Exact solution of the dynamics of the PASEP with open boundaries

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University of Melbourne

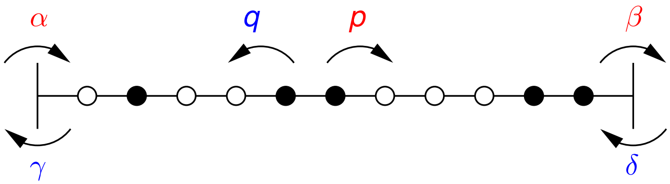
Non-Equilibrium Dynamics of Interacting Particle Systems, Cambridge 2006

In collaboration with Fabian Essler (Oxford)

Outline

1. PASEP
   - Definition
   - Why?
   - Spectral gap
2. Bethe's Ansatz
   - Advertisement
   - XXZ chain
   - Constraint
   - PASEP Bethe equations
3. TASEP dynamics
4. Conclusion

Transition rules



τ : 0 1 0 0 1 1 0 0 0 1 1

- Probability to find a configuration at time t: $P_\tau(t)$
- Transition matrix M
- Time evolution given by the Master Equation:

$$\frac{\mathrm{d}}{\mathrm{d}\,t} P_\tau(t) = -\sum_\sigma M_{\tau\sigma} P_\sigma(t)$$

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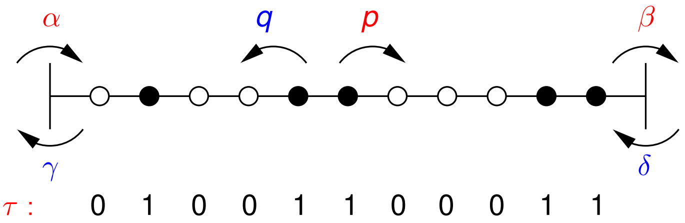

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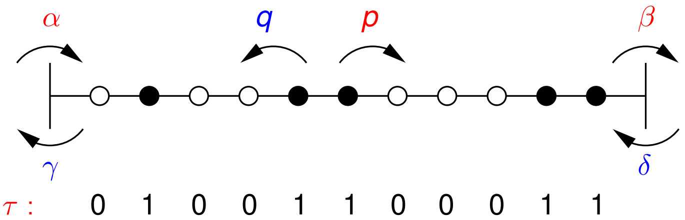

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- Theoretically: influence of boundary
- Phase transitions (even in 1D)
- Phase structure is well known for stationary state (Derrida, Evans, Hakim & Pasquier '93; Domany & Schütz '93)
- This talk: dynamics (long time)

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Approach to stationarity

Master Equation:

$$\frac{\mathrm{d}}{\mathrm{d}\,t} P_\tau(t) = -\sum_\sigma M_{\tau\sigma} P_\sigma(t)$$

Formal solution:

$$\mathbf{P}_t = \mathrm{e}^{-Mt}\,\mathbf{P}_{\text{initial}}$$

$$\mathbf{P}_{\text{initial}} = \sum_{n=0} a_n \psi_n, \qquad M\psi_n = \lambda_n \psi_n.$$

$$\mathbf{P}_t = \sum_{n=0} a_n\,\mathrm{e}^{-\lambda_n t}\,\psi_n$$

Because $\lambda_0 = 0$, $\lambda_n > 0$:

$$\mathbf{P}_t \approx a_0\psi_0 + a_1\,\mathrm{e}^{-\lambda_1 t}\,\psi_1 + \ldots \qquad (t \to \infty)$$

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- Transition matrix M is equivalent to the Hamiltonian of the XXZ quantum spin chain
- Use integrability (Bethe's Ansatz) to diagonalise M
- Works well for periodic boundary conditions (Gwa & Spohn '92 and Kim '95 in context of growth model)
- Open boundaries: no conserved charges
- Except for symmetric hopping $p = q$ (Stinchcombe & Schütz '95), Bethe Ansatz equations have been unknown so far . . .

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Example: Single diffusive particle

$$\begin{aligned}(H\psi)(x) &= p\psi(x-1) + q\psi(x+1) - (p+q)\psi(x) \\ &= \Lambda\psi(x)\end{aligned}$$

Ansatz:

$$\psi(x) = z^x$$

$$\Lambda = pz^{-1} + qz - p - q$$

Periodicity:

$$\psi(L+1) = \psi(1) \quad \Rightarrow \quad z^L = 1$$

L solutions $\Rightarrow$ L eigenvalues

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Results of Bethe's Ansatz:
Each eigenvalue can be expressed in roots of a system of high degree polynomials.

$$\lambda = \sum_{j=1}^{m} \lambda(z_j), \qquad P_L(z_j; z_1, \ldots, z_m) = 0 \quad (j = 1, \ldots, m).$$

- This year it is Bethe's 100th birthday (2 July 1906).
- It is 75 years ago since Bethe's paper on the XXX Heisenberg chain, where he used his famous Ansatz.
- JSTAT will publish a special issue with current research papers related to Bethe's Ansatz.
- Submissions are welcome.

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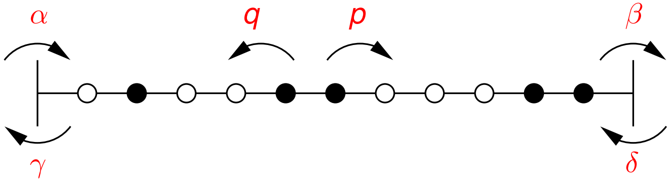

Parameters: $Q = \sqrt{\frac{q}{p}}, \alpha, \beta, \gamma, \delta$

Length: L

Mapping to XXZ

$$M \propto U^{-1} H U$$

where

$$U = \bigotimes_{j=1}^{L} \begin{pmatrix} 1 & 0 \\ 0 & Q^{j-1} \end{pmatrix}$$

$$H = -\frac{1}{2}\sum_{j=1}^{L-1}\Big[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \Delta\sigma_j^z\sigma_{j+1}^z + h(\sigma_{j+1}^z - \sigma_j^z) + \Delta\Big] + B_1 + B_L.$$

Δ and h are related to the PASEP bulk hopping rate by

$$\Delta = -\frac{1}{2}(Q + Q^{-1}), \quad h = \frac{1}{2}(Q - Q^{-1}), \quad Q = \sqrt{\frac{q}{p}}$$

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Mapping to XXZ

The boundary terms $B_{1,L}$ are

$$B_1 \propto \alpha + \gamma + (\alpha - \gamma)\sigma_1^z - 2\alpha\sigma_1^- - 2\gamma\sigma_1^+,$$

$$B_L \propto \beta + \delta - (\beta - \delta)\sigma_L^z - 2\beta Q^{-L+1}\sigma_L^+ - 2\delta Q^{L-1}\sigma_L^-.$$

H is integrable (Sklyanin '88)
(H satisfies Yang-Baxter and Reflection equations)

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- Integrability $\Rightarrow$ tools for exact diagonalisation.
- Exact diagonalisation of XXZ chain with general open boundaries by Bethe Ansatz is still an open problem.
- So what have we gained?
- Bethe equations for XXZ have been obtained under a constraint (Cao, Shi, Lin & Wang '03; Nepomechie '03)
- PASEP does not have most general XXZ open boundary conditions

Constraint in PASEP parameters:

$$(Q^{L+2k} - 1)(\alpha\beta - \gamma\delta Q^{L-2k-2}) = 0.$$

for any choice of $k \in \mathbb{Z}$, $|k| \leq L/2$.

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Structure of solution for arbitrary k

- Need two sets of Bethe equations, all eigenvalues are given by

$$E_1 = \sum_{j=1}^{L/2-k-1} \lambda(\{z_j\}), \qquad E_2 = \sum_{j=1}^{L/2+k} \lambda(\{w_j\}).$$

- Recall constraint:

$$(Q^{L+2k} - 1)(\alpha\beta - \gamma\delta Q^{L-2k-2}) = 0, \qquad |k| \leq L/2.$$

- Take $k = -L/2$
- One energy level $E_2 = 0$ (stationary state).
- All excited states described by E_1.

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- One energy level $E_2 = 0$ (stationary state).
- All excited states described by E_1.

Bethe equations

For all parameter values of the PASEP:

$$E(\{z_j\}) = \alpha + \beta + \gamma + \delta + \sum_{j=1}^{L-1} \frac{(Q^2-1)^2 z_j}{(Q-z_j)(Qz_j-1)}$$

$$\left[\frac{z_jQ-1}{Q-z_j}\right]^{2L} K(z_j) = \prod_{l\neq j}^{L-1} \left[\frac{z_jQ^2-z_l}{z_j-z_lQ^2}\right]\left[\frac{z_jz_lQ^2-1}{z_jz_l-Q^2}\right]$$

with $K(z) = \tilde{K}(z,\alpha,\gamma)\tilde{K}(z,\beta,\delta)$ and

$$\tilde{K}(z,\alpha,\gamma) = \frac{-\alpha z^2 + Qz(Q^2-1+\alpha-\gamma)+\gamma Q^2}{\gamma Q^2 z^2 + Qz(Q^2-1+\alpha-\gamma)-\alpha}$$

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XXZ chain

Constraint

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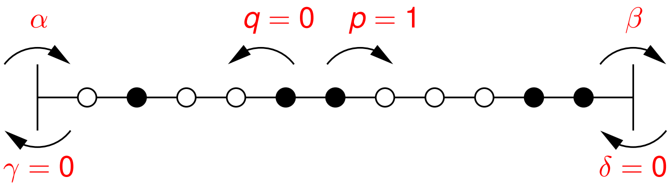
Total asymmetry

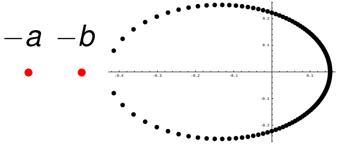
Reminder:



$\gamma = \delta = 0,\ Q \to 0$

$$
\begin{aligned}
E &= \alpha + \beta + \sum_{l=1}^{L-1} \frac{z_l}{z_l - 1}, \\
\left(\frac{(z_j - 1)^2}{z_j} \right)^L &= (z_j + a)(z_j + b) \prod_{l \neq j}^{L-1} (z_j - z_l^{-1}).
\end{aligned}
$$

For λ_1, excited state, Bethe roots lie on a simple curve



$$
z_{\rm c} = -\frac{1}{\sqrt{ab}}, \qquad a = \frac{1-\alpha}{\alpha}, \qquad b = \frac{1-\beta}{\beta}
$$

Integral equation

Taking log gives:

$$Y_L(z) = f(z) + \frac{1}{L} g(a,b,z) + \frac{1}{L} \sum_{j=1}^{L-1} \log(z - z_j^{-1})$$

Curve: $Y_L(z_j) = -\pi + \frac{2\pi j}{L}$

Euler-Maclaurin:

$$Y_L(z) = f(z) + \frac{1}{L}\tilde{g}(a,b,z) + \frac{1}{2\pi \mathrm{i}} \oint \log(z - w^{-1}) Y_L'(w) \mathrm{d}w + \mathcal{O}(L^{-2})$$

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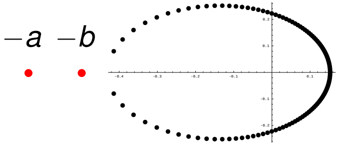
Low density I

$$-a, -b < z_{\mathrm{c}}$$



$$\lambda_1 = \alpha + \beta - \frac{2}{1+\sqrt{ab}} + \frac{\pi^2}{\sqrt{ab} - 1/\sqrt{ab}} L^{-2} + \mathcal{O}(L^{-3})$$

Finite gap $\Rightarrow$ exponential relaxation:

$$\mathbf{P}_t \approx a_0 \psi_0 + a_1 \, \mathrm{e}^{-\lambda_1 t} \, \psi_1 + \ldots \quad (t \to \infty)$$

Low density I

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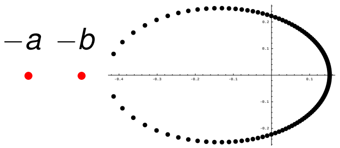

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Coexistence line:

$$\alpha = \beta : \qquad \lambda_1 = \frac{\pi^2 \alpha(1-\alpha)}{1-2\alpha} L^{-2}.$$

Diffusive relaxation

Domain wall theory (Dudzinski & Schütz '00)

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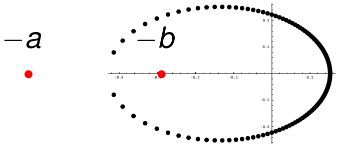
Diffusive relaxation

Domain wall theory (Dudzinski & Schütz '00)

Low density II

$$-a < z_c < -b$$



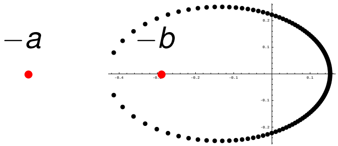
$$\lambda_1 = \alpha + \beta_c - \frac{2}{1+\sqrt{ab_c}} + \frac{4\pi^2}{\sqrt{ab_c} - 1/\sqrt{ab_c}} L^{-2} + \mathcal{O}(L^{-3})$$

$$b_c = -z_c = \frac{1}{\sqrt{ab_c}}$$

$$\beta_c = \left(1 + \left(\frac{\alpha}{1-\alpha}\right)^{1/3}\right)^{-1}$$

Low density II

$$-a < z_c < -b$$



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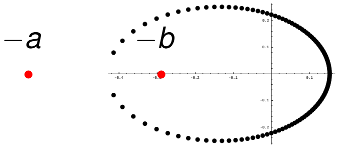

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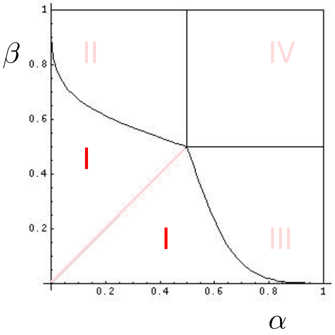
Phase diagram



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$\alpha = \beta$: $\frac{\pi^2 \alpha(1-\alpha)}{1-2\alpha} L^{-2}$

II: $\lambda_1 = \alpha + \beta_c - \frac{2}{1+\sqrt{ab_c}}$

III: $\lambda_1 = \alpha_c + \beta - \frac{2}{1+\sqrt{a_c b}}$

IV: $\lambda_1 \propto L^{-3/2}$

$a = (1-\alpha)/\alpha$, $b = (1-\beta)/\beta$,
$\beta_c = (1 + a^{-1/3})^{-1}$ and $\alpha_c = (1 + b^{-1/3})^{-1}$

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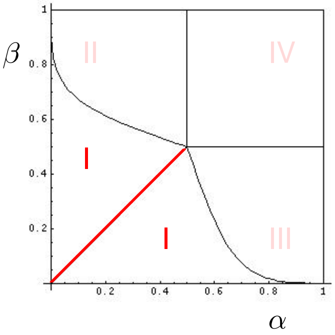

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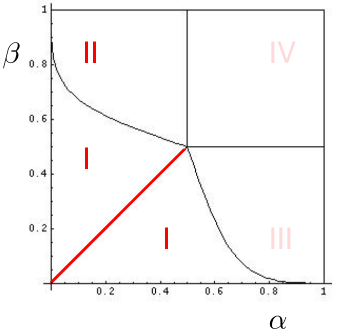

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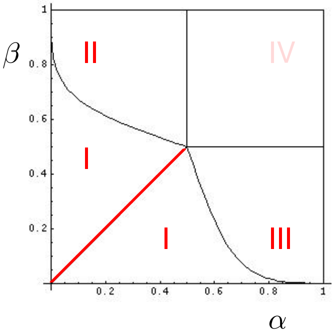

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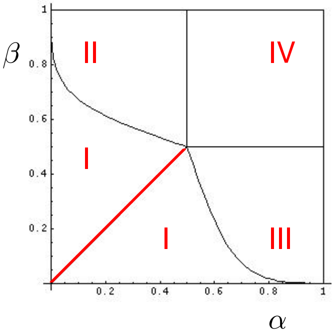

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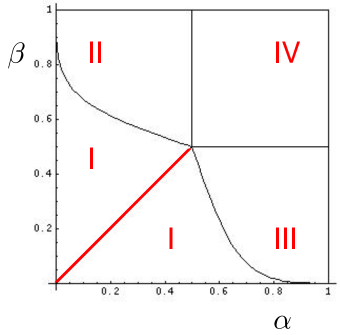

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Current fluctuations

Can we introduce λ to measure current fluctuations (Derrida)?

Constraint becomes:

$$(Q^{L+2k} - e^{\lambda})(\alpha\beta - \gamma\delta Q^{L-2k-2} e^{-\lambda}) = 0.$$

Same set of Bethe equations

For $Q \neq 1$ only a discrete set of values

Research problem: Bethe equations for general XXZ chain.

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- Boundary induced crossovers between diffusive (L^{-2}) and KPZ scaling ($L^{-3/2}$)
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