Spatial correlations of the 1D KPZ surface

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(based on joint work with T. Imamura)

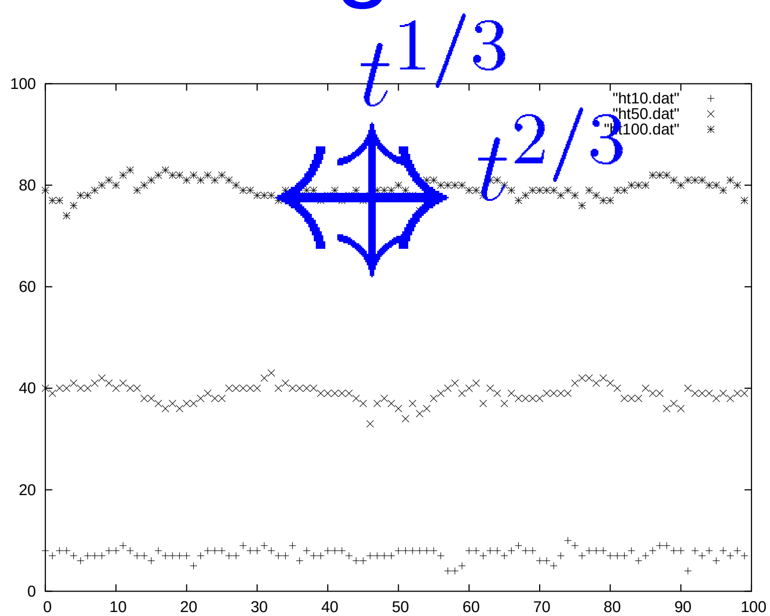
1. Introduction

1D growing surface

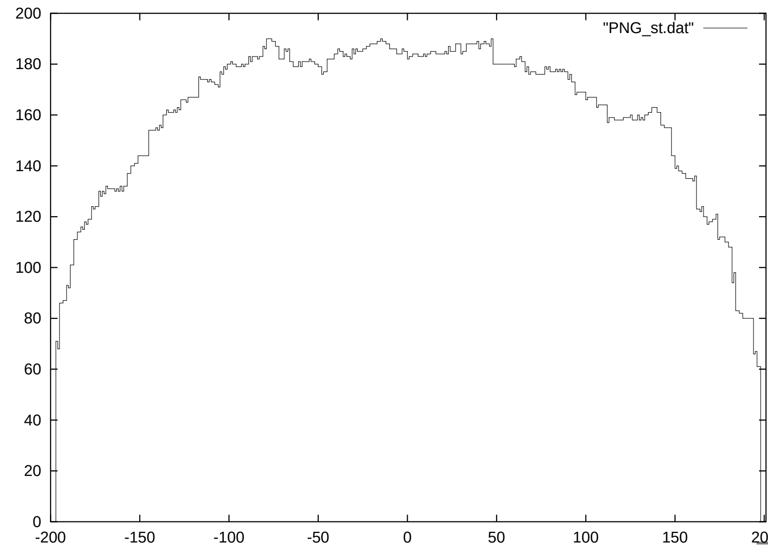
- percolation of ink into paper
- bacteria colony

simulation models

flat growth



droplet growth



KPZ equation, KPZ universality

1986 Kardar-Parisi-Zhang

$$\frac{\partial}{\partial t}h(x,t) = \nu\frac{\partial^2}{\partial x^2}h(x,t) + \frac{\lambda}{2}\left(\frac{\partial}{\partial x}h(x,t)\right)^2 + \eta(x,t)$$

where $h(x,t)$ is the height at position x at time t and

$$\langle\eta(x,t)\eta(x',t')\rangle = 2D\delta(x-x')\delta(t-t')$$

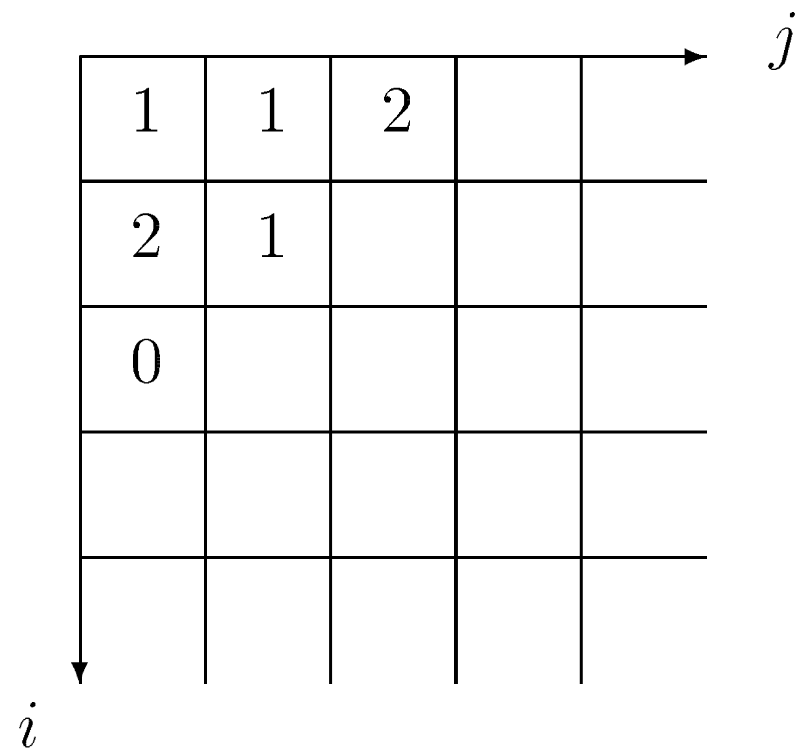
$\Rightarrow$ exponents $1/3, 2/3$

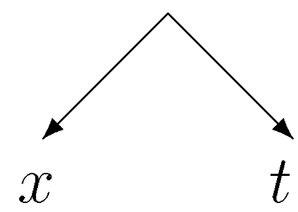
agree with simulation

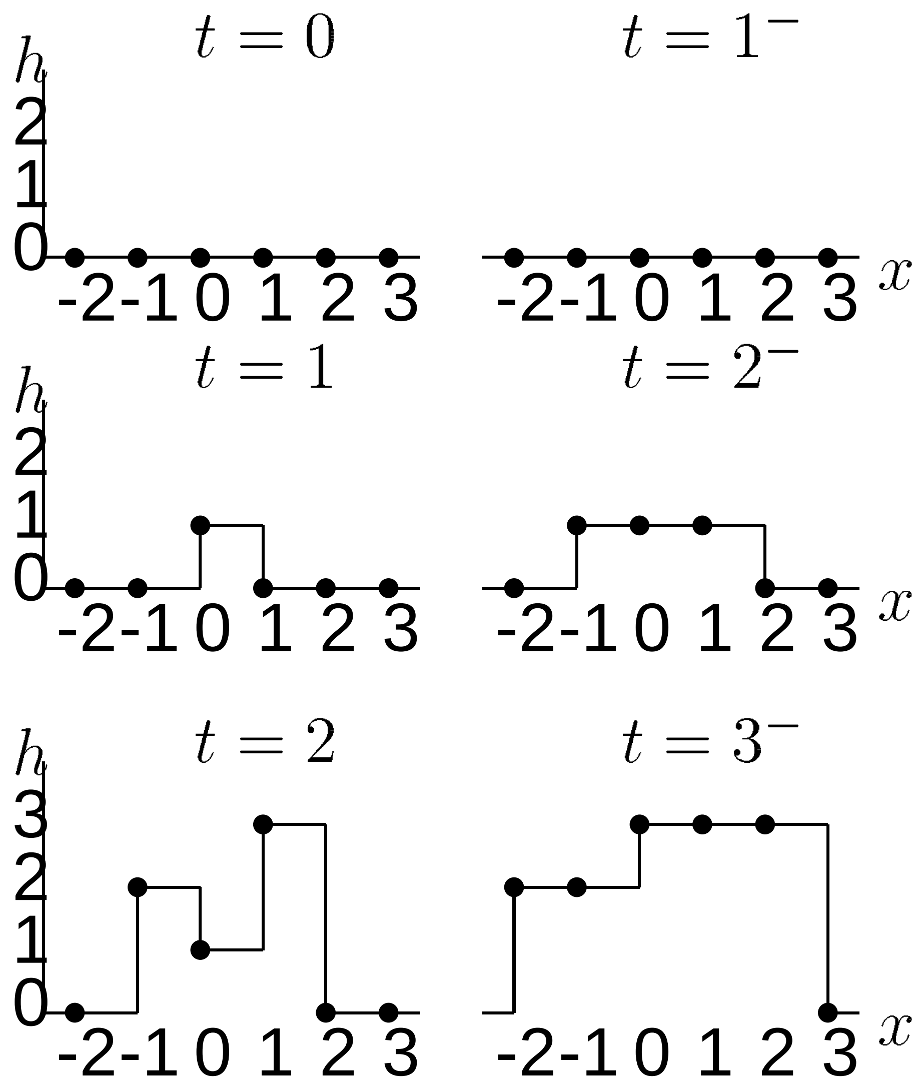
moments, distribution, correlation

2. PNG model

- PNG $=$ polynuclear growth
- $w_{i,j}$ on each lattice site (i,j)
  $\mathbb{P}(w_{i,j}=k)=(1-q)q^k\ (k=0,1,2,\dots)\quad (0<q<1)$

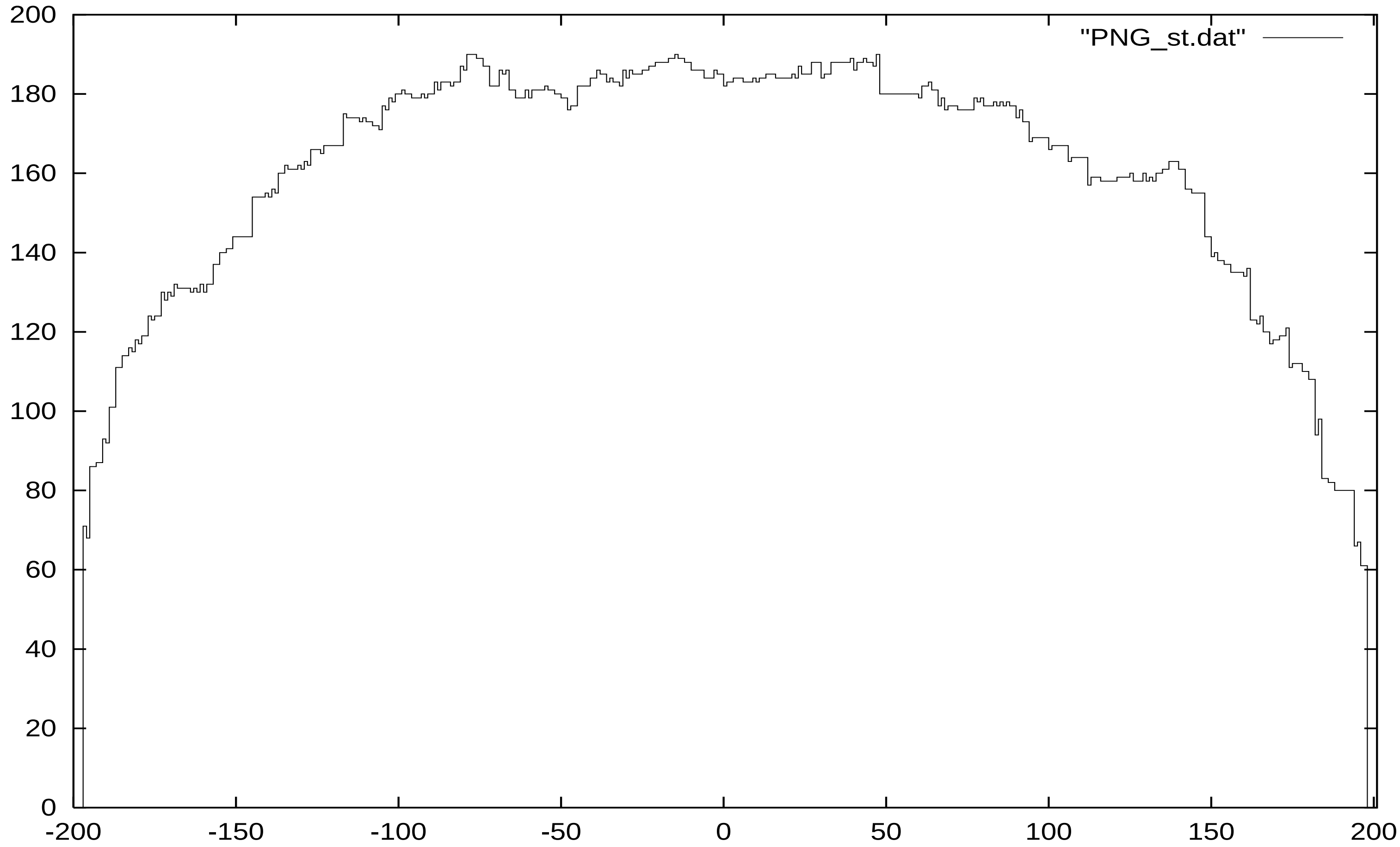





$t = 0$
$t = 1^-$
$t = 1$
$t = 2^-$
$t = 2$
$t = 3^-$
h
x
0
1
2
3
-2 -1 0 1 2 3

Simulation

h
(height)



x (space)

Statistics of $h(x,t)$?

3. Height Fluctuation at the origin

Johansson (2000)

$$\lim_{t\to\infty} \mathbb{P}\left[A_2(0) = \frac{h(x=0,t) - \frac{\sqrt{q}}{1-\sqrt{q}}t}{dt^{1/3}} \leq s\right] = F_2(s)$$

where $d = 2^{-1/3}q^{1/6}(1+\sqrt{q})^{1/3}/(1-\sqrt{q})$ and

$$F_2(s) = \exp[-\int_s^\infty (x-s)u(x)^2 dx]$$

with $u(x)$ being the solution to the Painlevé II eqution,

$$\frac{\partial^2}{\partial x^2}u = 2u^3 + xu, \quad u(x) \sim \mathrm{Ai}(x) \quad x \to +\infty$$

Random matrices

GUE (Gaussian Unitary Ensemble)

$$A = \begin{bmatrix} u_{11} & u_{12} + iv_{12} & \cdots & u_{1N} + iv_{1N} \\ u_{12} - iv_{12} & u_{22} & \cdots & u_{2N} + iv_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1N} - iv_{1N} & u_{2N} - iv_{2N} & \cdots & u_{NN} \end{bmatrix}$$

$$\prod_{j=1}^{N} \frac{1}{\sqrt{\pi}} e^{-u_{jj}^2} \prod_{j<k} \frac{2}{\pi} e^{-2u_{jk}^2 - 2v_{jk}^2}$$

Scaled distribution of the largest eigenvalue x_1

$$\lim_{N\to\infty} \mathbb{P}\left[(x_1 - \sqrt{2N})\sqrt{2}N^{1/6} < s \right] = F_2(s)$$

GOE (Gaussian Orthogonal Ensemble, $F_1(s)$)

$$A = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ u_{12} & u_{22} & \cdots & u_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{2N} & u_{2N} & \cdots & u_{NN} \end{bmatrix}$$

$$\prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-u_{jj}^2/2} \prod_{j<k} \frac{1}{\sqrt{\pi}} e^{-u_{jk}^2}$$

GSE (Gaussian Symplectic Ensemble, $F_4(s)$)

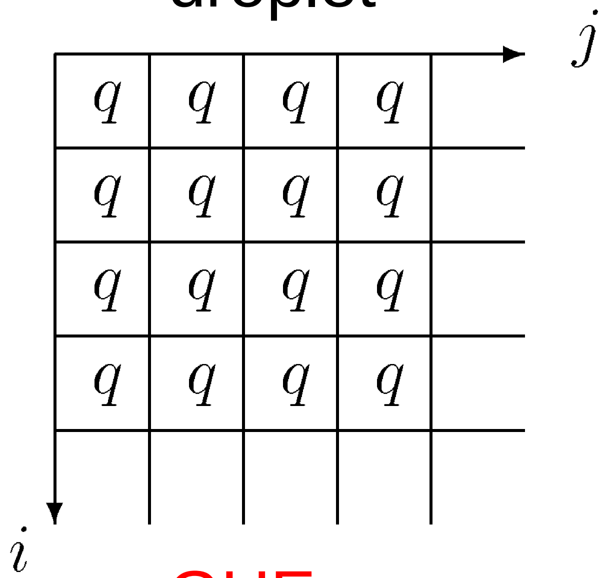
Time-dependent version $u_{jk} \to u_{jk}(t)$

tGUE, tGOE, tGSE

Generalisations

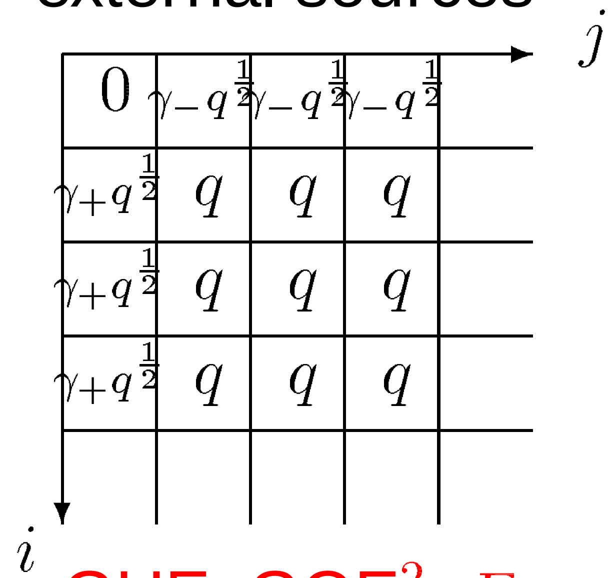
2000-2001 Prähofer-Spohn, Baik-Rains

droplet



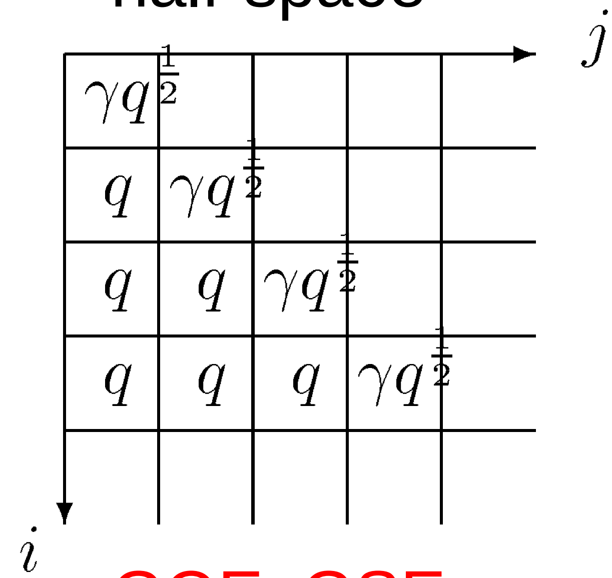
GUE

external sources



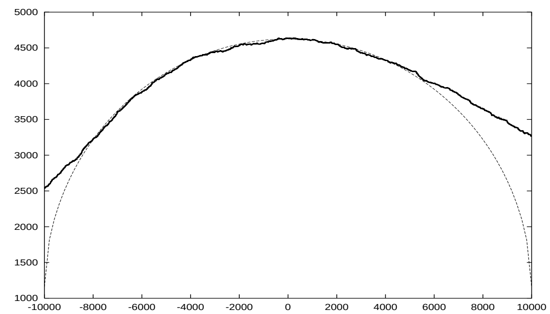
GUE, GOE^2, F_0

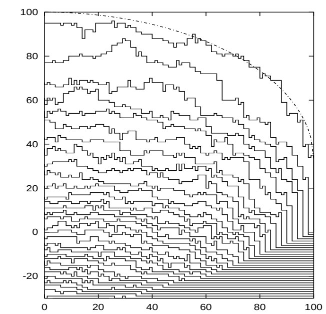
half-space



GOE, GSE

flat $\cdots$ GOE



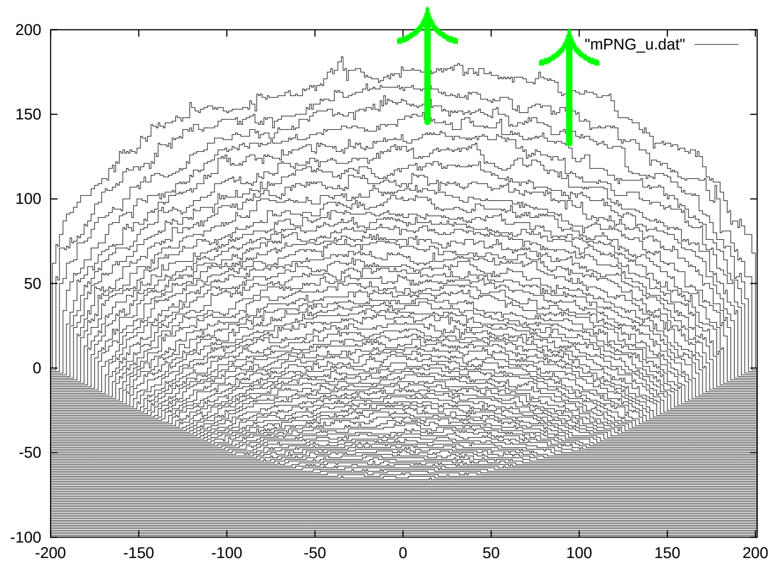


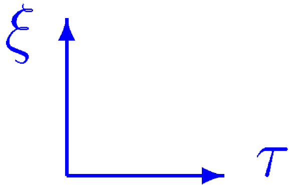
4. Spatial correlation $\cdots$ droplet

2002-2003 Prähofer-Spohn, Johansson

- "Roughness" of the surface at time t
- multi-layer PNG model (top layer $=$ original surface)

Scaled height $A_2(0)$ $A_2(\tau)$





$\sim$ vicous walks
(of watermelon type)

$\Rightarrow$ weight : productus of det

surface $\sim$ dynamics of the largest eigenvalue of tGUE

2pt joint distribution

Fredholm determinant

$$\mathbb{P}[A(0) < X_1, A(\tau) < X_2] = \det(1 - K_2)$$

$$K_2(\tau_1, \xi_1; \tau_2, \xi_2) = \tilde{K}_2(\tau_1, \xi_1; \tau_2, \xi_2) - \Phi_2(\tau_1, \xi_1; \tau_2, \xi_2)$$

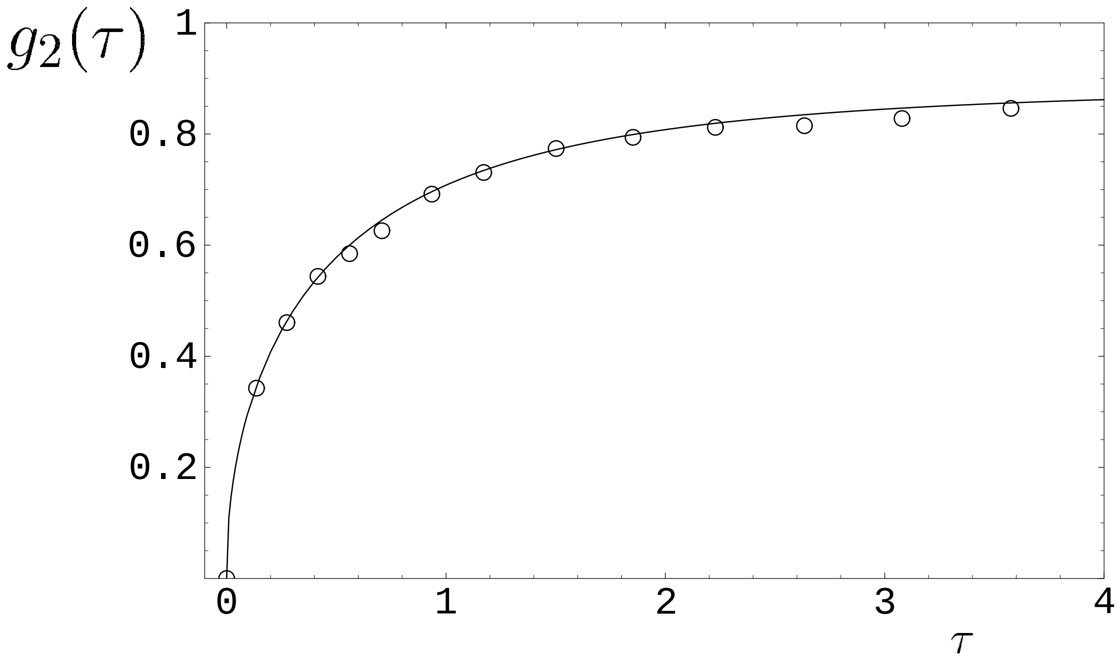
where

$$\tilde{K}_2(\tau_1, \xi_1; \tau_2, \xi_2) = \int_0^\infty d\lambda e^{-\lambda(\tau_1 - \tau_2)} \mathsf{Ai}(\xi_1 + \lambda) \mathsf{Ai}(\xi_2 + \lambda)$$

$$\Phi_2(\tau_1, \xi_1; \tau_2, \xi_2) = \begin{cases} \int_{-\infty}^\infty d\lambda e^{-\lambda(\tau_1 - \tau_2)} \mathsf{Ai}(\xi_1 + \lambda) \mathsf{Ai}(\xi_2 + \lambda) & \tau_1 < \tau_2 \\ 0 & \tau_1 \geq \tau_2 \end{cases}$$

2pt correlation

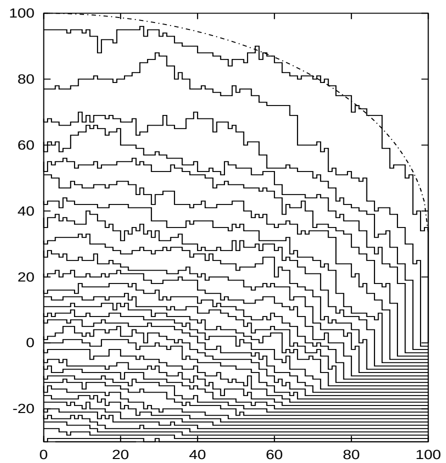
$$g_2(\tau) = \sqrt{\frac{\langle (A_2(\tau) - A_2(0))^2 \rangle}{2}}$$



Generalisations

mPNG, vicious walks, Fredholm determinant

half-space



(critical source: star)

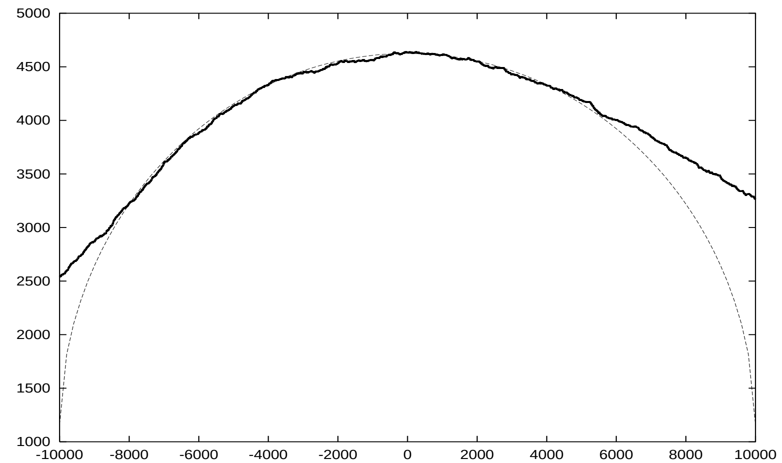
S & Imamura

Borodin-Rains

GOE,GSE(Origin)

GUE(Bulk)

external sources



(anglerfish)

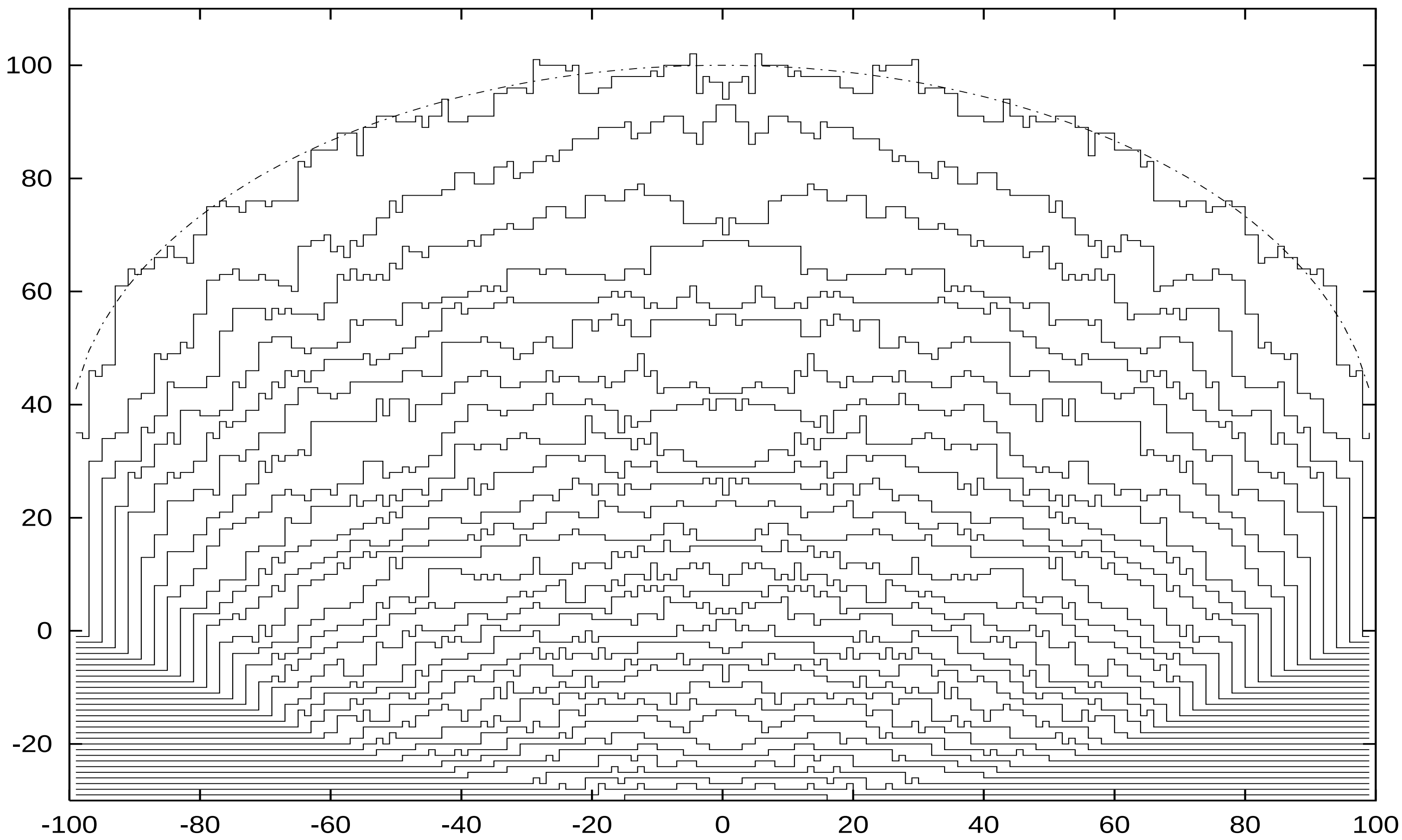
Imamura & S

GOE$^2, F_0$ (Intersection)

GUE,Gauss(Bulk)

Symmetry Transitions

No source



5. Spatial correlation $\cdots$ flat case

Situation (2004 P. Ferrari)

- mPNG $\cdots$ GOE eigenvalues
- Relation to tGOE($=$ ($\beta = 1$) Calogero model) ?

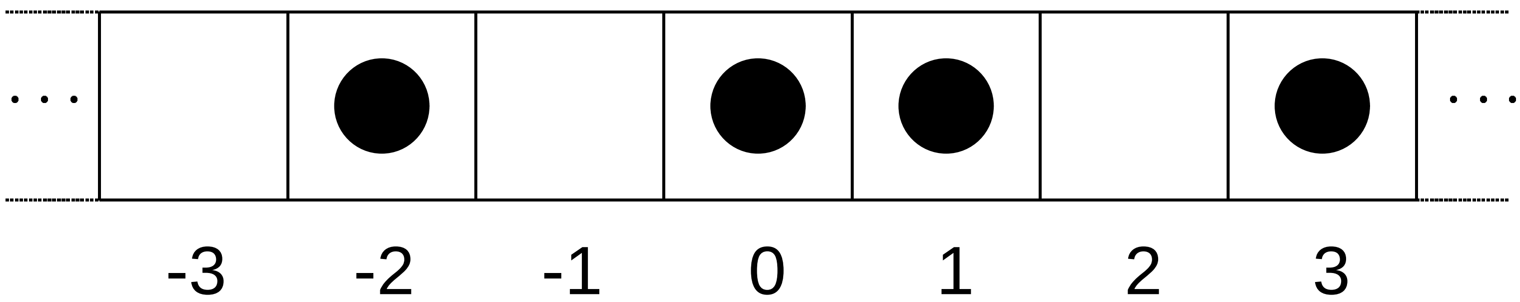
	droplet	flat
1pt	GUE	GOE
2pt	tGUE	tGOE?

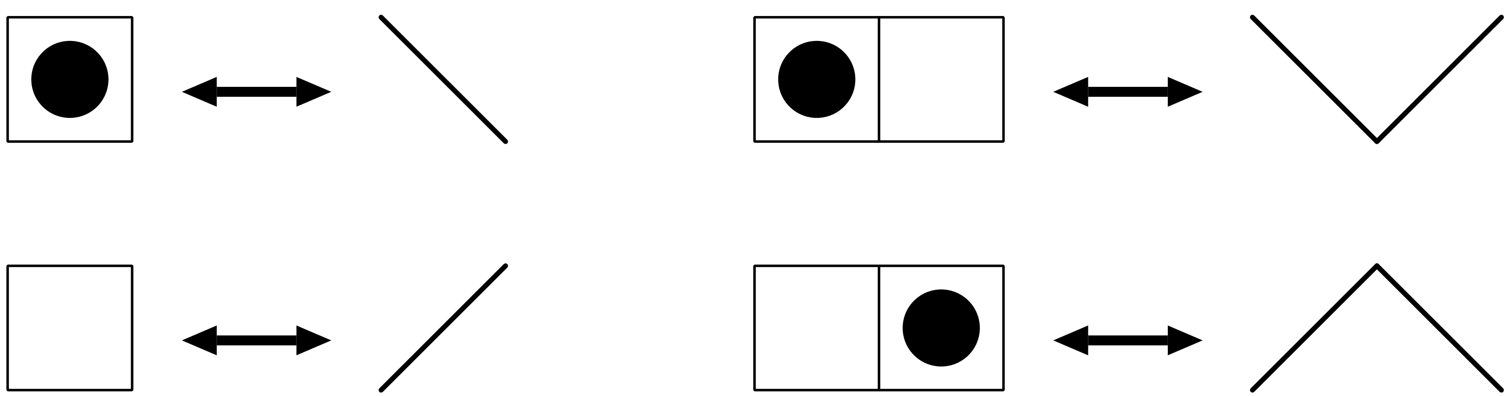
My results

- Compute correlation using the surface model related to TASEP
- Joint distribution is again Fredholm determinant
- Relation to tGOE still unknown

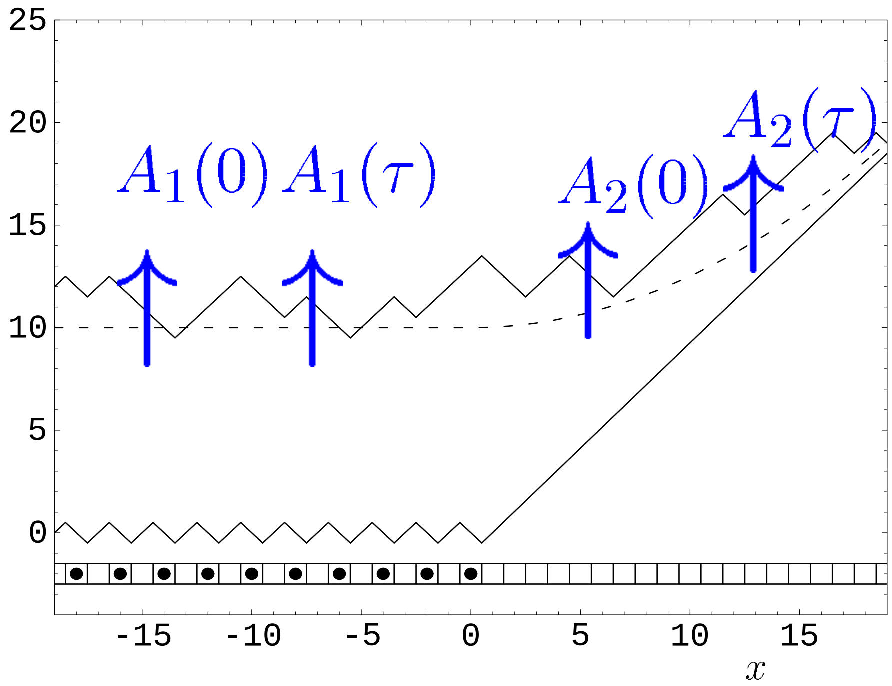
TASEP and surface growth

TASEP = Totally asymmetric simple exclusion process





”Half flat” initial condition



- Each particle cannot affect the particles on its right
- Same as flat case deep inside the negative ($x < 0$) region

surface height $\Leftrightarrow$ particle positions

Determinatal Green's function

Schütz 1997

Probability that N particles starting from $y_1, y_2, \cdots, y_N$ ($y_N < \cdots < y_1$) are on $x_1, x_2, \cdots, x_N$ ($x_N < \cdots < x_1$) at time t

$$G(x_1, x_2, \cdots, x_N; t | y_1, y_2, \cdots, y_N; 0) \quad (= G(x_1, x_2, \cdots, x_N; t))$$

$$= \det[F_{k-j}(x_{N-k+1} - y_{N-j+1}; t)]_{j,k=1,\cdots,N}$$

$$= \begin{vmatrix} F_0(x_N - y_N; t) & F_{-1}(x_N - y_{N-1}; t) & \cdots & F_{-N+1}(x_N - y_1; t) \\ F_1(x_{N-1} - y_N; t) & F_0(x_{N-1} - y_{N-1}; t) & \cdots & F_{-N+2}(x_{N-1} - y_1; t) \\ \vdots & \vdots & & \vdots \\ F_{N-1}(x_1 - y_N; t) & F_{N-2}(x_1 - y_{N-1}; t) & \cdots & F_0(x_1 - y_1; t) \end{vmatrix}$$

where $F_n(x;t) = e^{-t} \frac{t^x}{x!} \sum_{k=0}^{\infty} (-1)^k \frac{(n)_k}{(x+1)_k} \frac{t^k}{k!}$

Interpretation in terms of vicious walk

vicious walk type weight

weight

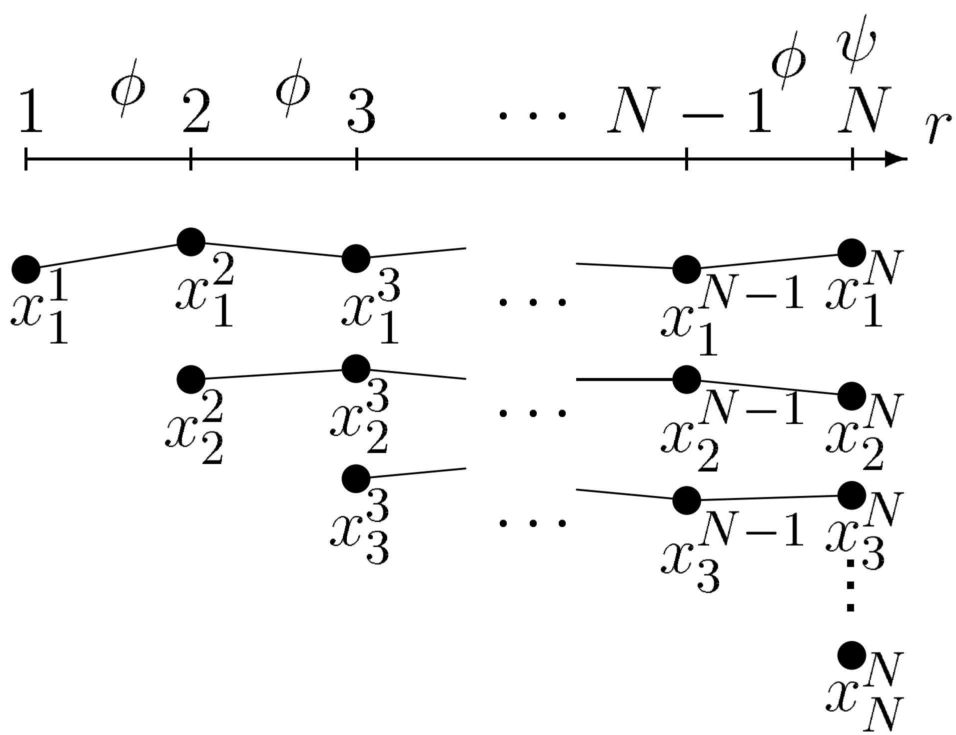
$$\prod_{r=1}^{N-1} \det[\phi(x_j^r, x_k^{r+1})]_{j,k=1}^{r+1} \cdot \det[\psi_j^{(N)}(x_{k+1}^N)]_{j,k=0}^{N-1}$$

$$= \begin{vmatrix} \phi(x_1^1, x_1^2) & \phi(x_1^1, x_2^2) \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \phi(x_1^2, x_1^3) & \phi(x_1^2, x_2^3) & \phi(x_1^2, x_3^3) \\ \phi(x_2^2, x_1^3) & \phi(x_2^2, x_2^3) & \phi(x_2^2, x_3^3) \\ 1 & 1 & 1 \end{vmatrix} \cdots$$

$$\times \begin{vmatrix} \psi_0^{(N)}(x_1^N) & \psi_0^{(N)}(x_2^N) & \cdots & \psi_0^{(N)}(x_N^N) \\ \psi_1^{(N)}(x_1^N) & \psi_1^{(N)}(x_2^N) & \cdots & \psi_1^{(N)}(x_N^N) \\ \vdots & \vdots & & \vdots \\ \psi_{N-1}^{(N)}(x_1^N) & \psi_1^{(N)}(x_{N-1}^N) & \cdots & \psi_{N-1}^{(N)}(x_N^N) \end{vmatrix}$$

$$\phi(x_1, x_2) = \begin{cases} 0 & (x_1 > x_2) \\ -1 & (x_1 \le x_2) \end{cases}$$

$$\psi_j^{(r)}(x) = (-1)^{r-1-j} F_{-r+1+j}(x - y_{j+1}; t)$$



vicious walk interpretation of G

Let $\mathbb{P}$ denote the corresponding measure. We have

$$G(x_1, \cdots, x_N; t) = \mathbb{P}[x_1^r = x_r \ (r = 1, \cdots, N)]$$

TASEP particle positions $\sim$ dynamics of the 1st walker

<u>ex. $N = 2$</u>

$$\mathbb{P}[x_1^1 = x_1, x_2^1 = x_2] = \sum_{x_2^2 (> x_1^2)} \begin{vmatrix} \phi(x_1^1, x_1^2) & \phi(x_1^1, x_2^2) \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \psi_0^{(2)}(x_1^2) & \psi_0^{(2)}(x_2^2) \\ \psi_1^{(2)}(x_1^2) & \psi_1^{(2)}(x_2^2) \end{vmatrix}$$

$$\begin{cases} x_1^1 = x_1 > x_2 = x_1^2 \Rightarrow \phi(x_1^1, x_1^2) = 0 \\ x_2^2 < x_1^1 \text{ then} \phi(x_1^1, x_2^2) = 0 \\ x_2^2 \geq x_1^1 \text{ then } \phi(x_1^1, x_2^2) = -1 \end{cases}$$

$$
\begin{aligned}
&= \sum_{x_2^2 = x_1^1}^{\infty} \begin{vmatrix} -F_{-1}(x_1^2 - y_1; t) & -F_{-1}(x_2^2 - y_1; t) \\ F_0(x_1^2 - y_2; t) & F_0(x_2^2 - y_2; t) \end{vmatrix} \\
&= \sum_{x_2^2 = x_1}^{\infty} \begin{vmatrix} F_0(x_2 - y_2; t) & F_0(x_2^2 - y_2; t) \\ F_{-1}(x_2 - y_1; t) & F_{-1}(x_2^2 - y_1; t) \end{vmatrix} \\
&= \begin{vmatrix} F_0(x_2 - y_2; t) & F_1(x_1 - y_2; t) \\ F_{-1}(x_2 - y_1; t) & F_0(x_1 - y_1; t) \end{vmatrix} \\
&= G(x_1, x_2; t)
\end{aligned}
$$

2pt joint distribution

$$\mathbb{P}[A_j(0) < X_1, A_j(\tau) < X_2] = \det(1 - K_j) \quad (j = 1, 2)$$

- positive ($x > 0$) region $\cdots$ K_2
- negative ($x < 0$) region $\cdots$ K_1

$$K_1(\tau_1, \xi_1; \tau_2, \xi_2) = \tilde{K}_1(\tau_1, \xi_1; \tau_2, \xi_2) - \Phi_1(\tau_1, \xi_1; \tau_2, \xi_2)$$

where

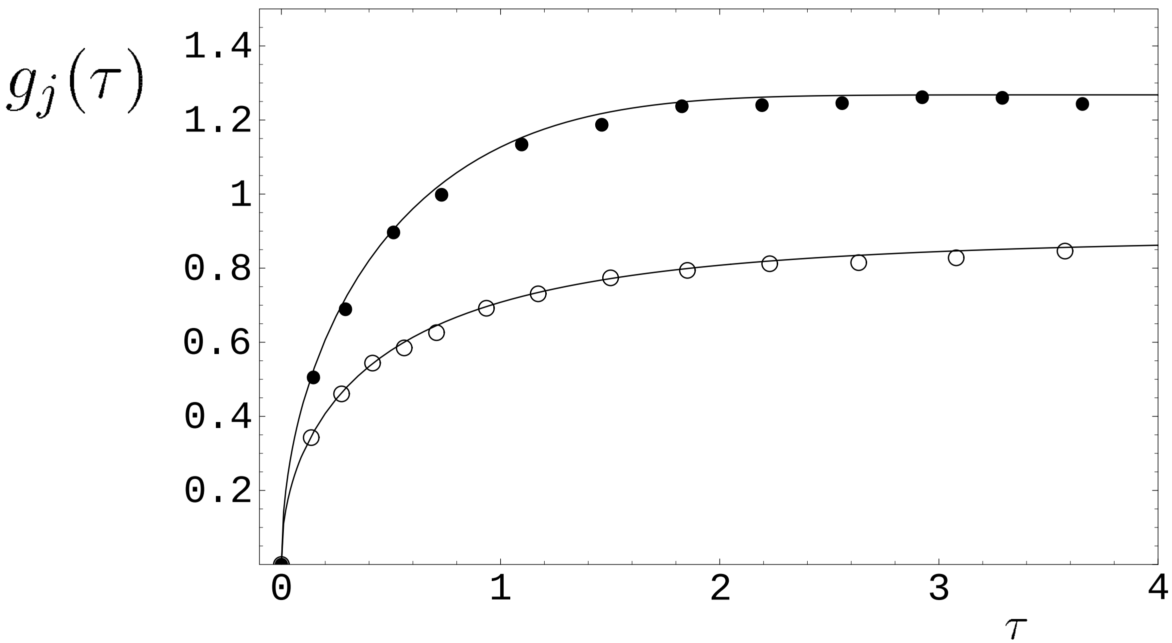
$$\tilde{K}_1(\tau_1, \xi_1; \tau_2, \xi_2) = \frac{1}{2} e^{(\tau_2 - \tau_1)(\xi_1 + \xi_2)/4 + (\tau_2 - \tau_1)^3/12} \mathrm{Ai}\left(\frac{\xi_1 + \xi_2}{2} + \frac{(\tau_2 - \tau_1)^2}{4} \right.$$

$$\Phi_1(\tau_1, \xi_1; \tau_2, \xi_2) = \begin{cases} \frac{1}{\sqrt{8\pi(\tau_2 - \tau_1)}} \exp\left[-\frac{(\xi_2 - \xi_1)^2}{8(\tau_2 - \tau_1)} \right] & \tau_1 < \tau_2 \\ 0 & \tau_1 \geq \tau_2 \end{cases}$$

Relation to tGOE still unknown

2pt correlation

$$g_j(\tau) = \sqrt{\frac{\langle (A_j(\tau) - A_j(0))^2 \rangle}{2}} \quad (j = 1, 2)$$



New formula for F_1

$\tau_1 = \tau_2 (= \tau)$

$$K_1(\tau, \xi_1; \tau, \xi_2) = \frac{1}{2}\text{Ai}\left(\frac{\xi_1 + \xi_2}{2}\right)$$

Since 1pt fluctuation is F_1 (GOE),

$$F_1(s) = \det(1 - K_1)$$

proved directly by Ferrari, Spohn

6. Summary

- Fluctuation properties of the 1D KPZ surface
- 1pt height fluctuation $\Leftrightarrow$ largest eigenvalue fluctuation of random matrices
- Multi-point height fluctuation have been computed for various cases

Future problems

- Relation to tGOE
- two-time correlation
- finite system

References

[1](PNG in half-space) T. Sasamoto and T. Imamura, J. Stat. Phys. **115** (2004) 749-803.

[2](Green's function by Schütz and random matrix) T. Nagao and T. Sasamoto, Nucl. Phys. B **699** (2004) 487–502.

[3](PNG with external soureces) T. Imamura and T. Sasamoto, Nucl. Phys. B **699** (2004) 503–544.

[4](PNG with source and RM with source) T. Imamura and T. Sasamoto, Phys. Rev. E **71**(2005) 041606.

[5](flat PNG) T. Sasamoto, J. Phys. A 38(2005)L549-L556.