Stationary results on condensation in two-species zero-range processes

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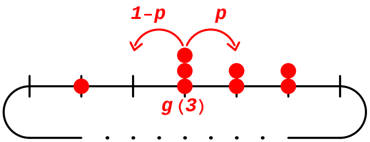
The single species ZRP

Lattice: $\Lambda_L = \{1, \ldots, L\}$ with pbc

State space: $X_L = \{0, 1, ..\}^{\Lambda_L}$

$$\boldsymbol{\eta} = (\eta(x))_{x \in \Lambda_L}$$



Jump rates: $g : \{0, 1, ..\} \to [0, \infty)$

$$g(k) = 0 \;\Leftrightarrow\; k = 0$$

Jump probability: $p \in [0, 1]$, drift $m(p) = 2p - 1$

[Spitzer (1970), Andjel (1982)]

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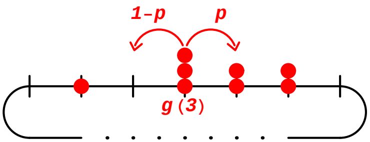

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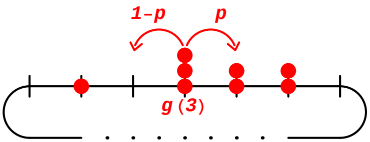

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e.g. $g(k) = k \quad \Rightarrow$ independent particles
in general: $g(k) \nearrow \quad \Rightarrow$ homogeneous invariant measure

[Spitzer (1970), Andjel (1982)]

Condensation

$g(k)$ decreasing in $k \quad \Rightarrow \quad$ effective attraction

e.g. $g(k) \simeq 1 + \dfrac{b}{k^\gamma}\,, \quad b, \gamma > 0$ [Evans (2000)]

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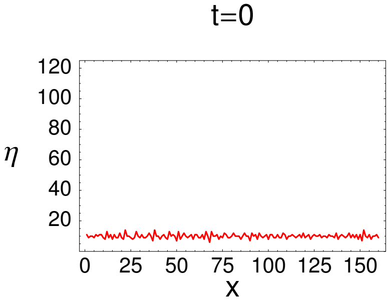

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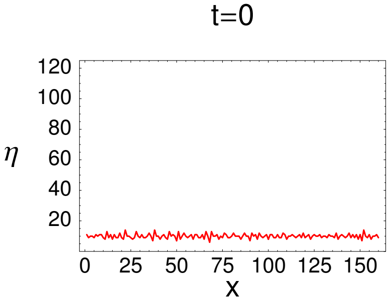

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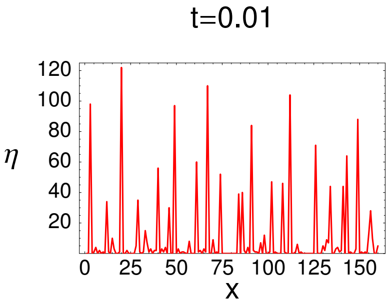

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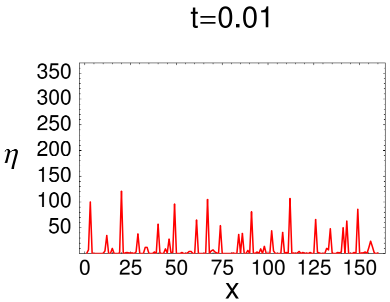

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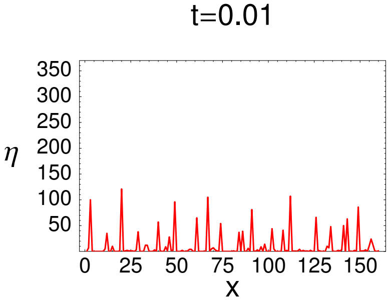

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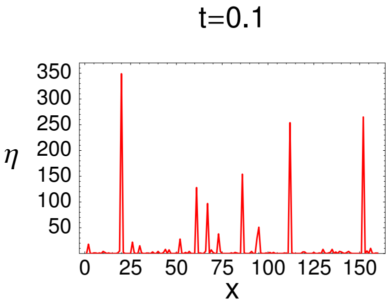

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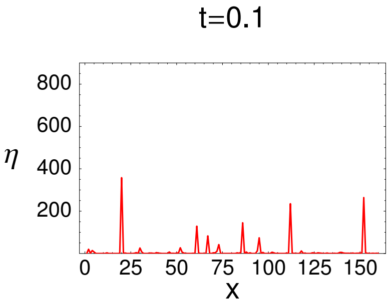

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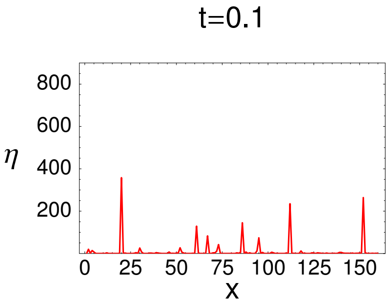

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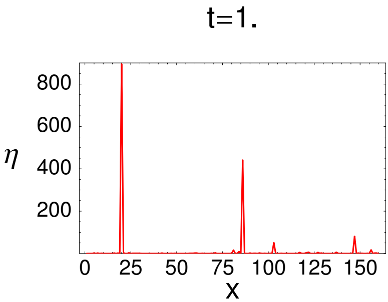

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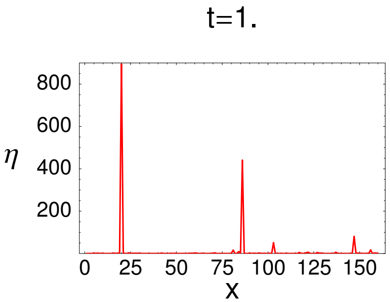

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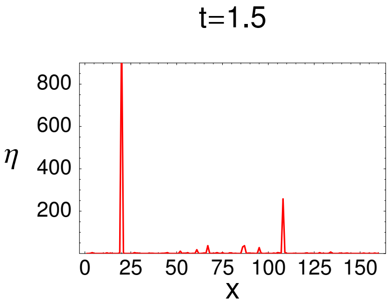

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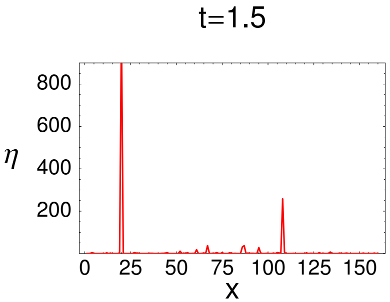
1. relaxation dynamics (coarsening regime)
2. stationary measure (long time limit)



[Godrèche (2003), G., Schütz, Spohn (2003)]

[Godrèche, Luck (2005)]

Motivation

Applications [Evans, Hanney (2005)]

- bosonic lattice gas with condensation
  robust (spatial disorder in the jump rates, geometry)
- growing and rewiring networks, bus route model,
  shaken granular gases

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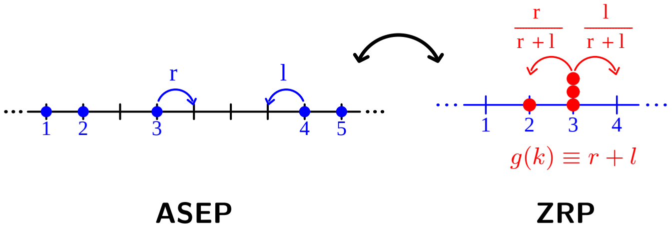
- growing and rewiring networks, bus route model, shaken granular gases
- connection to exclusion models [Kafri et al. (2002)]



Stationary measures

Stationary product weight

$$w^L(\boldsymbol{\eta}) = \prod_{x \in \Lambda_L} w(\eta(x)) \qquad \text{with} \qquad w(k) = \prod_{i \leq k} g(i)^{-1}$$

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corresponding ZRP $\quad g(k) = \dfrac{w(k-1)}{w(k)}\ , \quad w(k) = \prod_{i\leq k} g(i)^{-1}$

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1-species ZRP $\quad g(k) = \dfrac{w(k-1)}{w(k)}\ , \quad w(k) = \prod_{i \le k} g(i)^{-1}$

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Corresponding 2-species ZRP [Evans, Hanney (2003); G., Spohn (2003)]

$$g_1(k_1,k_2) = \frac{w(k_1-1,k_2)}{w(k_1,k_2)} \qquad g_2(k_1,k_2) = \frac{w(k_1,k_2-1)}{w(k_1,k_2)}$$

1-species ZRP $g(k) = \dfrac{w(k-1)}{w(k)}$, $w(k) = \prod_{i\leq k} g(i)^{-1}$

Stationary measures

$$\Sigma_L(\boldsymbol{\eta}) = \sum_{x\in\Lambda_L} \boldsymbol{\eta}(x) \quad \text{is conserved for each species}$$

$\Rightarrow$ family of stationary measures $\quad w^L(\boldsymbol{\eta})\ g\big(\Sigma_L(\boldsymbol{\eta})\big)$

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$\Rightarrow$ family of stationary measures $w^L(\boldsymbol{\eta})\, g\big(\Sigma_L(\boldsymbol{\eta})\big)$

Canonical ensemble $g \sim \delta(\Sigma_L, \mathbf{N})$, $\mathbf{N} = (N_1, N_2)$

measures on $X_{L,\mathbf{N}} = \big\{\boldsymbol{\eta}\,\big|\,\Sigma_L(\boldsymbol{\eta}) = \mathbf{N}\big\}$ with fixed particle number

$$\pi_{L,\mathbf{N}}(\boldsymbol{\eta}) = \frac{1}{Z(L,\mathbf{N})}\, w^L(\boldsymbol{\eta})\, \delta\big(\Sigma_L(\boldsymbol{\eta}), \mathbf{N}\big)$$

unique on each $X_{L,\mathbf{N}}$, extremal measures on X_L

Stationary measures

Grand canonical ensemble $g \sim \exp(\Sigma_L \cdot \boldsymbol{\mu})$

product measures on X_L with chemical potential $\boldsymbol{\mu} = (\mu_1, \mu_2)$

$$\nu^L_{\boldsymbol{\mu}}(\boldsymbol{\eta}) = \frac{1}{z(\boldsymbol{\mu})^L} \prod_{x \in \Lambda_L} w(\boldsymbol{\eta}(x))\, e^{\boldsymbol{\mu} \cdot \boldsymbol{\eta}(x)} \qquad z(\boldsymbol{\mu}) = \sum_{\mathbf{k} \in \mathbb{N}^2} w(\mathbf{k})\, e^{\boldsymbol{\mu} \cdot \mathbf{k}}$$

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density $\quad R_i(\boldsymbol{\mu}) = \langle \eta_i(x) \rangle_{\nu^L_{\boldsymbol{\mu}}} = \partial_{\mu_i} \log z(\boldsymbol{\mu}) \qquad$ (free energy)

$$\text{domain } D_\mu \neq \emptyset \quad \Leftrightarrow \quad \limsup_{\mathbf{k} \to \infty} \frac{1}{|\mathbf{k}|} \log w(\mathbf{k}) < \infty$$

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$D_\mu \subseteq \mathbb{R}^2$ is convex, $\log z(\boldsymbol{\mu})$ is strictly convex on D_μ
$\Rightarrow \quad \mathbf{R} : D_\mu \to D_\rho$ is invertible, $\mathbf{M} : D_\rho \to D_\mu$.

Equivalence of ensembles

$\boldsymbol{\rho} \in D_\rho \qquad \Rightarrow \qquad \pi_{L,[\boldsymbol{\rho}L]} \to \nu_{\mathbf{M}(\boldsymbol{\rho})}$ as $L \to \infty$.

$\boldsymbol{\rho} \in (0,\infty)^2 \setminus D_\rho$ convergence to a mixture (phase separation)

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Relative entropy

$$H(\mu;\nu) = \sum_{\omega\in\Omega} \mu(\omega) \log \frac{\mu(\omega)}{\nu(\omega)} = 0 \quad \Leftrightarrow \quad \mu = \nu$$

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Specific relative entropy

$$h_L(\boldsymbol{\rho},\boldsymbol{\mu}) := \frac{1}{L}\, H\Big(\pi_{L,[\boldsymbol{\rho}L]}\,;\, \nu^L_{\boldsymbol{\mu}}\Big) = -\frac{1}{L}\, \log \nu^L_{\boldsymbol{\mu}}\big(\{\Sigma_L = [\boldsymbol{\rho}L]\}\big)$$

[Csiszár, Körner (1981), Lewis et al (1994)]

Equivalence of ensembles

Specific relative entropy

$$h_L(\boldsymbol{\rho}, \boldsymbol{\mu}) = -\frac{1}{L} \log Z\big(L, [\boldsymbol{\rho} L]\big) - \qquad \big[\boldsymbol{\rho} \cdot \boldsymbol{\mu} - \log z(\boldsymbol{\mu})\big]$$

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$s(\boldsymbol{\rho})$ has a unique maximizer $\overline{\mathbf{M}}(\boldsymbol{\rho})$

$$\begin{array}{lll} \overline{\mathbf{M}}(\boldsymbol{\rho}) = \mathbf{M}(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho}\in D_\rho & \text{local maximum} \\ \overline{\mathbf{M}}(\boldsymbol{\rho}) \in \partial D_\mu \cap D_\mu & \text{for } \boldsymbol{\rho}\notin D_\rho & \text{boundary maximum} \end{array}$$

Equivalence of ensembles

Theorem 1.

For all $\boldsymbol{\rho} \in (0, \infty)^2$, $\quad \lim_{L \to \infty} h_L\big(\boldsymbol{\rho}, \overline{\mathbf{M}}(\boldsymbol{\rho})\big) = 0$.

Equivalence of ensembles

Theorem 1.

For all $\boldsymbol{\rho} \in (0,\infty)^2$, $\lim_{L\to\infty} h_L\big(\boldsymbol{\rho}, \overline{\mathbf{M}}(\boldsymbol{\rho})\big) = 0$.

Consequences

- convergence of entropy $\lim_{L\to\infty} \frac{1}{L} \log Z\big(L, [\boldsymbol{\rho} L]\big) = -s(\boldsymbol{\rho})$
- weak convergence $\big\langle f \big\rangle_{\pi_{L,[\boldsymbol{\rho} L]}} \to \big\langle f \big\rangle_{\nu_{\overline{\mathbf{M}}(\boldsymbol{\rho})}}$ as $L \to \infty$

  for bounded, local observables $f \in C_b(X_n)$
- if $\boldsymbol{\rho} \in D_\rho$ convergence for $f \in L^2\big(X_n, \nu_{\overline{\mathbf{M}}(\boldsymbol{\rho})}\big)$ [Kipnis, Landim]

Condensation transition

Phase separation for $\boldsymbol{\rho} \notin D_\rho$

- homogeneous background phase with distribution $\nu_{\overline{\mathbf{M}}(\boldsymbol{\rho})}$
  density $\boldsymbol{\rho}_c(\boldsymbol{\rho}) = \mathbf{R}\big(\overline{\mathbf{M}}(\boldsymbol{\rho})\big) \in \partial D_\rho$
- condensed phase with volume $o(L)$
  containing $\big(\boldsymbol{\rho} - \boldsymbol{\rho}_c(\boldsymbol{\rho})\big)L$ excess particles

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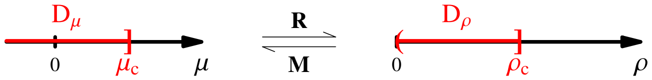
Phase diagram for a single species



Continuous PT, phase coexistence (no symmetry breaking)

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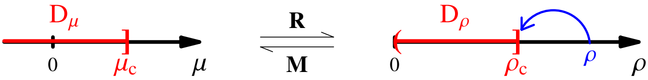

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Condensation transition

Two species

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Condensation transition

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D_μ convex $\quad\Rightarrow\quad$ ∂D_μ is a.e. differentiable

Phase diagram $\quad A_i = \big\{\boldsymbol{\rho}\in(0,\infty)^2 \big| \rho_{c_i}(\boldsymbol{\rho}) < \rho_i\big\}$

$$PD = \big\{D_\rho, A_1 \setminus A_2, A_2 \setminus A_1, A_1 \cap A_2\big\}$$

Example

$$w(k_1, k_2) \simeq k_1^{-b} \left(\frac{k_1+1}{k_1+2}\right)^{k_2}, \qquad \mu_1, \mu_2 \leq 0$$

[Evans, Hanney (2003)]

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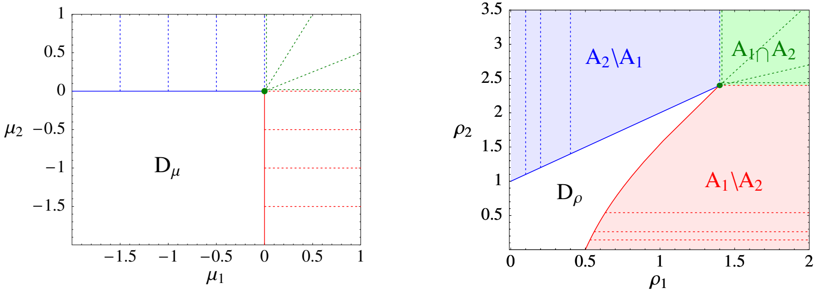
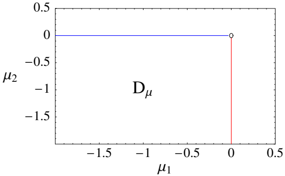


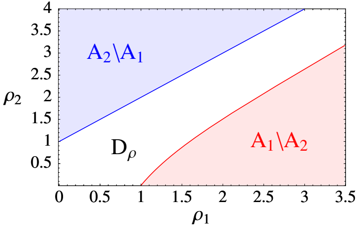
$$b = 4$$

$$g_1 = \left(\frac{1+1/(k_1+1)}{1+1/k_1}\right)^{k_2} \left(1+\frac{b}{k_1}\right) \qquad g_2 = 1 + \frac{1}{k_1+1}$$

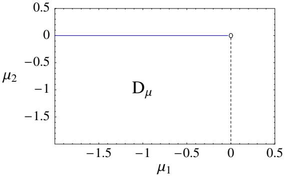
Example

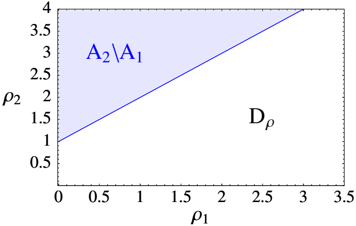
$2 \le b < 3$





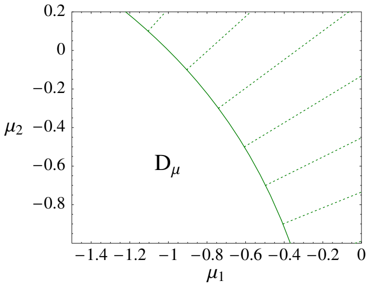
$b \le 2$

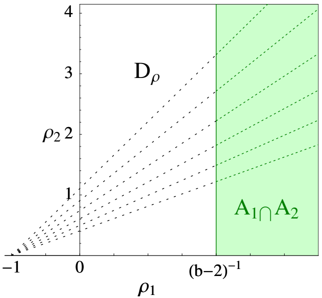




Example

$$w(k_1,k_2) \simeq k_1^{-b}\ \frac{(k_1+1)^{k_2}}{k_2!}\ , \qquad \partial D_\mu = \left\{\boldsymbol{\mu}\middle| e^{\mu_2}+\mu_1=0\right\}$$

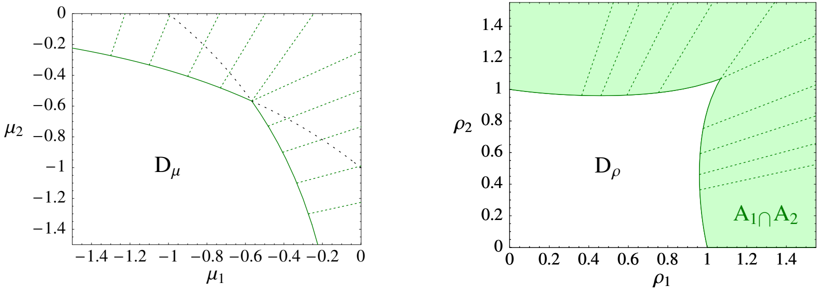




$$g_1 = \left(\frac{k_1}{1+k_1}\right)^{k_2}(1+b/k_1) \qquad g_2 = \frac{k_2}{1+k_1}$$

Example

$$w(k_1, k_2) = k_1^{-b} \frac{(k_1+1)^{k_2}}{k_2!} + k_2^{-b} \frac{(k_2+1)^{k_1}}{k_1!}$$



Structure of the condensed phase

For $\boldsymbol{\rho} \in A_i$ the background distribution $\nu^{(i)}_{\mathbf{M}(\boldsymbol{\rho})}$ of species i has a subexponential tail $\liminf_{k\to\infty} -\frac{1}{k} \log \nu^{(i)}_{\mathbf{M}(\boldsymbol{\rho})}(k) = 0$.

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If $\boldsymbol{\rho} \in A_i$ and $\nu^{(i)}_{\mathbf{M}(\boldsymbol{\rho})}(k) \simeq k^{-b}$, $b > 2$ as $k \to \infty$, then

$$\frac{1}{L} \max_{x\in\Lambda_L} \eta_i(x) \overset{\pi_{L,[\boldsymbol{\rho} L]}}{\longrightarrow} (\rho_i - \rho_{c_i}) , \quad \text{as } L \to \infty .$$

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Interpretation

The $(\rho - \rho_c)L$ excess particles typically condense at a single, randomly located site.

Remarks

Theorem 2. conditional form of the canonical measure

$$\pi_{L,[\boldsymbol{\rho}L]}(A) = \nu^{L}_{\mathbf{M}(\boldsymbol{\rho})}\Big(A \,\Big|\, \{\Sigma_L = [\boldsymbol{\rho}L]\}\Big) = \frac{\nu^{L}_{\mathbf{M}(\boldsymbol{\rho})}\big(A \cap \{\Sigma_L = [\boldsymbol{\rho}L]\}\big)}{\nu^{L}_{\mathbf{M}(\boldsymbol{\rho})}\big(\{\Sigma_L = [\boldsymbol{\rho}L]\}\big)}$$

large deviations for heavy-tailed distributions [Jeon et al. (2000)]

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Theorem 1. $h_L\big(\boldsymbol{\rho}, \overline{\mathbf{M}}(\boldsymbol{\rho})\big) = -\frac{1}{L}\,\log \nu^L_{\mathbf{M}(\boldsymbol{\rho})}\big(\{\Sigma_L = [\boldsymbol{\rho}L]\}\big)$

requires regularity assumption on the tail, e.g.

for all k_j $\quad \lim\limits_{k_i} \frac{1}{k_i}\,\log w(k_1, k_2) \in \mathbb{R}$ exists.

Summary

Results on the condensation transition

1. Equivalence of ensembles
2. Structure of the phase diagram
3. Volume of the condensed phase

Further questions

- central limit theorem for the condensate
- non-product weight $W_L(\boldsymbol{\eta})$
- rigorous results on the dynamics (hydrodynamic limit)