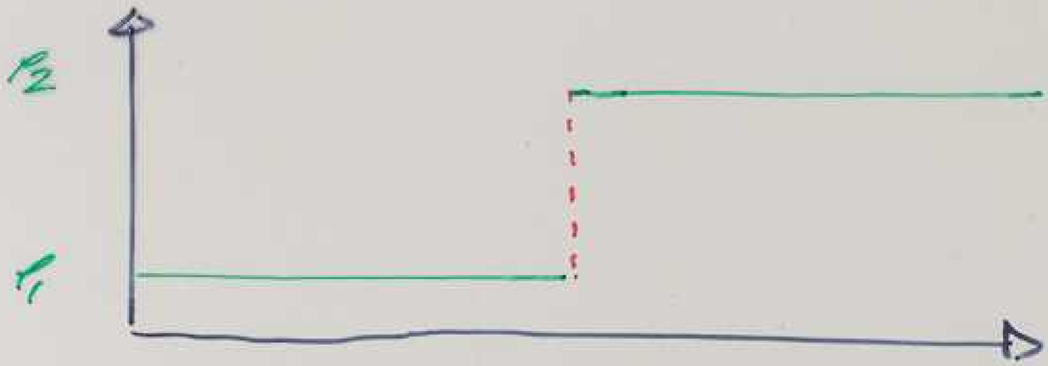
Exponential Decay

Exponential Decay

1st Order Transition Line

$$\alpha = \beta < \frac{1}{2}(1-q)$$

$\equiv$ Superposition of Shocks



At Arbitrary position

$$\rho_1(1-\rho_1) = \rho_2(1-\rho_2)$$

Lee Yang Theory

RA Blythe & MR Evans PRL 2002

- Consider normalisation $Z_N(\alpha,\beta)$ as partition function

$$Z_N = \sum_{p=1}^{N} \frac{p(2N-1-p)!}{N!(N-p)!} \frac{(1/\beta)^{p+1} - (1/\alpha)^{p+1}}{(1/\beta) - (1/\alpha)}$$

- Generalise real rates α, β to complex parameters and consider zeros of Z_N in e.g. complex α plane

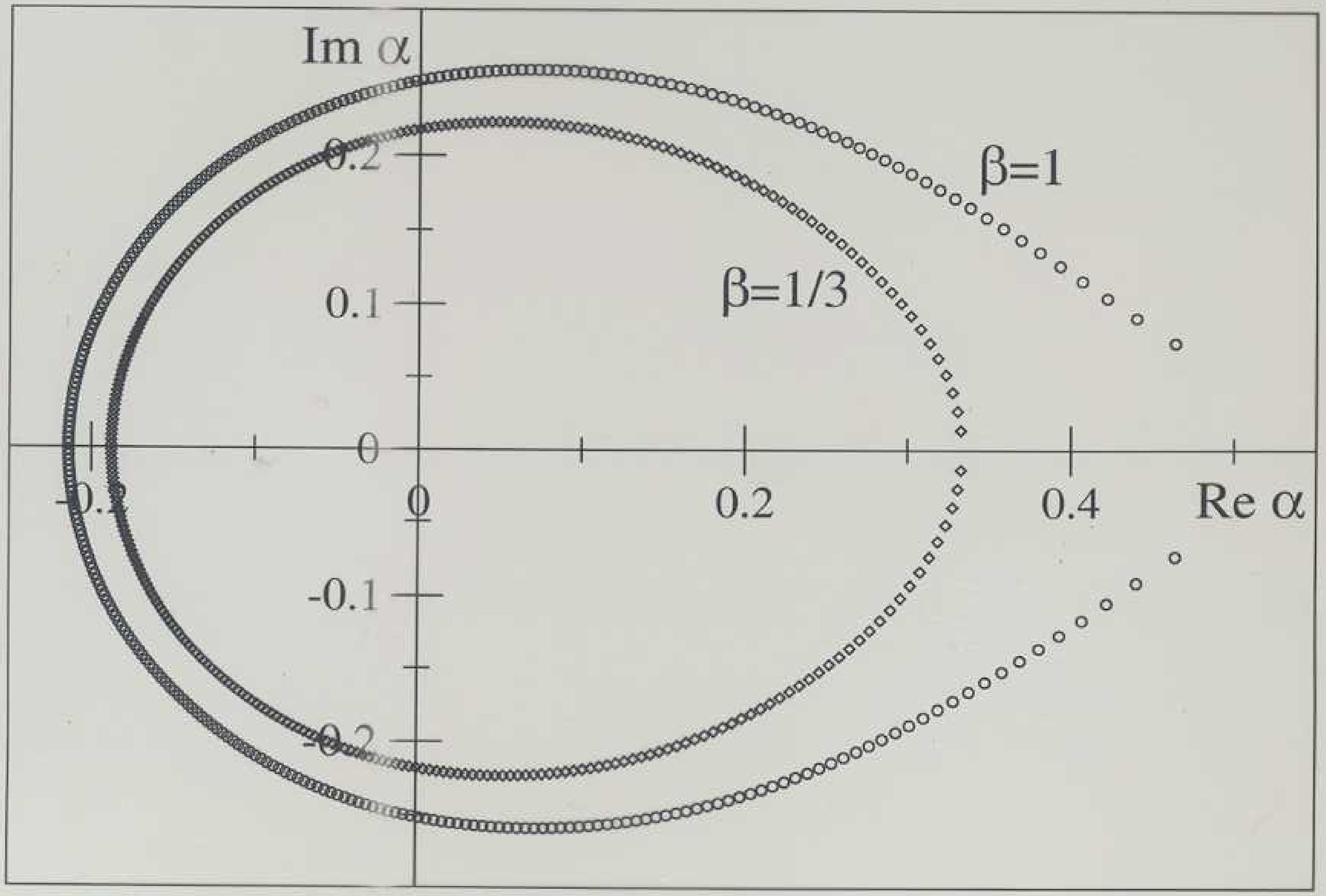
- Phase transitions occur when zeros of Z_N pinch real axis as $N \to \infty$

  1st order line: finite density of zeros pinch at angle $\pi/2$

  2nd order lines: vanishing density pinch at angle $\pi/4$

N.B. $\lim_{N\to\infty} \frac{1}{N} \ln Z_N = \ln J$

Zeros of Z_{300} in complex alpha plane



Molecular Motors

- motor proteins can perform directed motion along 1-d filaments
- head of protein couples to filament; tail carries load to be transported.
- head can detach as well as attach
  → exchange between filament and cytoplasm
  cargoes transported from one part of cell to another
- commonly work in large 'ensembles' to accomplish cellular functions

examples

myosins	involved in muscles
dyneins	drives beating of cilia
kinesins	move organelles along microtubules in cells

along actin filaments and towards the plus end. Kinesins and dyneins move along microtubules, kinesins move towards the plus end and dyneins towards the minus end, see Fig. 1.

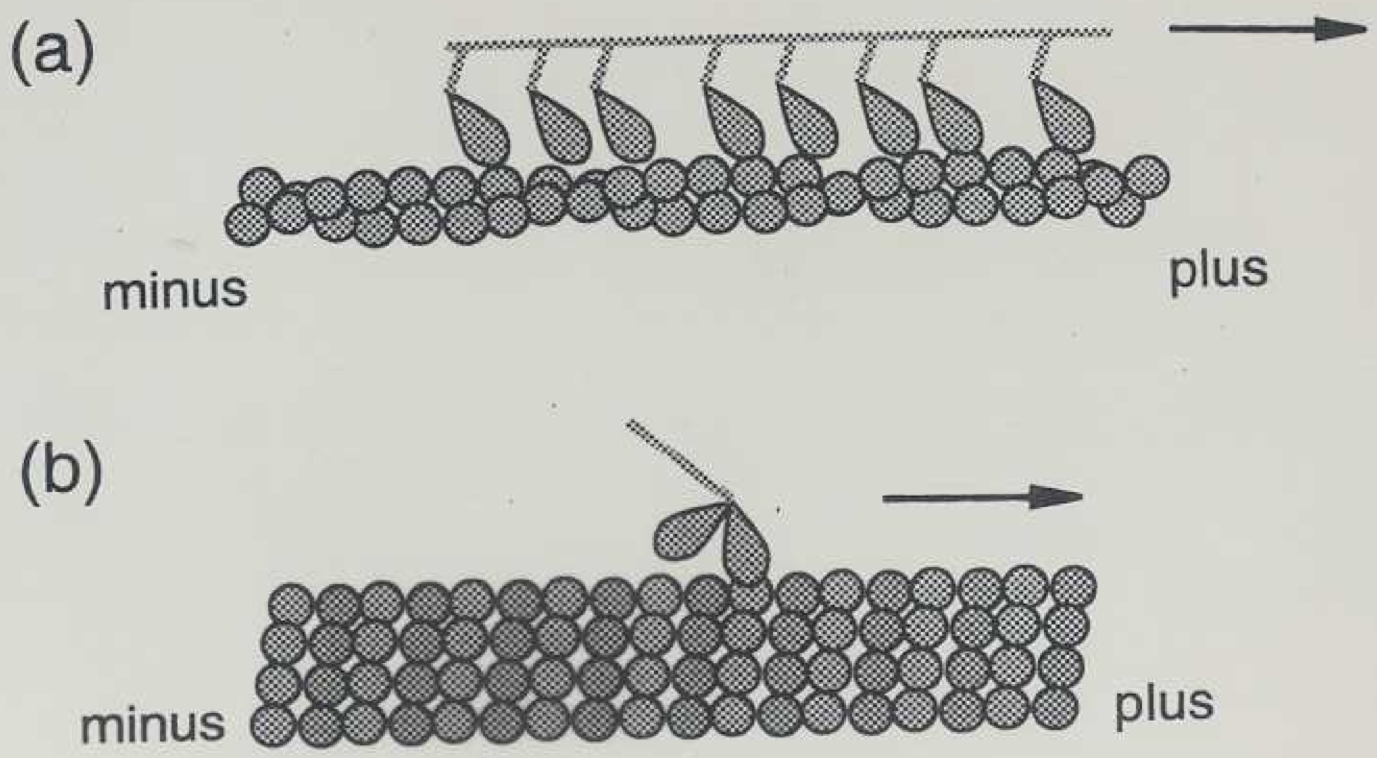
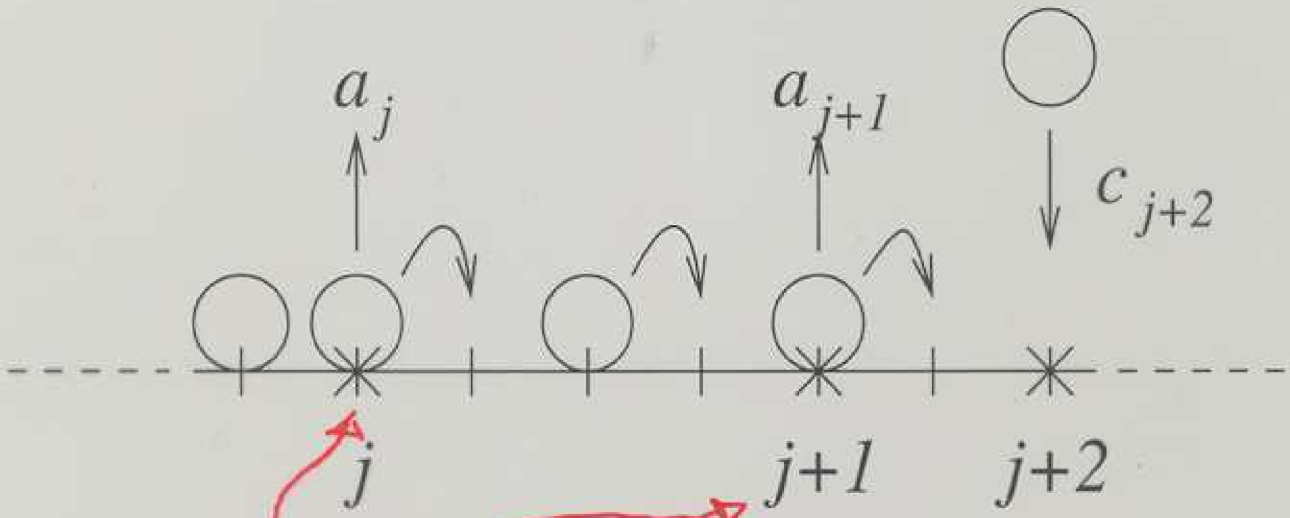


Fig. 1. Schematic representation of molecular motors and track filaments. (a) myosin interacting with actin filaments. (b) kinesin moving along a microtubule. Both types of filaments are polar and periodic, their two different ends are denoted "plus" and

Asep with creation/annihilation at 'disorder' sites

Evans Hanney Kafri PRE (2004)



Each site is disorder site with probability $1-p$

(quenched disorder)

disorder sites motivated by 'promoter' and 'termination' sites on DNA

Solvable Cases

I $\quad a_j = a \qquad c_j = c \qquad \forall j$

Steady State: All configurations with M particles have equal probability P_M

$$P_M = \frac{c^M a^{N-M}}{(a+c)^N}$$

$\rho = \frac{c}{a+c}$ is 'Langmuir Density'

and detailed balance w.r.t. creation/annihilation

$c P_M = a P_{M+1} \qquad\qquad a P_M = c P_{M-1}$

Solvable Cases

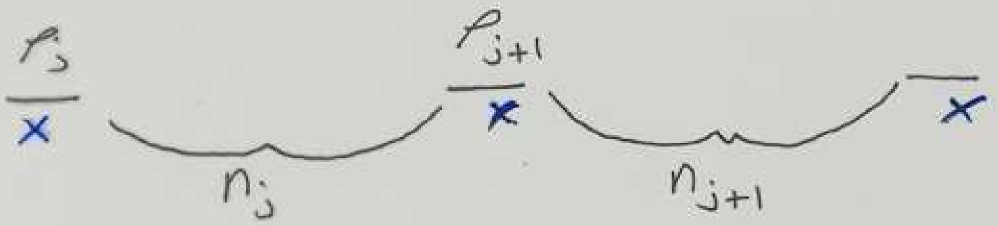
II $\quad a_j, c_j \to \infty$

- Density at disorder sites determined by

$$(1-\rho_j)c_j = \rho_j a_j$$

$$\Rightarrow \quad \rho_j = \frac{c_j}{a_j + c_j}$$

- System factorises into Conserving Domains



- Each domain is an open boundary TASEP

  with $\quad \alpha = \rho_j$

  $\qquad\quad \beta = 1 - \rho_{j+1}$

  length $\; n_j$

Consequences of Factorisation

I Relaxation Dynamics

- if system factorises into (maximal current) conserving domains with Poisson distributed length n

- relaxation of domain of length n

$$\delta\rho_n = \rho_n - 1/2 \sim e^{-\Delta_n t} \qquad \Delta_n = \frac{\text{Const.}}{n^z}$$

- relaxation of whole system

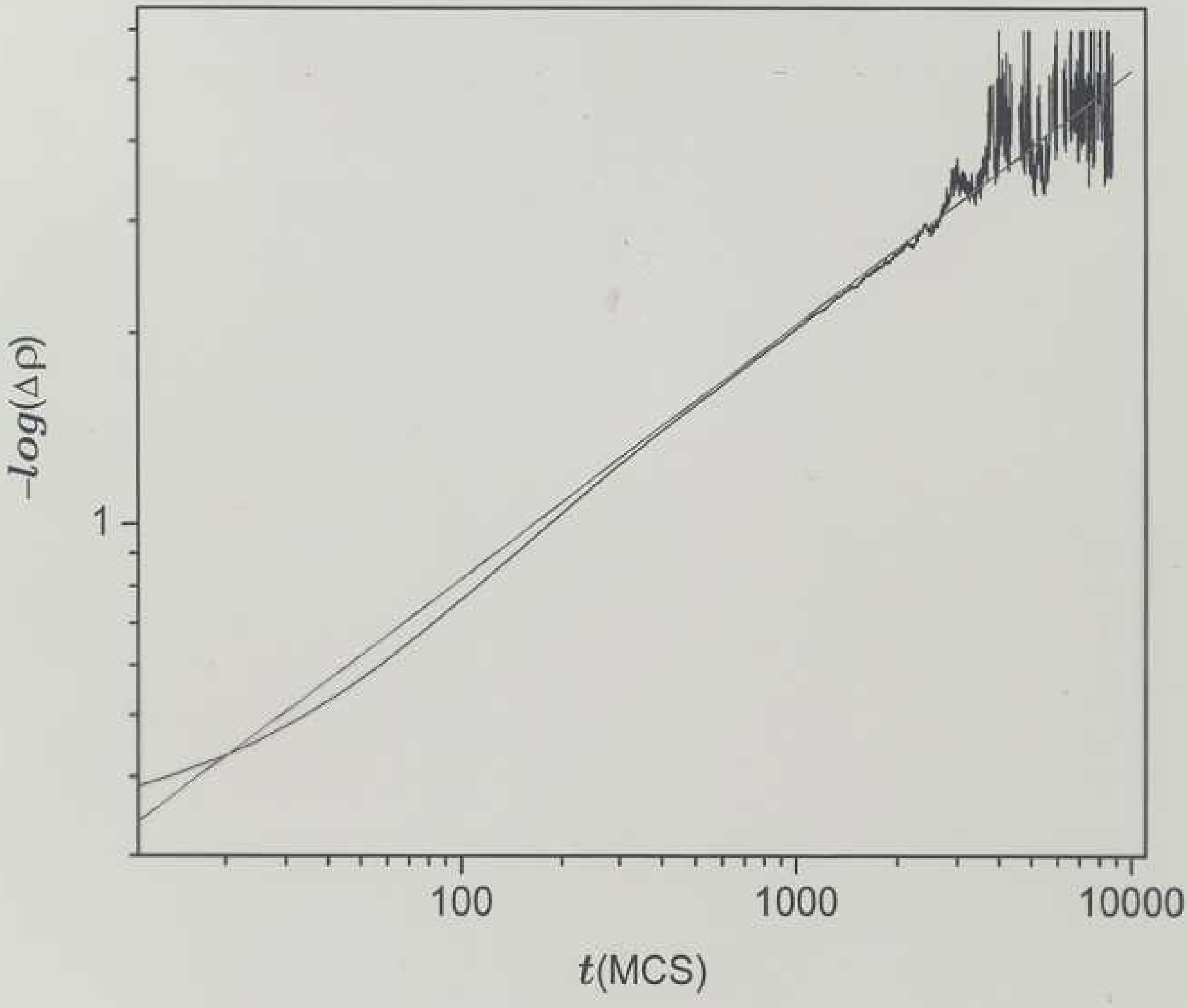
$$\delta\rho \sim \sum_n p^n e^{-\Delta_n t}$$

dominated at large time t by $n \sim t^{1/z+1}$ and

where
$$\boxed{\begin{aligned} &\delta\rho \sim \exp - \text{Const}\, t^{\phi} \\ &\phi = \frac{1}{1+z} = 2/5 \text{ for } z = 3/2. \end{aligned}}$$

$\ln \Delta\rho$ vs t

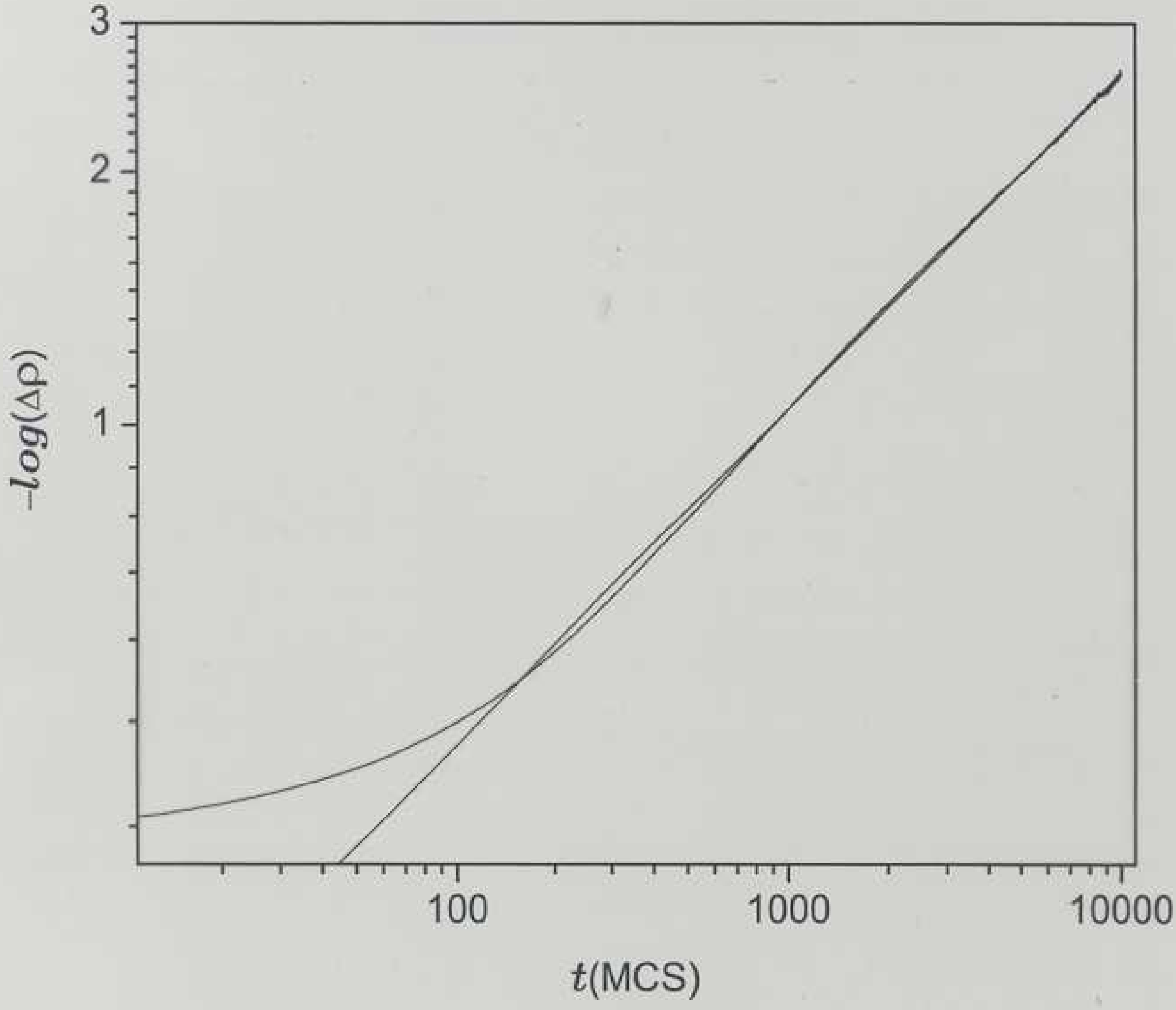
(ln-ln plot)



$a_j = c_j = 1 \quad \forall j$

$\Rightarrow \rho \to 1/2$

$\ln \Delta\rho$ vs t



$a_j = c_j = 10 \quad \forall j$

$\Rightarrow \varphi \to 1/2$

Consequences of Factorisation

case II : Normalisation

$$Z_N = \langle \rho_1 | C^{n_1} | 1-\rho_2 \rangle \langle \rho_2 | C^{n_2} | 1-\rho_3 \rangle \cdots \langle \rho_\ell | C^{n_\ell} | 1-\rho_1 \rangle$$

- Average over disorder sites

simple example $\quad \rho_j = u \quad$ prob. q

$\qquad\qquad\qquad\;\; = v \quad$ prob. $1-q$

Then $\displaystyle \lim_{N\to\infty} \frac{\langle \ln Z_N \rangle}{N} =$

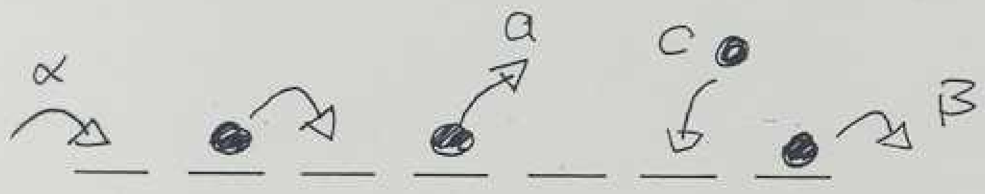
$$(1-p)^2 \sum_{n=0}^{\infty} p^n \Big\{ q^2 \ln Z_n(u,1-u) + q(1-q) \ln Z_n(u,1-v) + q(1-q) \ln Z_n(v,1-u) + (1-q)^2 \ln Z_n(v,1-v) \Big\}$$

$\Rightarrow$ exponentially supressed singularities at e.g. $v = 1/2$.

"Griffiths singularities"

Open Boundary Case (Parmeggiani, Franosch & Frey) PRL 2003

non conservation at all sites



- Continuum mean field description $\quad 0 \le x \le 1$

$$\frac{\partial \rho}{\partial t} + \frac{1}{N}\frac{\partial}{\partial x}[\rho(1-\rho)] = c(1-\rho) - a\rho$$

∴ choose $\quad a = \dfrac{\omega_a}{N} \qquad c = \dfrac{\omega_c}{N}$

Steady state

$$(1-2\rho)\frac{d\rho}{dx} - \omega_a[K-(1+K)\rho] = 0$$

$$K = \frac{\omega_c}{\omega_a}$$

Solution of mean field steady state

Evans Juhasz Santen 2003
Parmeggiani Frey Franosch 2004

K=1 $$(1-2\rho)\left(\frac{d\rho}{dx}-\omega\right)=0$$

$\omega_a = \omega_d$

- piecewise linear solutions

$$\rho = \frac{1}{2} \qquad \rho = \omega x + b$$

K≠1 $$(1-2\rho)\frac{d\rho}{dx} = \omega_a\left[K-(1+K)\rho\right]$$

integrate from left $$x = \frac{1}{\omega_a}\int_{\alpha}^{\rho_\ell(x)} d\rho \,\frac{1-2\rho}{K-(1+K)\rho}$$

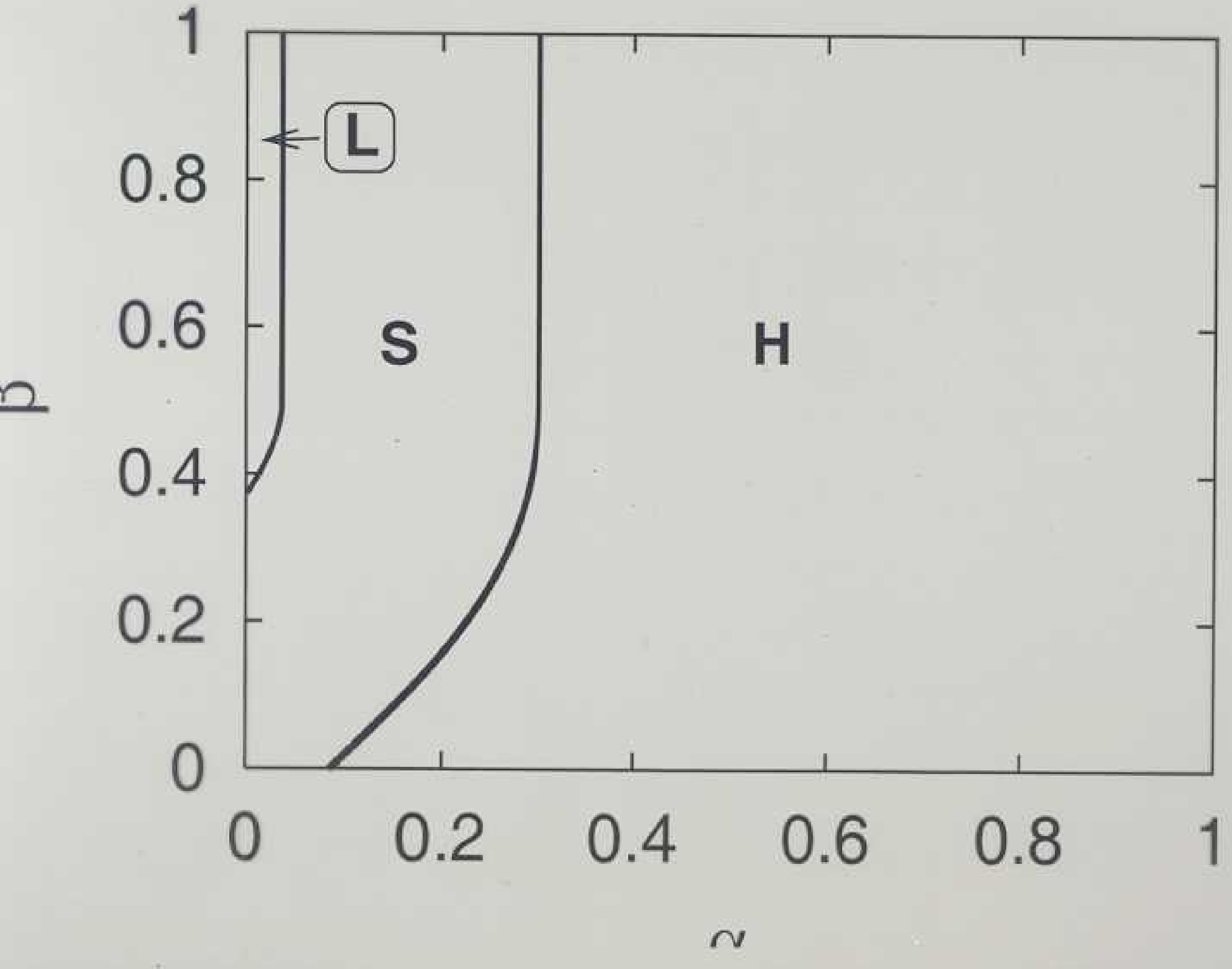
integrate from right $$1-x = \frac{1}{\omega_a}\int_{\rho_r(x)}^{1-\beta} d\rho \,\frac{1-2\rho}{K-(1+K)\rho}$$

- match at shock (stationary) $$J_\ell = \rho_\ell(1-\rho_\ell) = \rho_r(1-\rho_r) = J_r$$

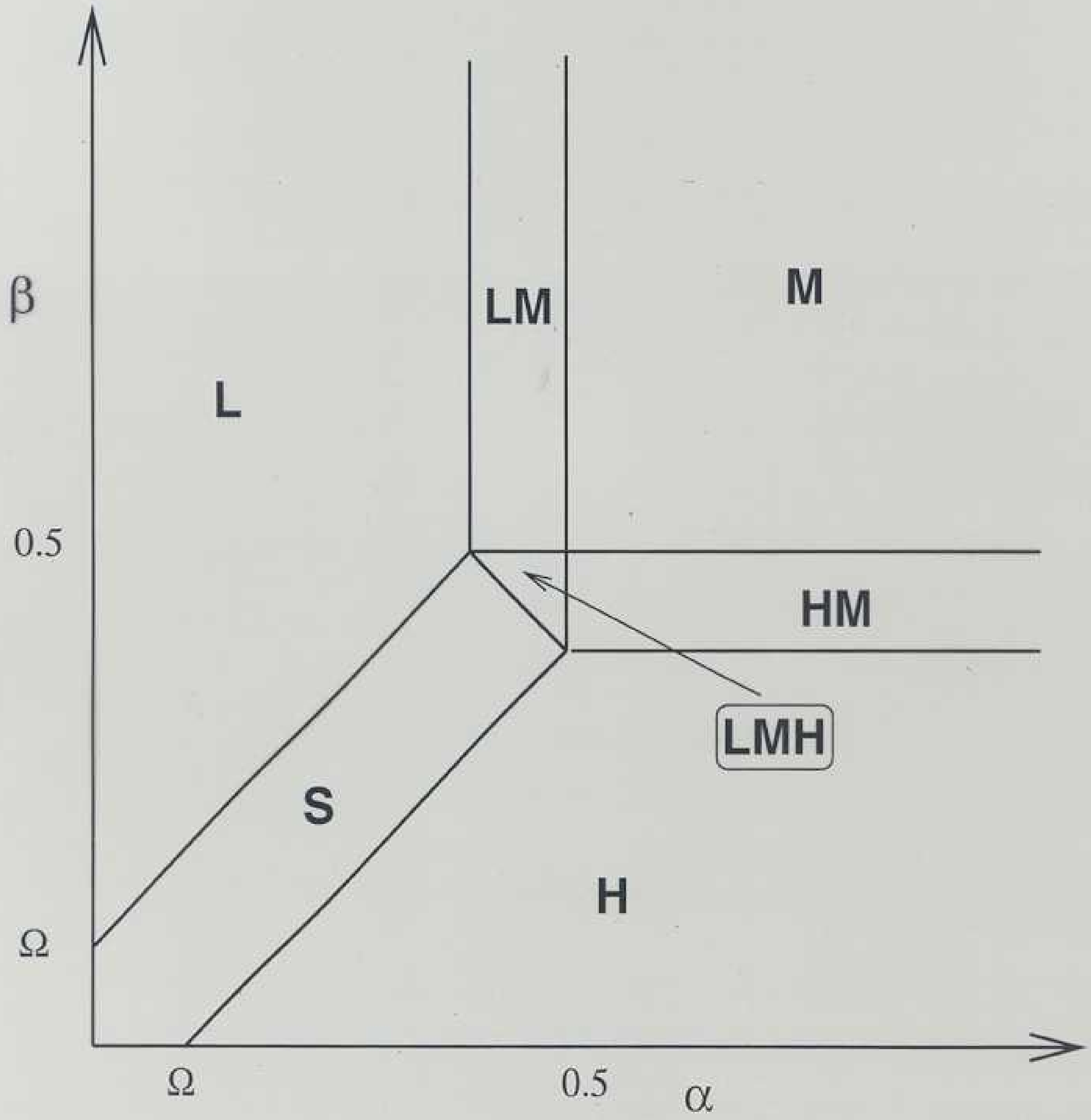
$$\Rightarrow \quad 1-\rho_\ell = \rho_r$$

- else one of ρ_ℓ ρ_r controls whole system

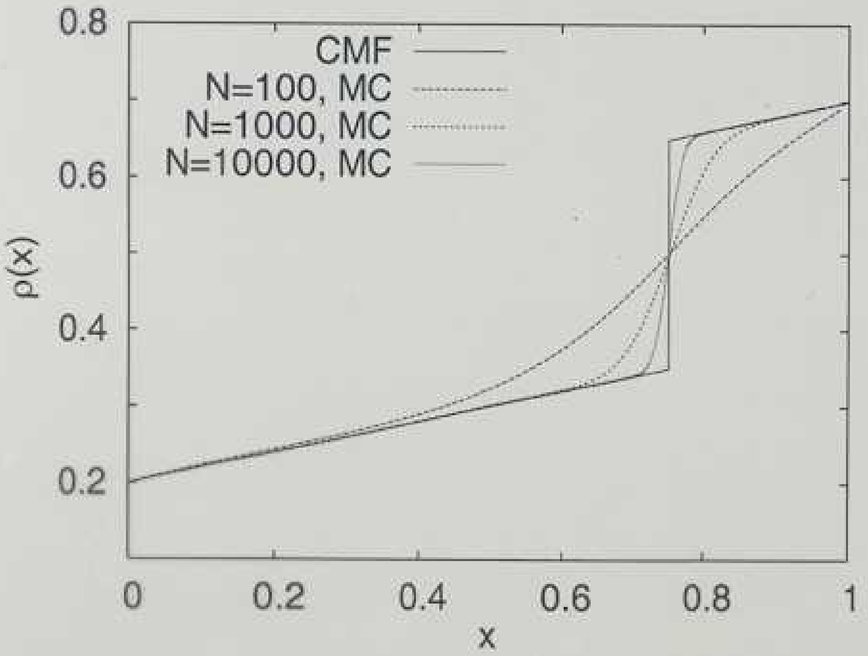
Phase Diagram K=3



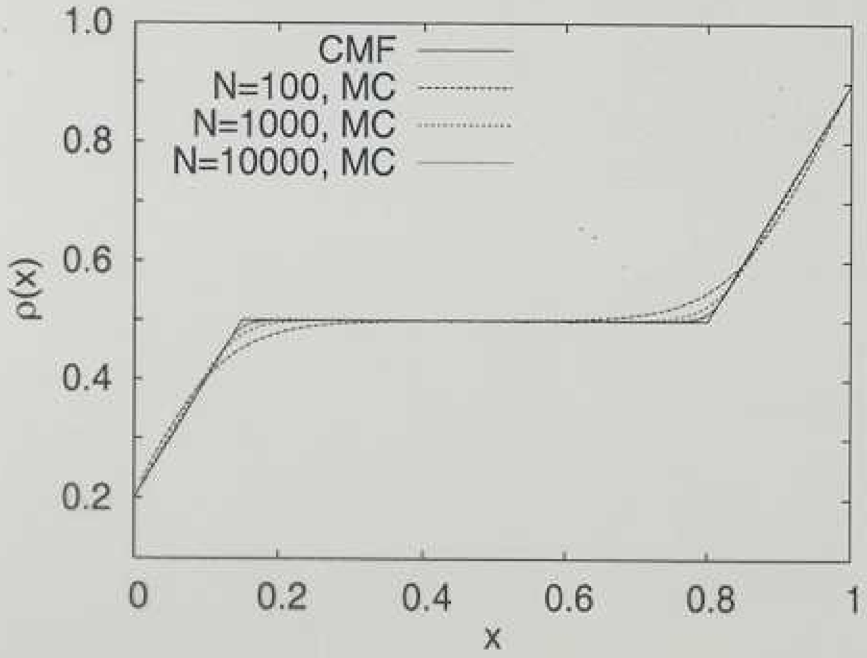
Phase Diagram $K=1$ $\omega < 1/2$



K=1 'Shock' solution



K=1 '3 phase coexistence'



K=3