Multiclass Hammersley-Aldous-Diaconis process and $M/M/1$ queues

Pablo A. Ferrari

São Paulo

James B. Martin

Oxford

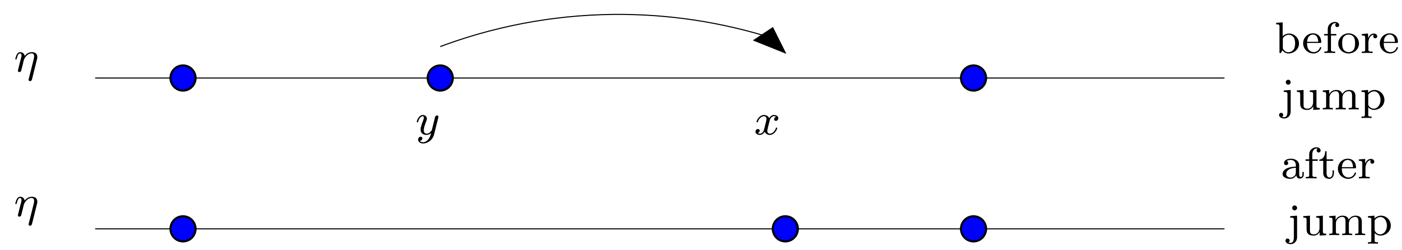
http://www.ime.usp.br/~pablo

Summary:

- Hammersley-Aldous-Diaconis process (HAD) in $\mathbb{R}$.
- Also works for TASEP in $\mathbb{Z}$.
- Discrete HAD and other transport processes.

- Basic coupling induces a *multiclass* process. Shock measures
- Invariant measures of the multiclass systems are the same for all processes.
- Output process of a system of multiclass queues in tandem;
- New coupling between stationary versions of the processes called a *multi-line process*
- *Dual points*: when the *graphical construction* is used to construct a trajectory of the TASEP or HAD process as a function of a Poisson process in $\mathbb{Z} \times \mathbb{R}$,

Hammersley Aldous Diaconis Process



- Hammersley-Aldous-Diaconis: generator

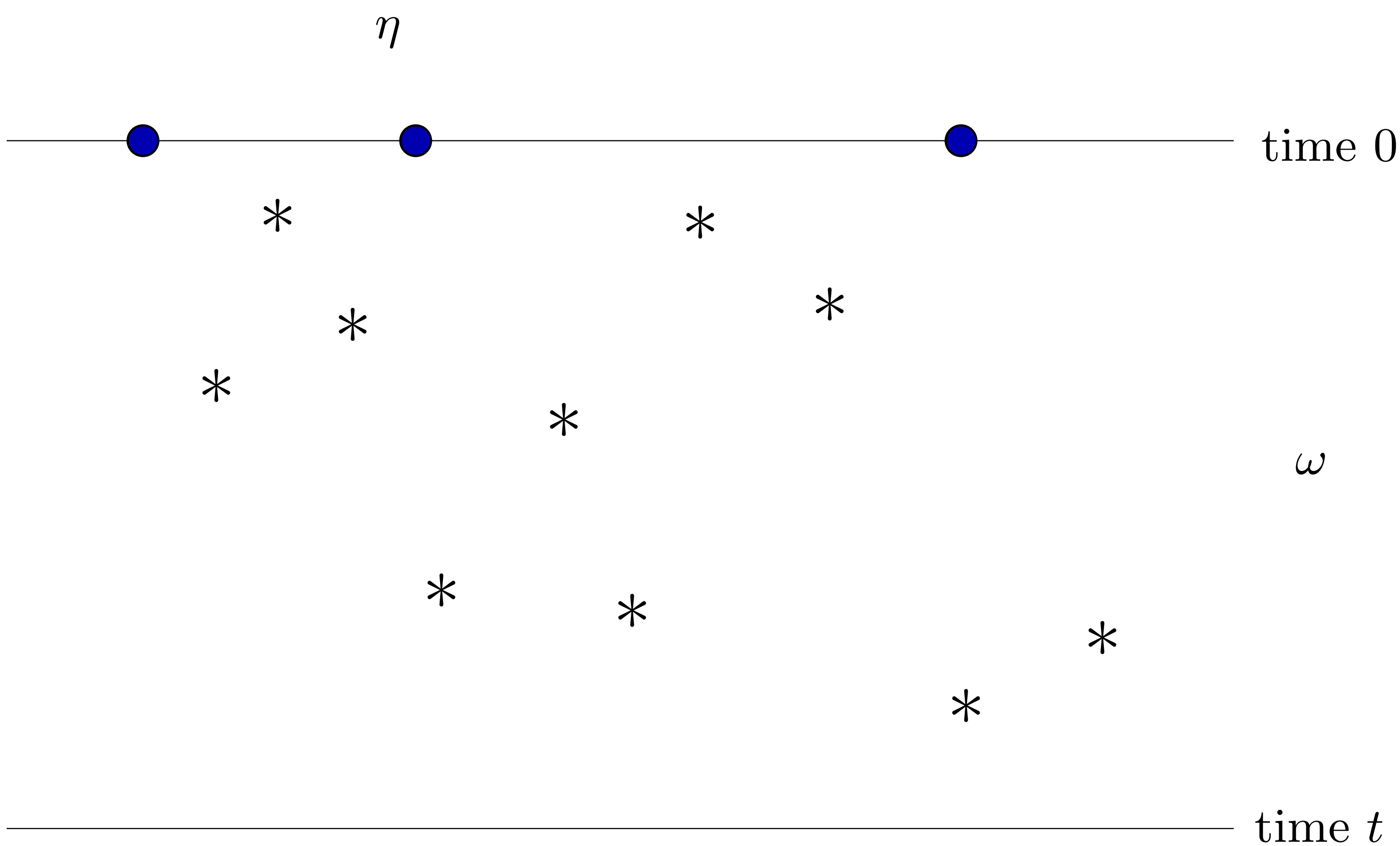
$$L_H f(\eta) = \int_{\mathbb{R}} [f(J(\eta, x)) - f(\eta)] dx \tag{1}$$

$$J(\eta, x) = \eta \cup \{x\} \setminus \{y\} \tag{2}$$

where $y = y(\eta, x) = \sup\{\eta \cap (-\infty, x)\}$ closest particle of η to the left of x.

Harris graphical construction

Poisson process of rate 1 on $\mathbb{R}^2$. * represent Poisson events.



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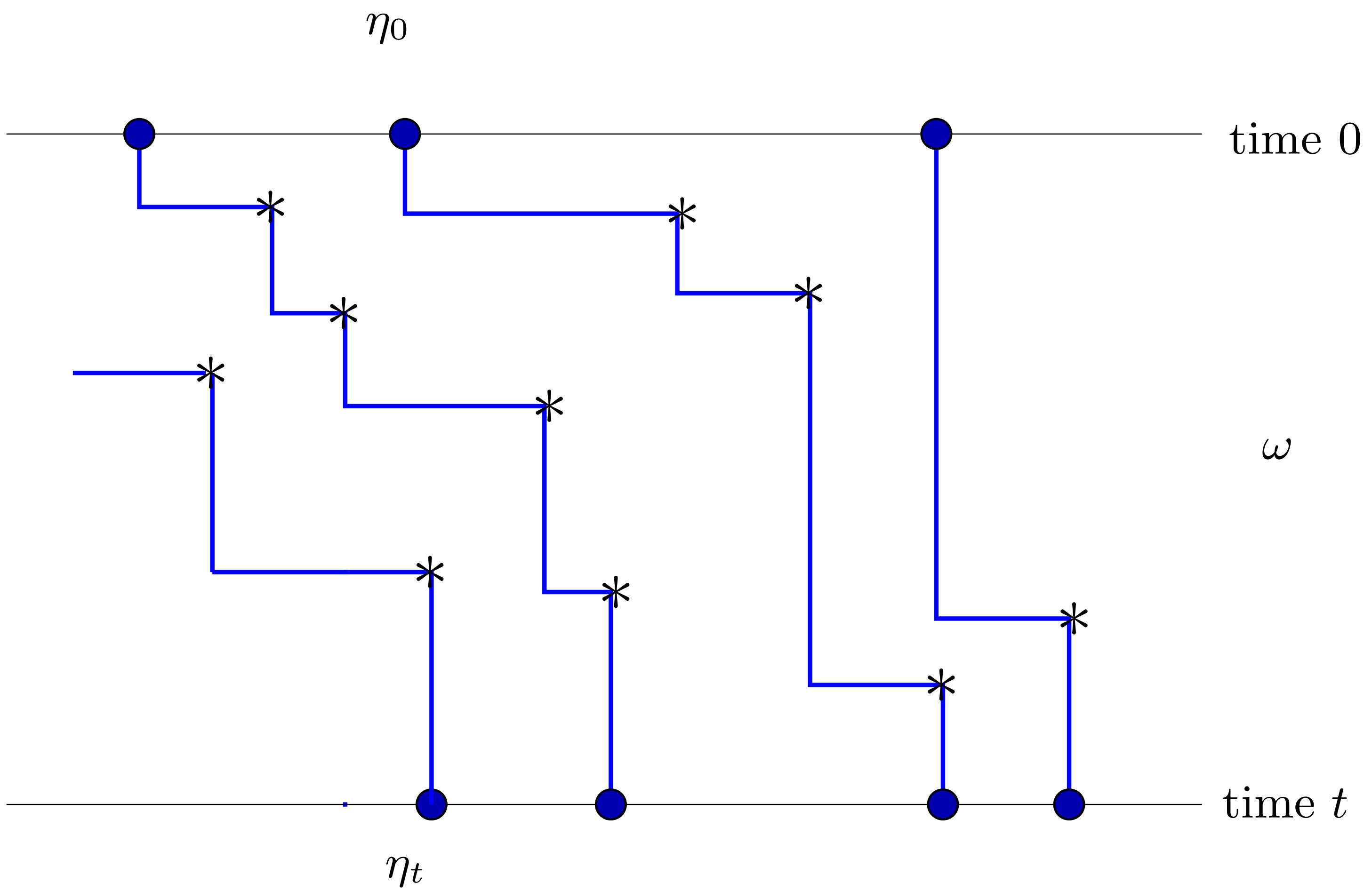


- Seppäläinen (1996): OK in state space

$$\mathcal{X} = \{\eta : \lim_{s\to\infty} |\eta \cap [0, s]|^2/s = \infty\}$$

- ω a realization of Poisson points

$$(t, \omega, \eta_0) \mapsto \phi(t, \omega, \eta_0) \tag{3}$$

$(\eta_t, t \geq 0)$ *governed by* ω *with initial condition* η_0.

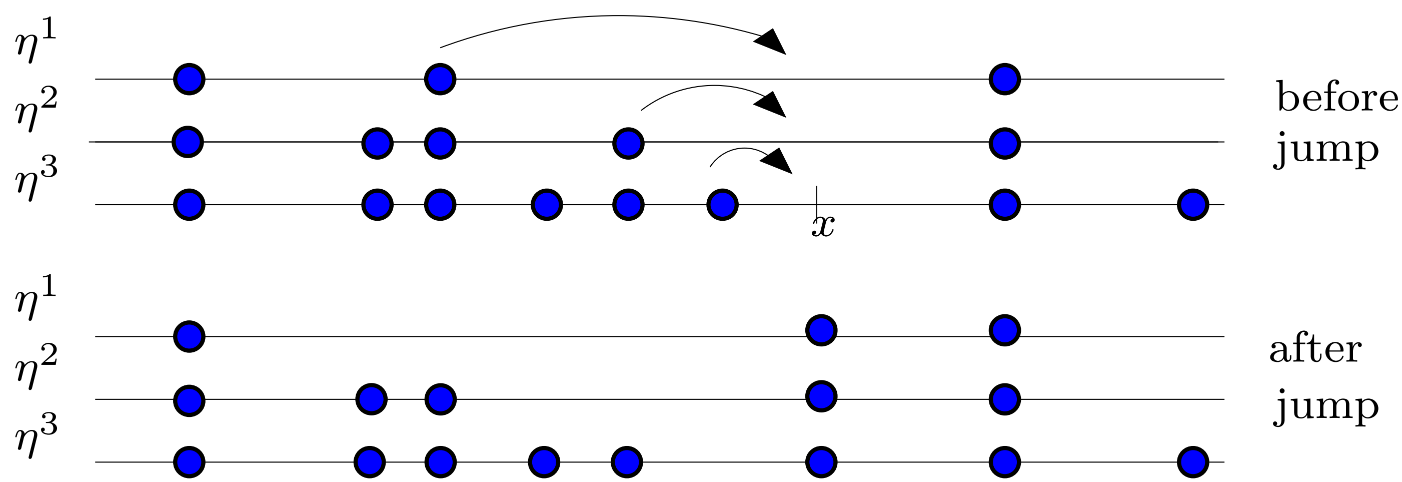
$$\eta_t = \phi(t - s, \tau_s\omega, \eta_s)$$

for all $0 \leq s < t < \infty$. (τ_s time translation by s)

- Extremal invariant measures: 1-d Poisson processes ν^ρ.

Coupling Initial configurations $\eta_0^1 \subset \ldots \subset \eta_0^n$ with same bells ω:

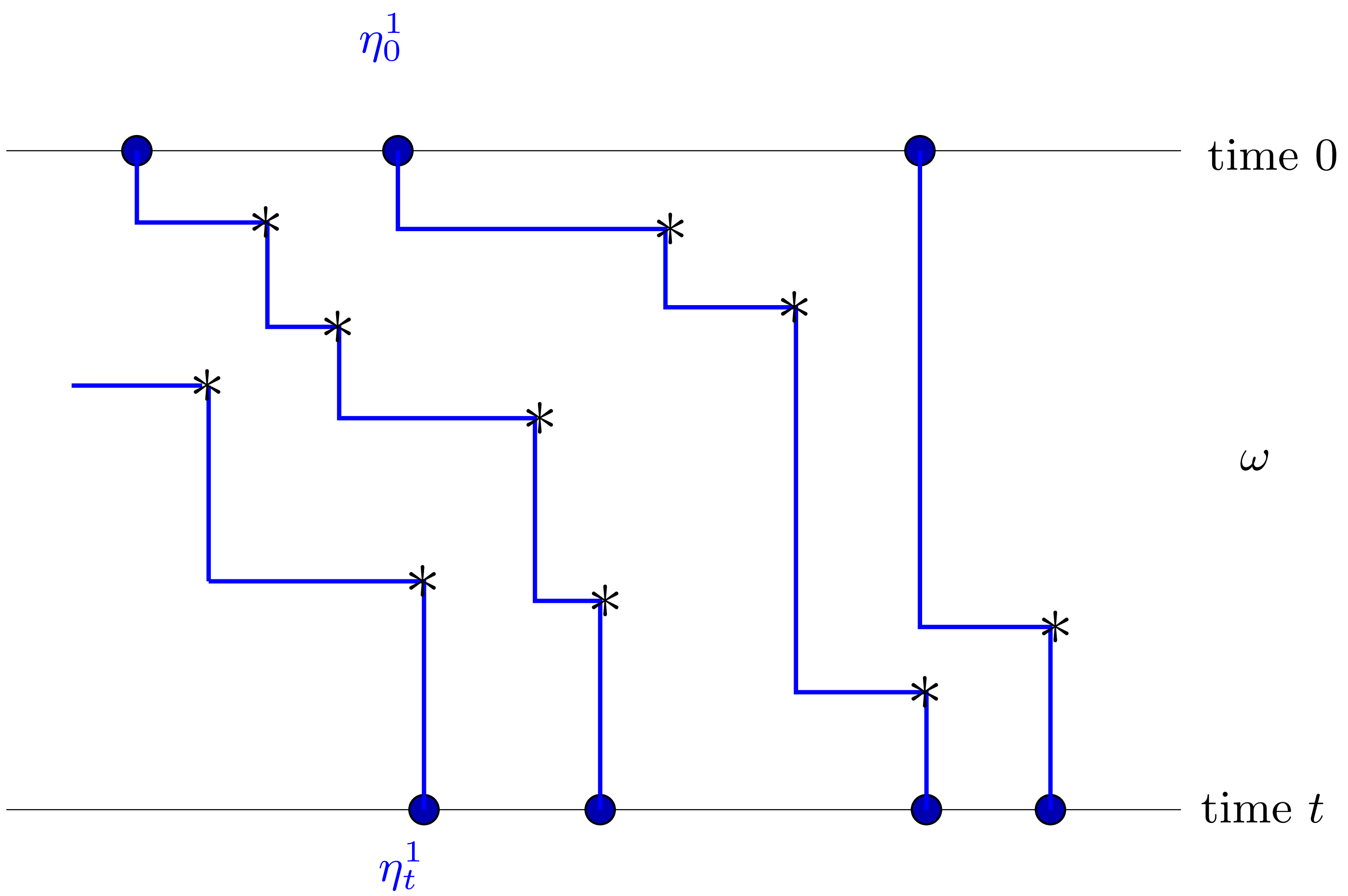
$$\eta_t^k = \phi(t, \omega, \eta_0^k), \qquad k = 1, \ldots, n$$



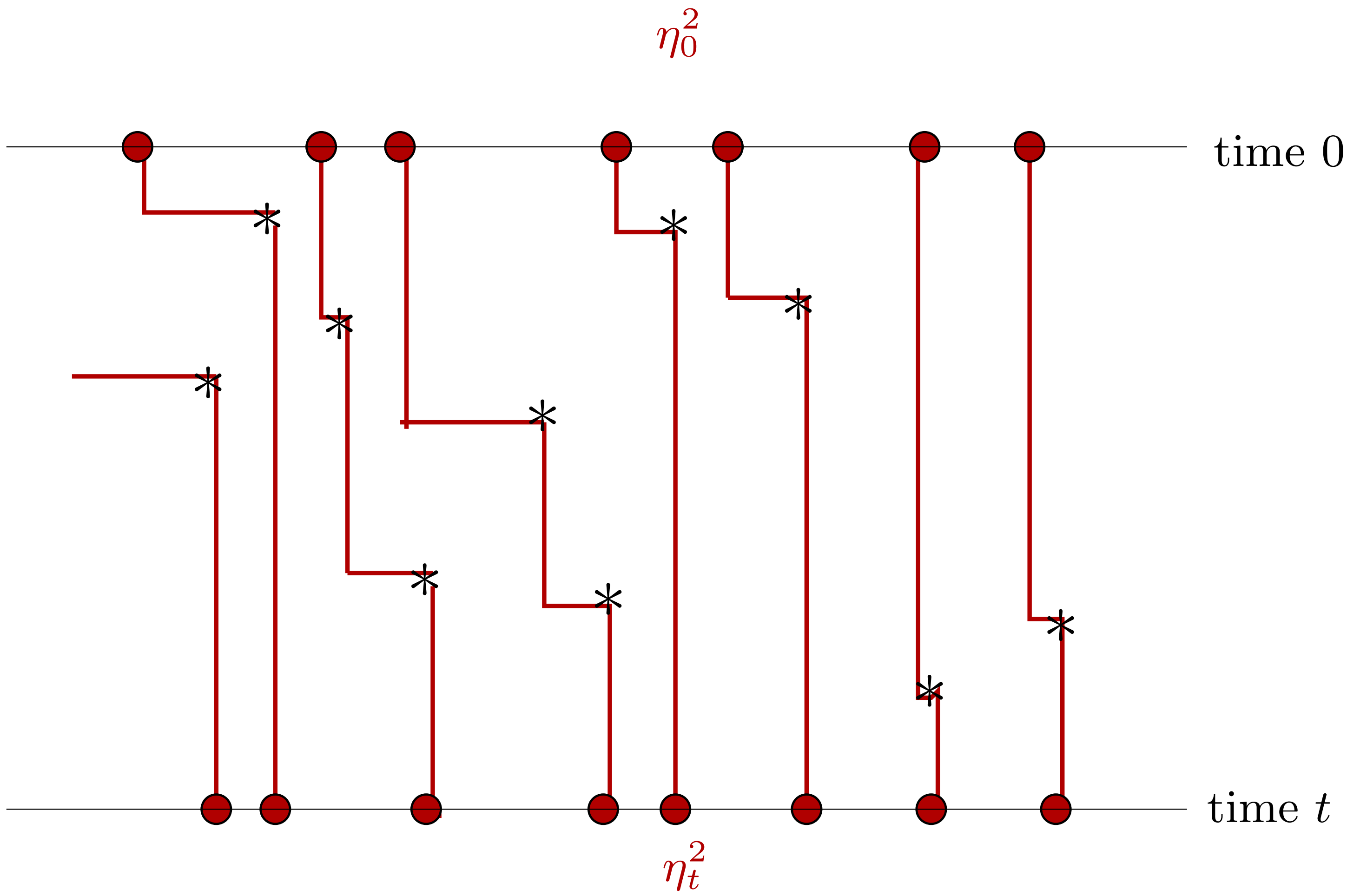
Generator of coupled process:

$$L_C f(\eta) = \int_{\mathbb{R}} [f(J(\eta, x)) - f(\eta)] \tag{4}$$

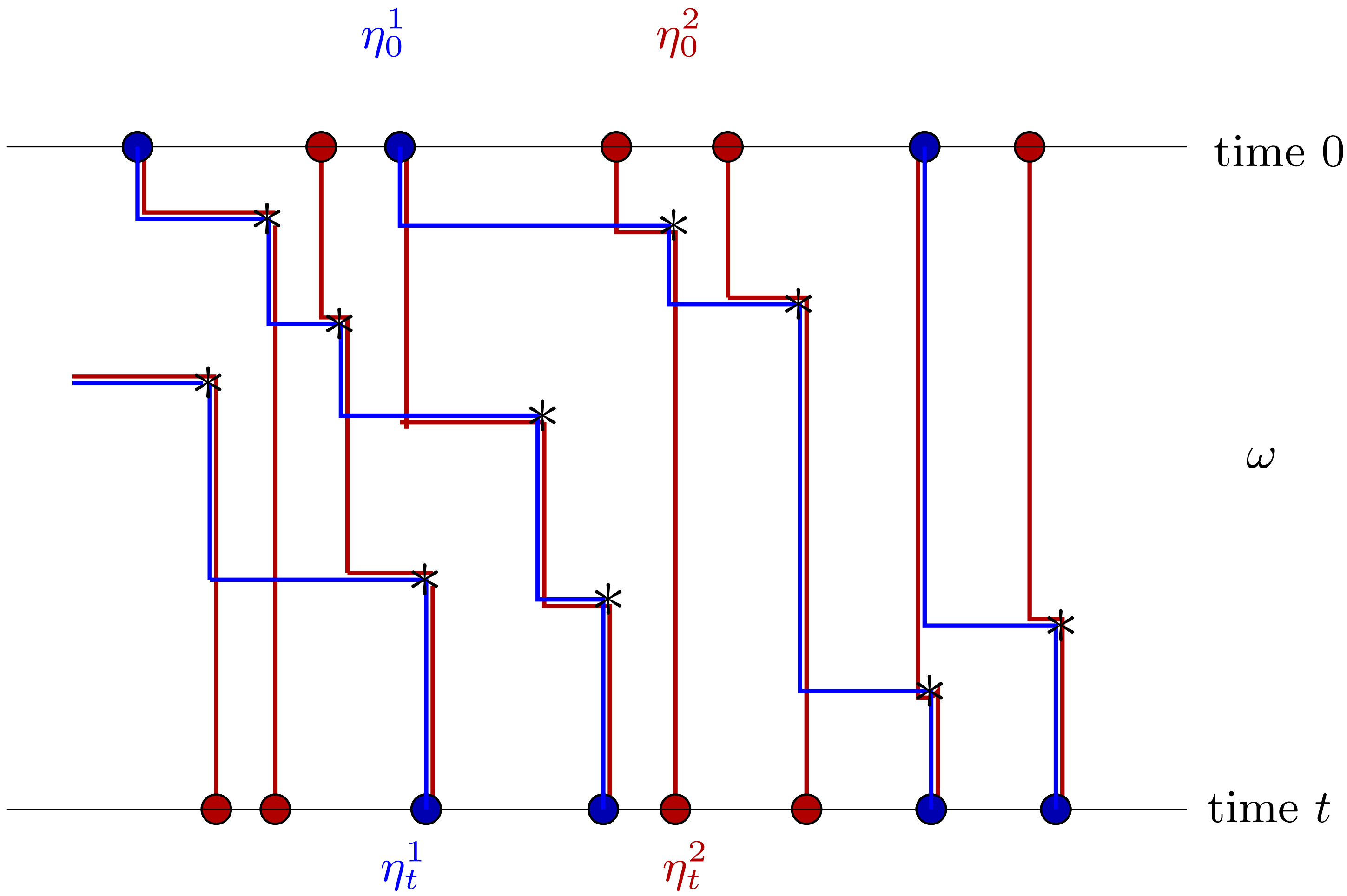
where $J(\eta, x) = (J(\eta^1, x), \ldots, J(\eta^n, x))$.



η_0^1
time 0
ω
time t
η_t^1

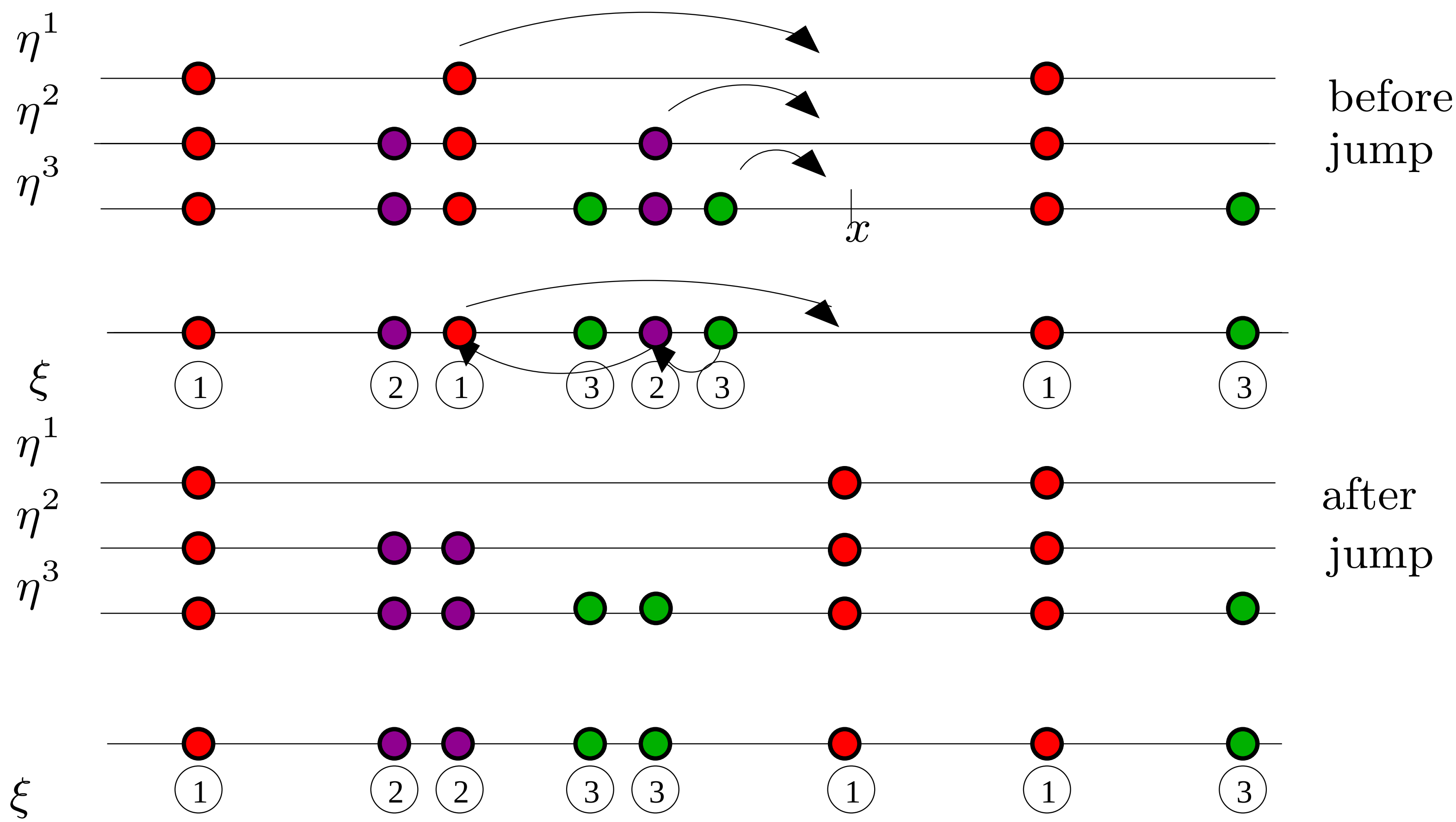


η_0^2
time 0
time t
η_t^2



η_0^1
η_0^2
time 0
ω
time t
η_t^1
η_t^2

Multiclass process $\xi^k = \eta^k \setminus \eta^{k-1}$. Map $\xi = R\eta$.



Coupled and multiclass processes. Effect of a bell at x

The *multiclass process* $\xi_t := R\eta_t$.

Problem:

Find invariant measure for the coupled process
Find invariant measure for the multiclass process

- Marginals must be Poisson.
- If densities are ordered, marginals must be ordered.

Plan:

Construct a measure and show it is invariant for the processes.

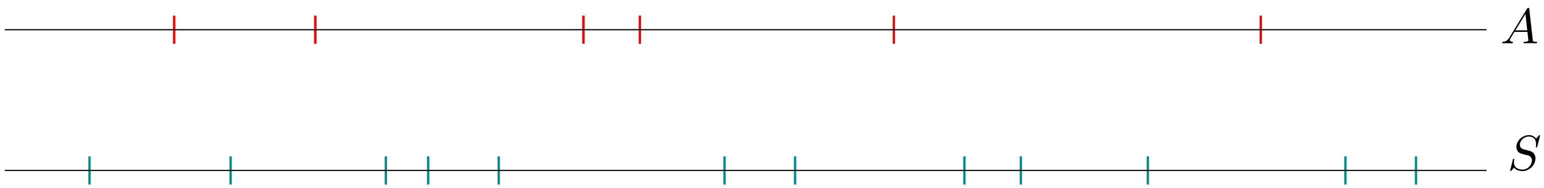
Related to $M/M/1$ queue.

On the way we find "multiline process" and "dual points".

Asymmetric random walk

A and S are Poisson processes of parameters $\lambda_1 < \lambda_2$ in $\mathbb{R}$

$A(t) - A(s)$ number of points in $[s, t]$.



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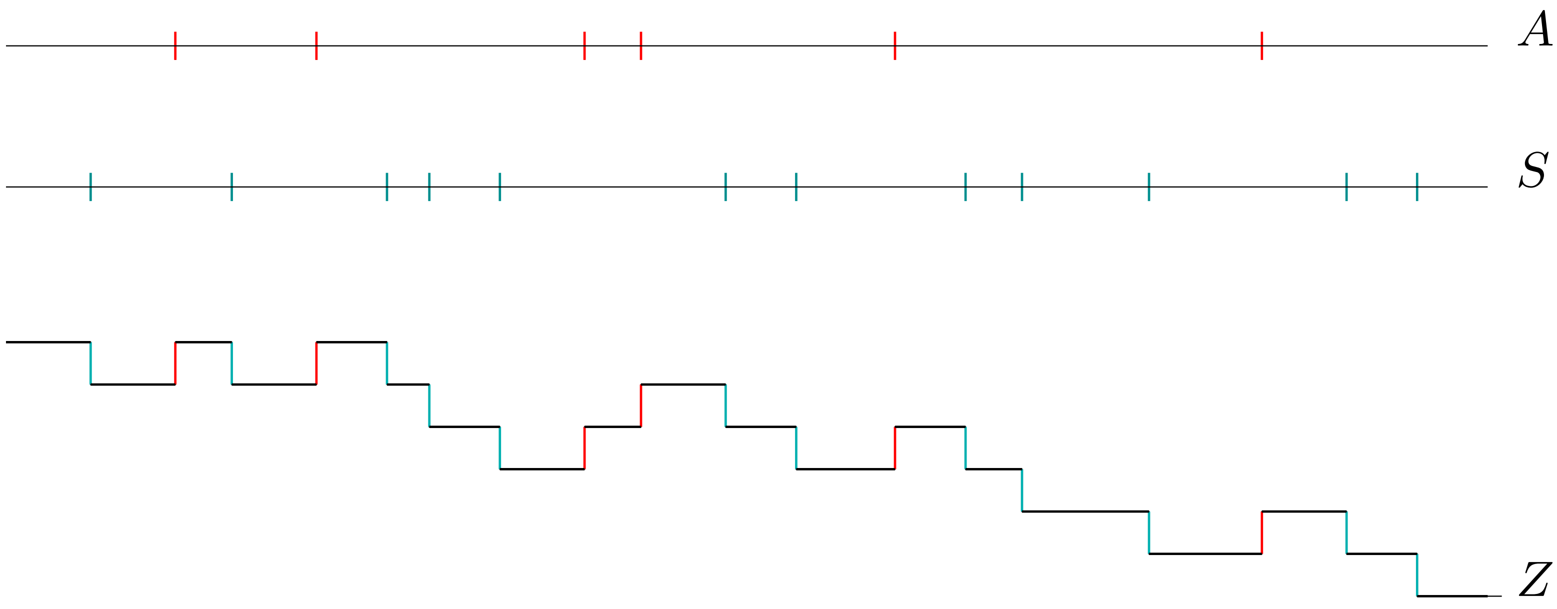
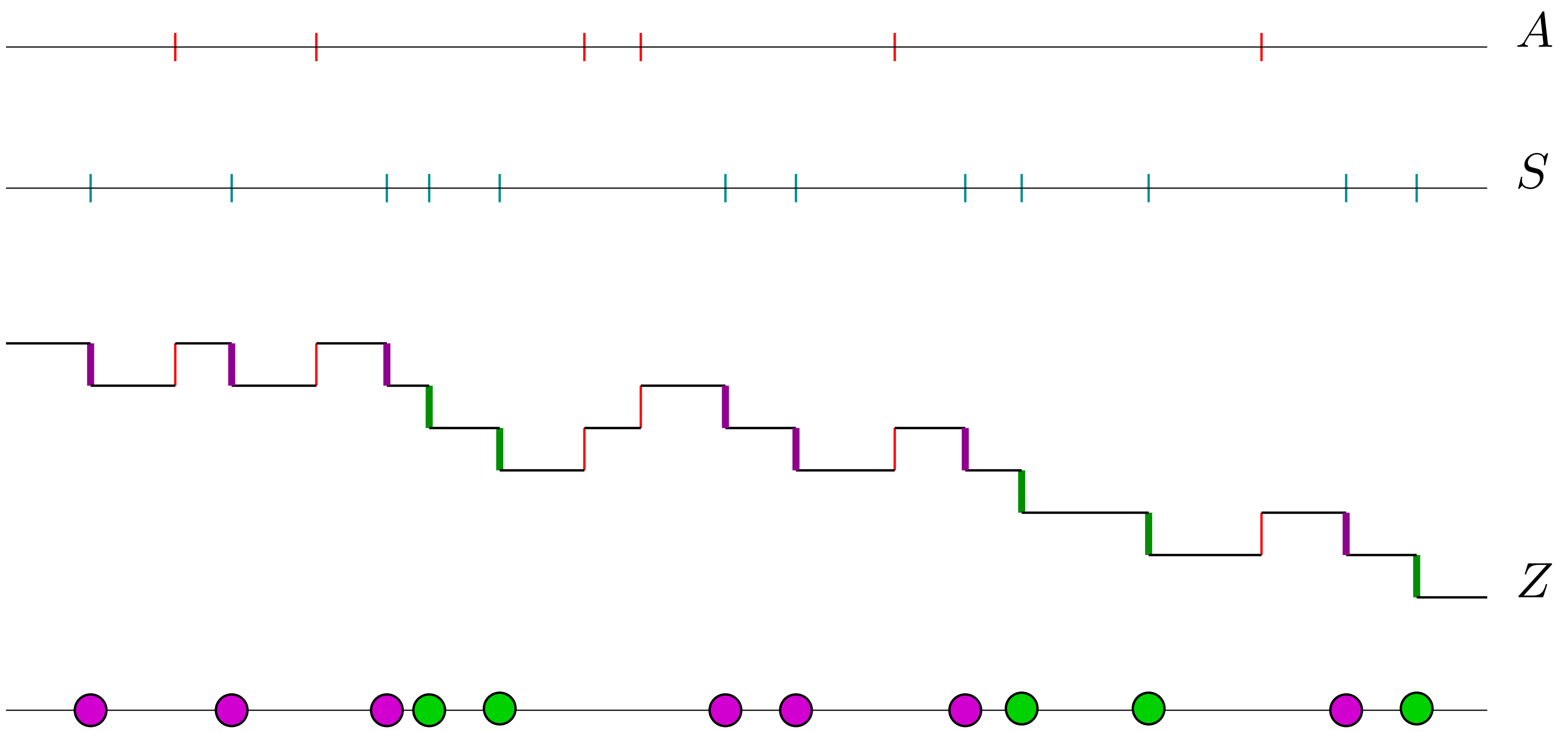


$Z(t) - Z(s) := A(t) - A(s) - (S(t) - S(s))$

Asymmetric continuous time random walk; stationary increments.

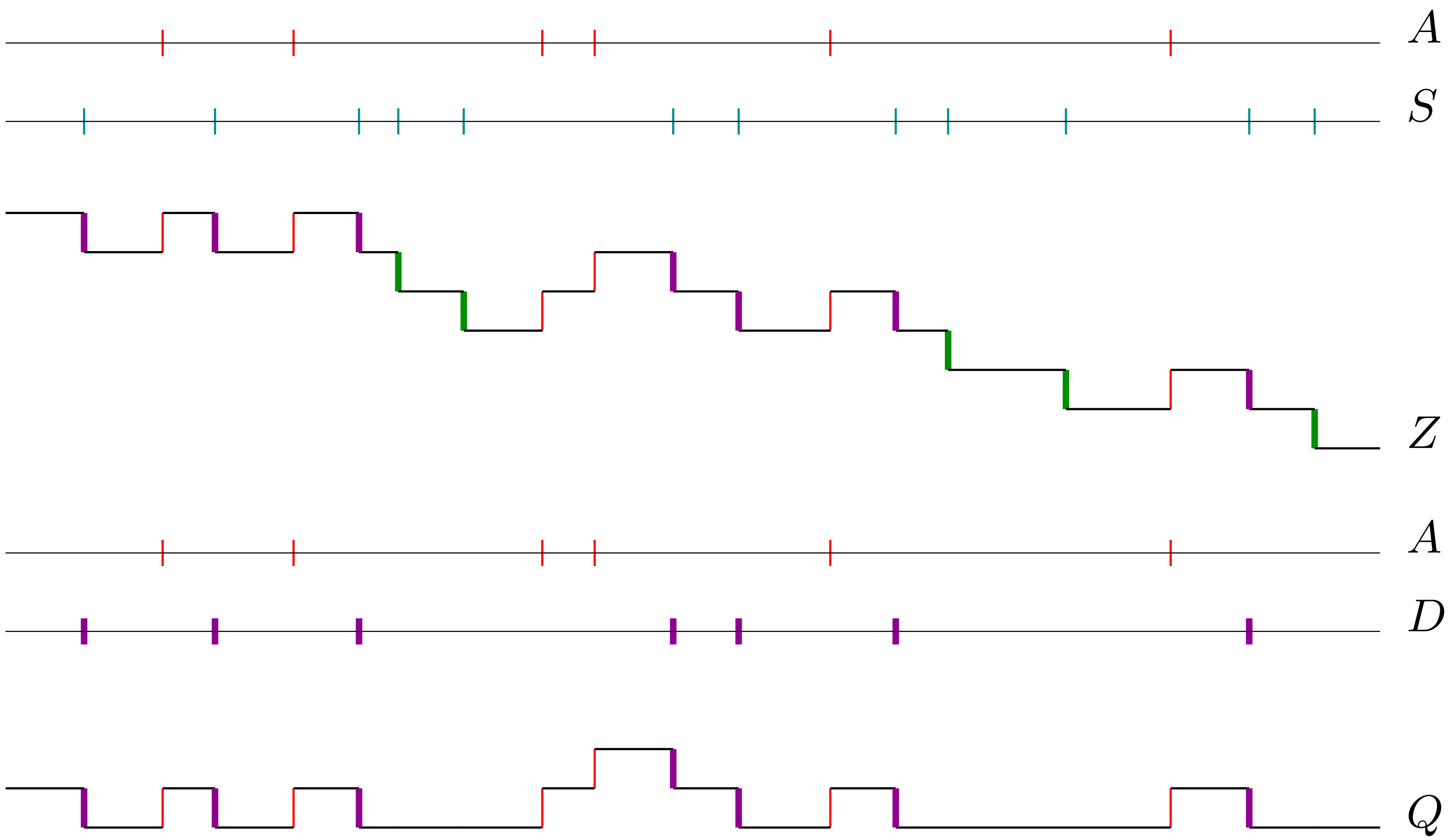
Down records



Records down: $U = U(A, S)$ (green)

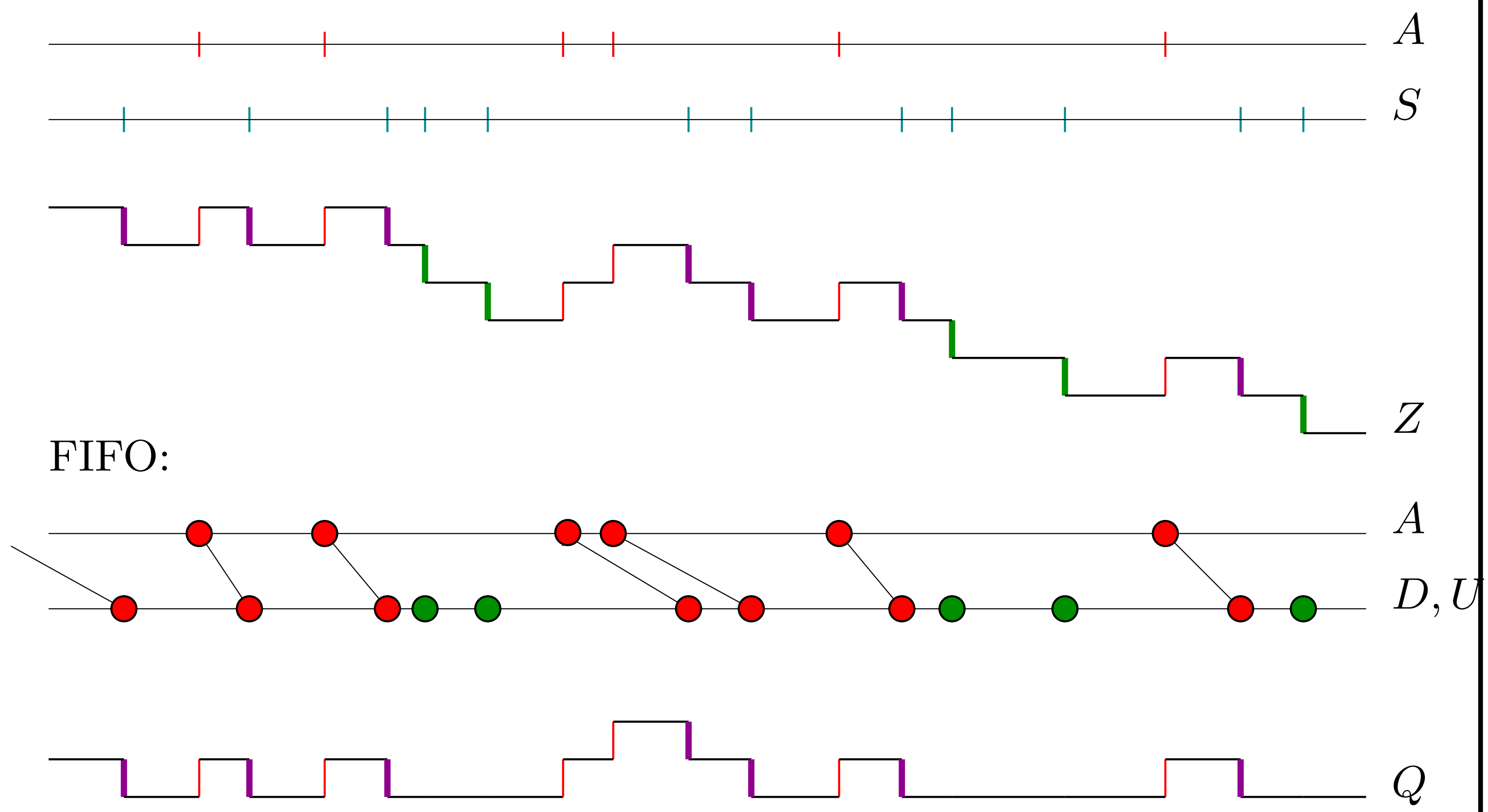
$D = D(A, S) := S \setminus U$ (violet)

Stationary $M/M/1$ queue



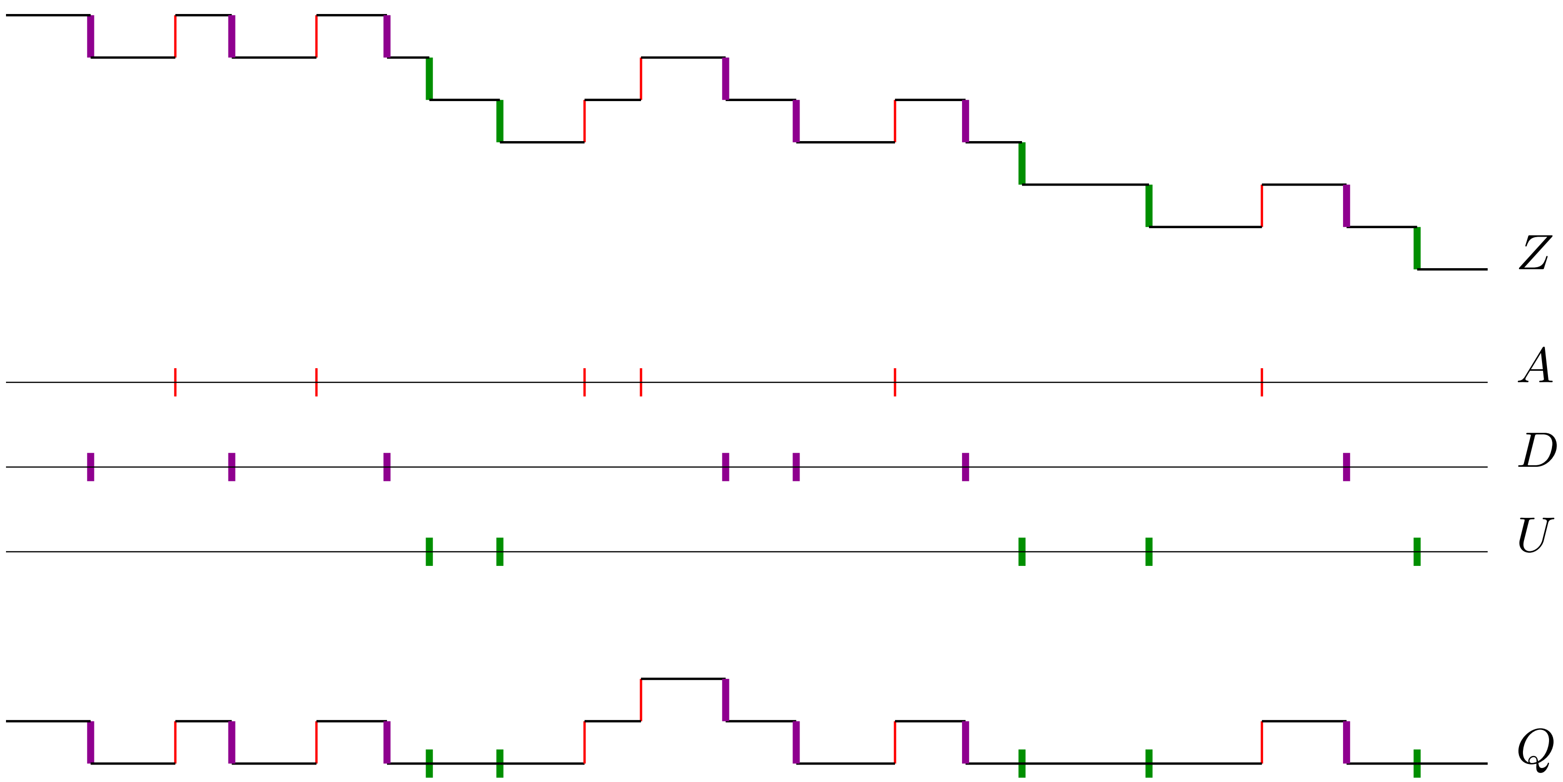
$M/M/1$ queue: $Q(t) - Q(s) = A(t) - A(s) - (D(t) - D(s))$

Stationary $M/M/1$ queue



$M/M/1$ queue: $Q(t) - Q(s) = A(t) - A(s) - (D(t) - D(s))$

Departures and Unused services

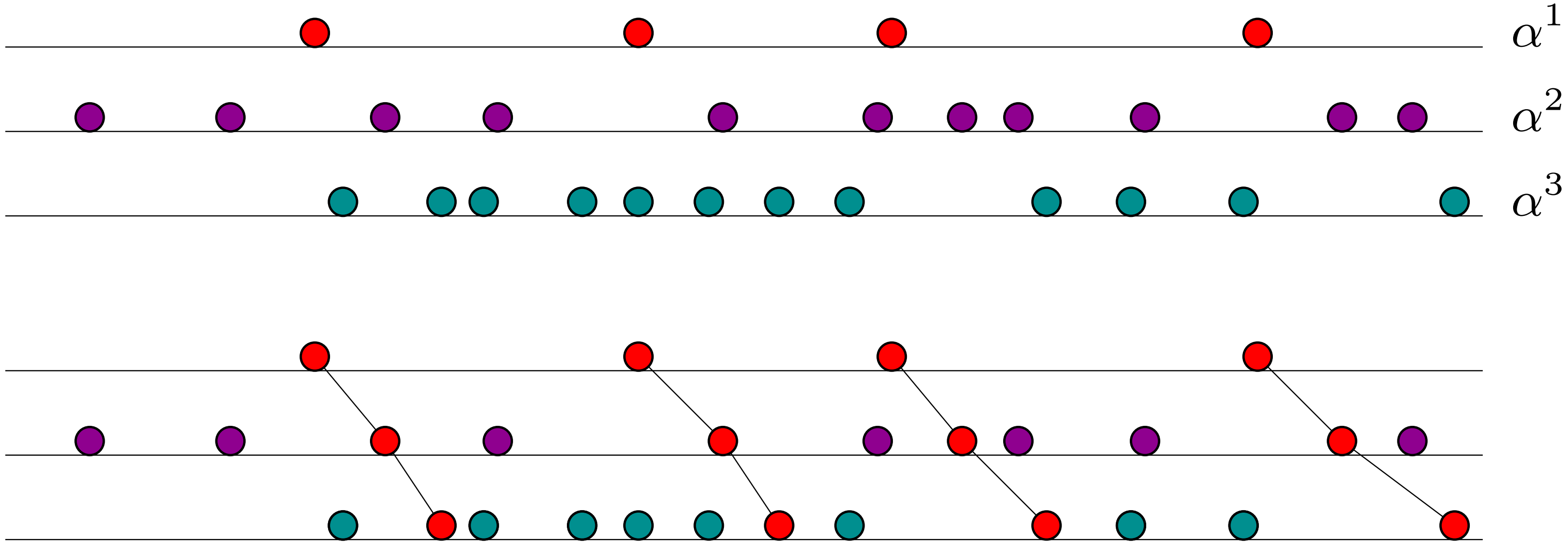


Departures: $D = D(A, S)$. Unused services: $U = U(A, S)$.

$D \cup U = S$; D is also Poisson(λ_1) (Burke)

Multiclass invariant measures

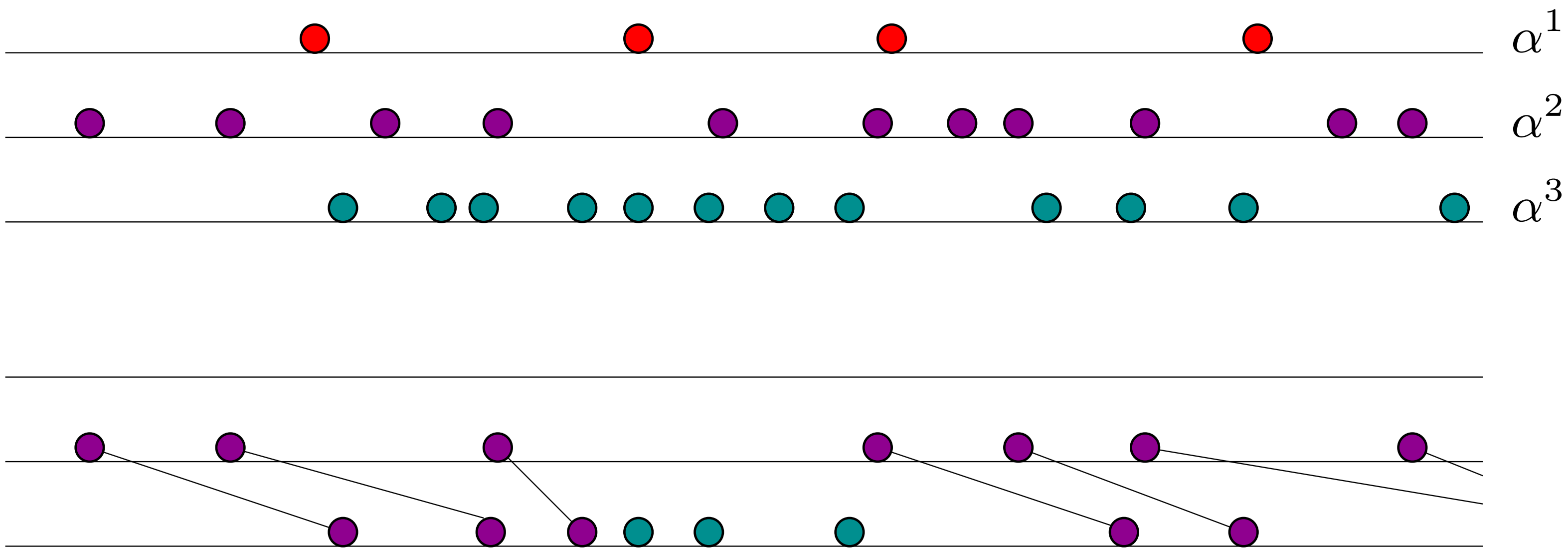
$(\alpha^1, \ldots, \alpha^n)$ iid Poisson, $\rho^1 < \ldots < \rho^n$.



$\xi^1 = D(D(\alpha^1, \alpha^2), \alpha^3)$

Multiclass invariant measures

$(\alpha^1, \ldots, \alpha^n)$ iid Poisson, $\rho^1 < \ldots < \rho^n$

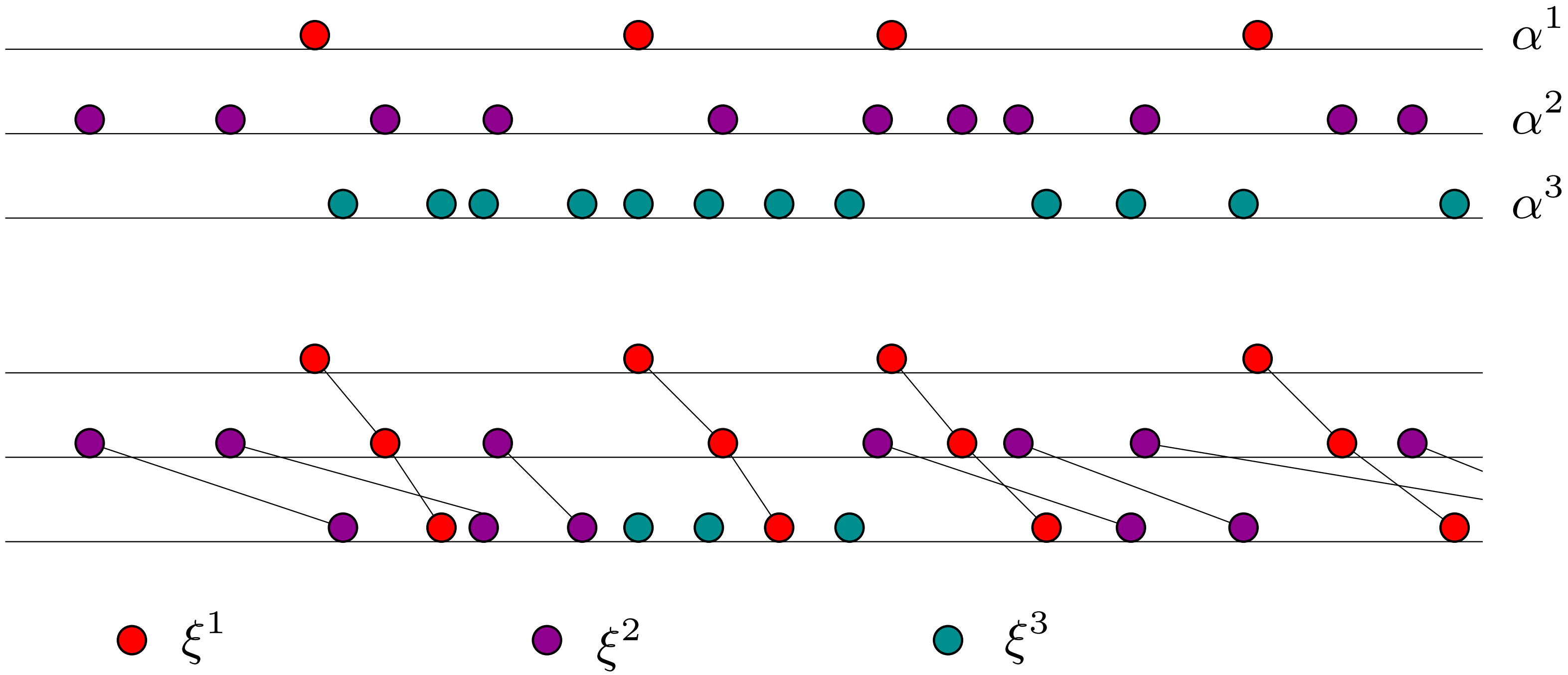


$\xi^2 = D(U(\alpha^1, \alpha^2), \alpha^3 \setminus \xi^1)$

$\xi^3 = U(U(\alpha^1, \alpha^2), \alpha^3 \setminus \xi^1)$

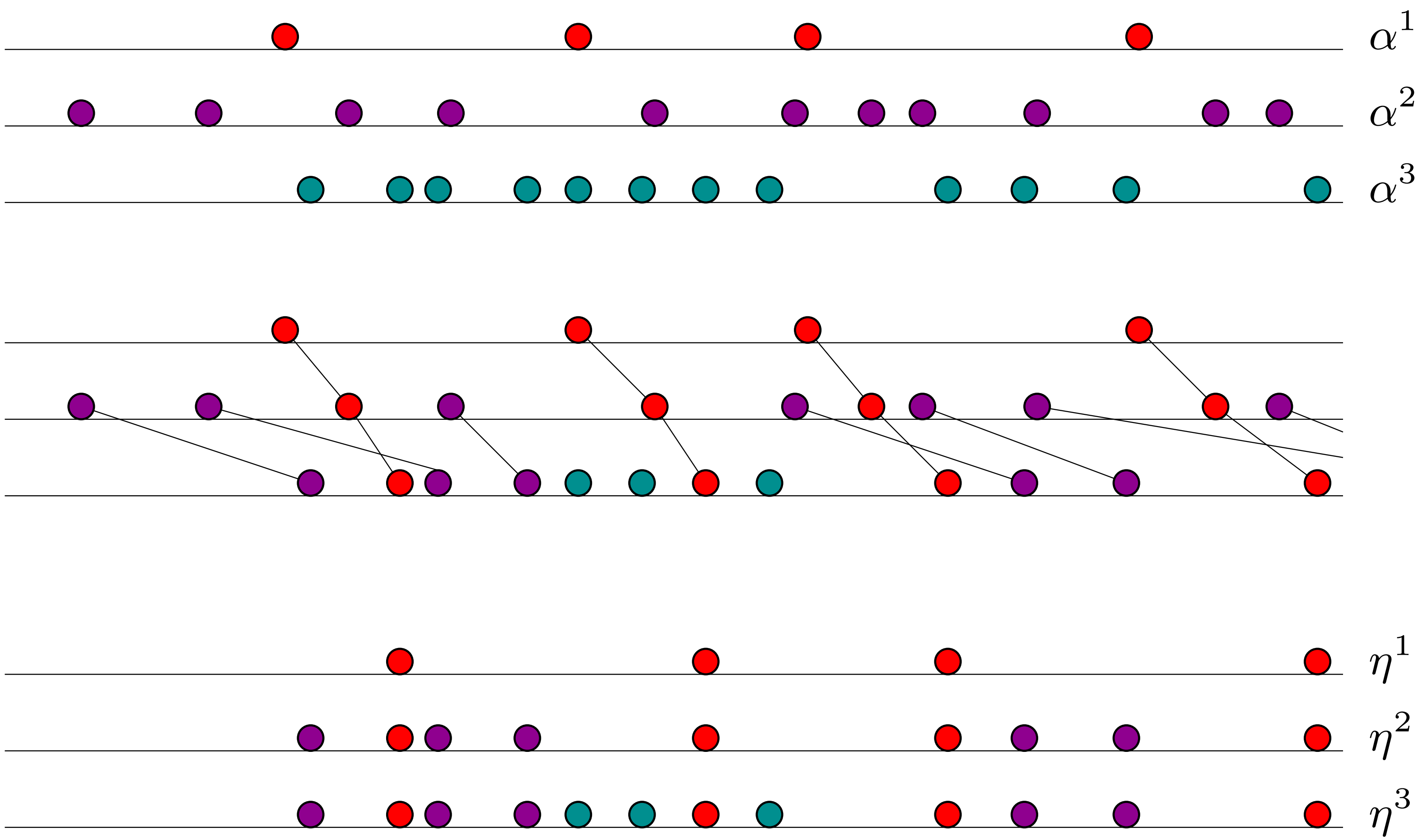
Multiclass invariant measures

$(\alpha^1, \ldots, \alpha^n)$ iid Poisson, $\rho^1 < \ldots < \rho^n$



- Map $M(\alpha) = \xi$
- $\xi^1 \cup \xi^2 \cup \xi^3 = \alpha^3$.

Coupled measure



Coupled measure

- $\eta^k := \xi^1 \cup \ldots \cup \xi^k$.
- $\eta^1 \subset \ldots \subset \eta^n$.
- for each k, η^k has marginal distribution ν^{ρ^k} (Burke).
- Map $T\alpha = \eta$.
- Define $\pi = T\nu$ the induced distribution of η.

The case $n = 2$

- Computed by Derrida, Janowsky, Lebowitz and Speer (1993).
- F. Fontes Kohayakawa (1994), Speer (1994), Duchi and Schaeffer (2005).
- Angel (2005) for the invariant measure of the two-class TASEP.
- Queueing interpretation F. and Martin (2005).

Invariance of μ

Theorem 1

Let $\alpha = (\alpha^1, \ldots, \alpha^n)$ have law ν, product of Poisson with densities $\rho^1 < \ldots < \rho^n$. Then

- *π, law of $T\alpha$, is invariant for coupled HAD process (η_t)*
- *μ, law of $M\alpha$, is invariant for multiclass HAD process (ξ_t).*

Sketch of proof The statements are equivalent ($M = RT$).

New dynamics $\alpha_t = (\alpha_t^1, \ldots, \alpha_t^n)$ *multi-line process*, and then show:

1) The product measure ν is invariant for the multi-line process α_t.

2) $T\alpha_t$ is the coupled process η_t. $\square$

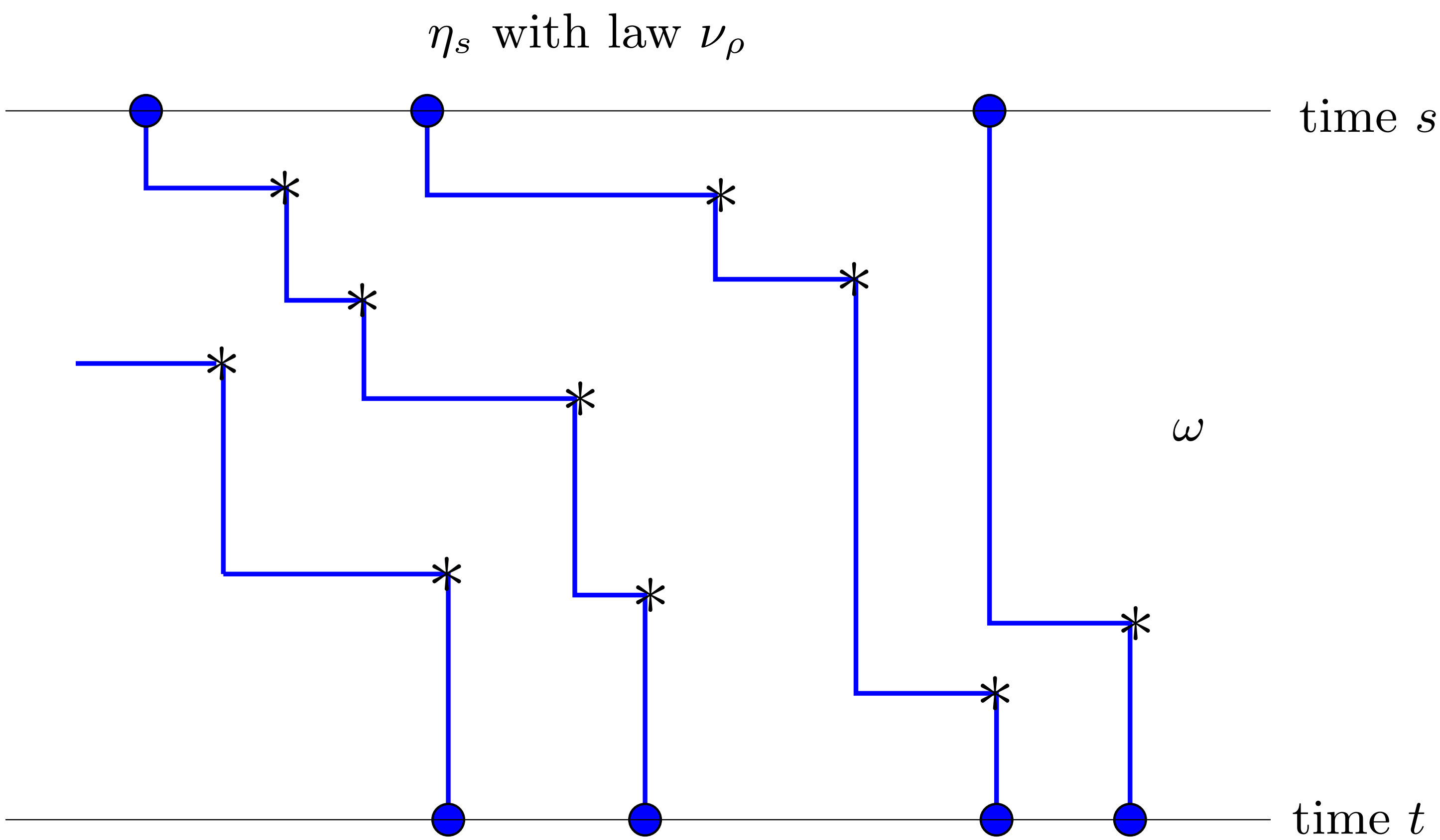
Dual points

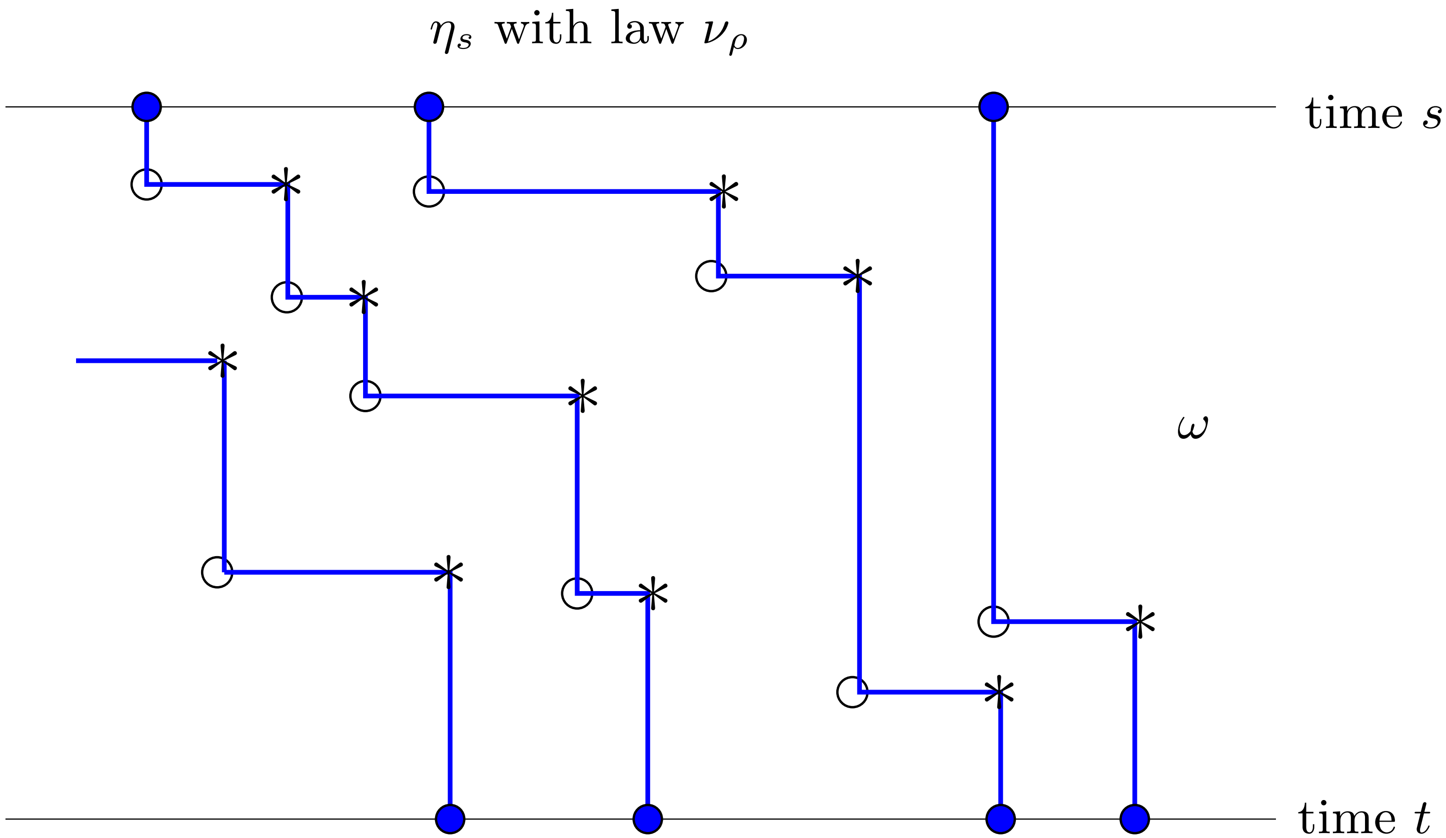
Stationary HAD trajectory $(\eta_t,\, t \in \mathbb{R})$ governed by ω with time-marginal ν^ρ.

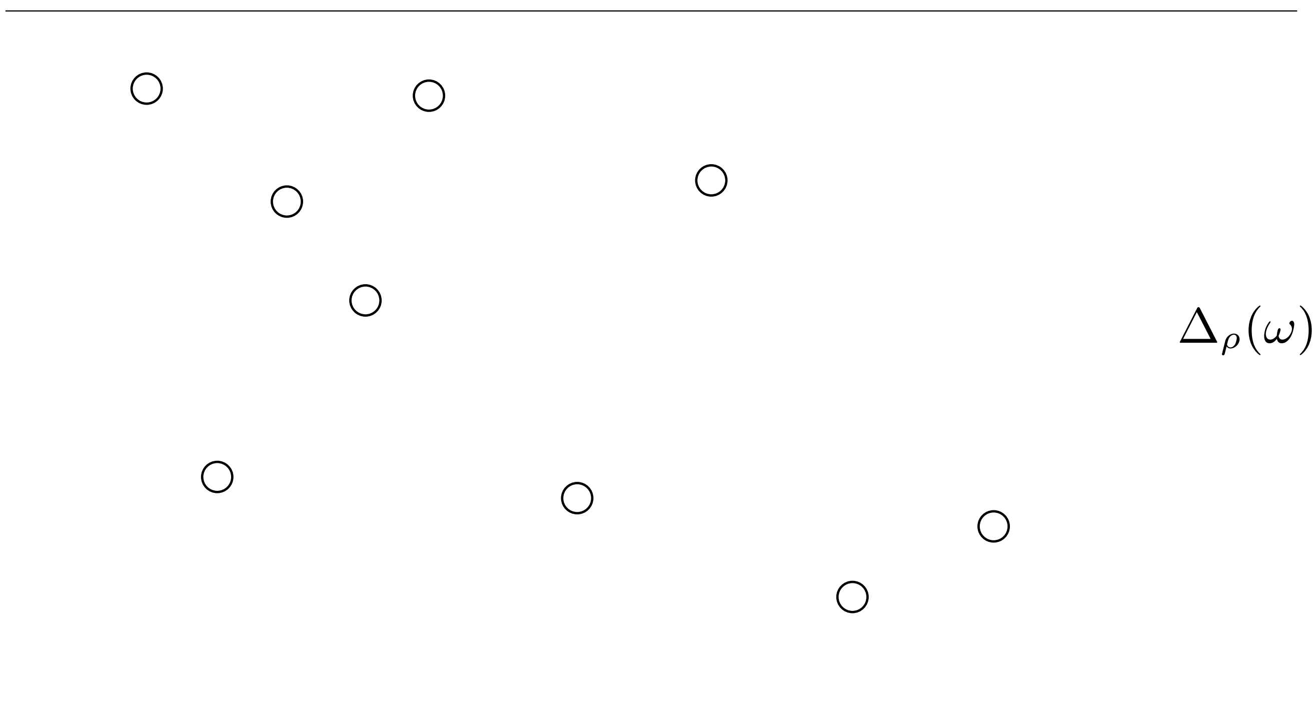
$\Delta_\rho(\omega)$, *dual points*, positions of the particles just before jumps:

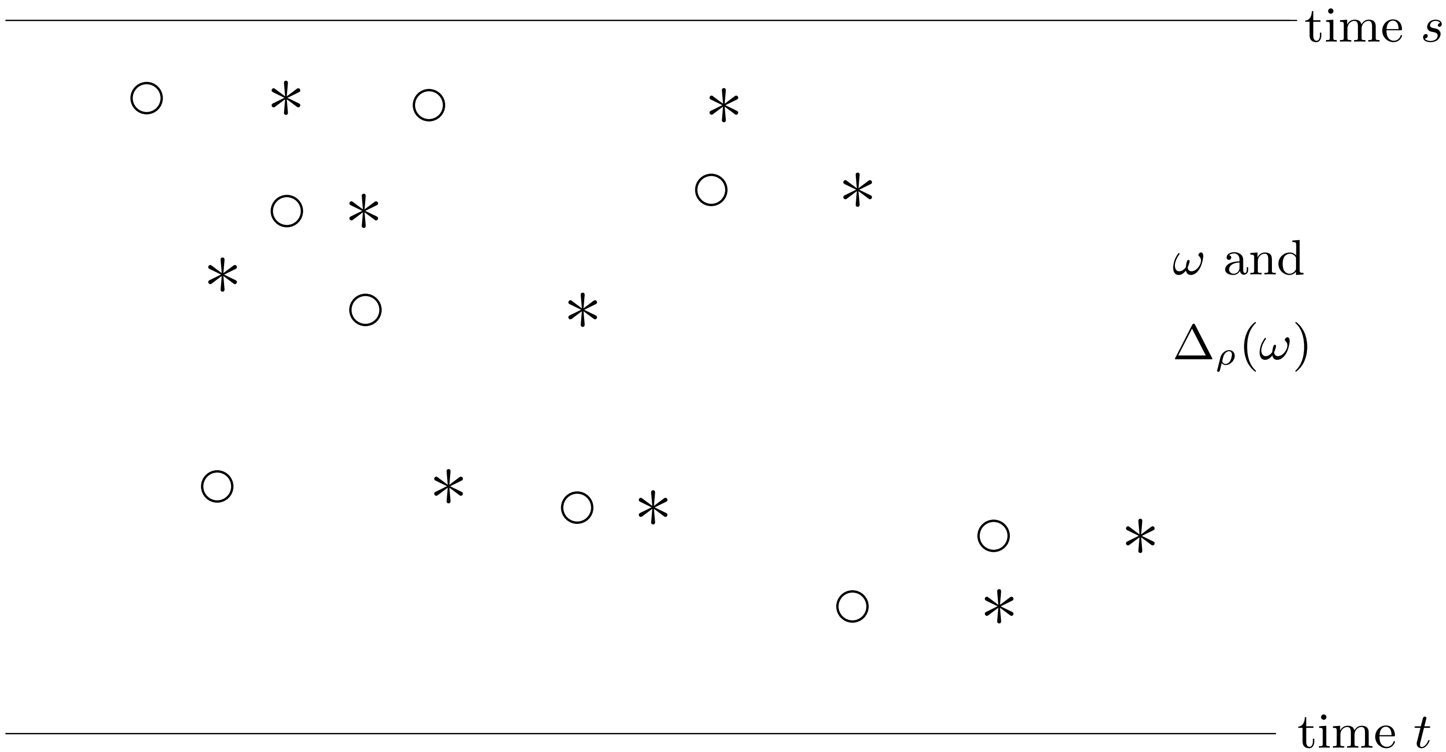
$$\Delta_\rho(\omega) := \{(y(\eta_{t-}, x), t) :\; (x, t) \in \omega\} \tag{5}$$

where $y(\eta, x)$ is closest η particle to the left of x


η_s with law ν_ρ
time s
ω
time t


η_s with law ν_ρ
time s
ω
time t


$\Delta_\rho(\omega)$



time s
ω and
$\Delta_\rho(\omega)$
time t

Important properties of dual points

• Dual points govern the time-reversal of the trajectory.

• Time reversed trajectory is Hammersley with jumps to left.

• Hence the law of the dual points $\Delta_\rho(\omega)$ is also Poisson

• Furthermore $\{(x, s) \in \Delta_\rho(\omega) : s < t\}$, the set of dual points in the past of t, is independent of the configuration η_t.

Proof: Along Reich proof of Burke, Cator and Groeneboom (2005).

Multi-line HAD process

$\alpha_t = (\alpha_t^1, \ldots, \alpha_t^n)$ in $\mathcal{X}^n$ governed by ω.

Let $\rho^1 < \ldots < \rho^n$.

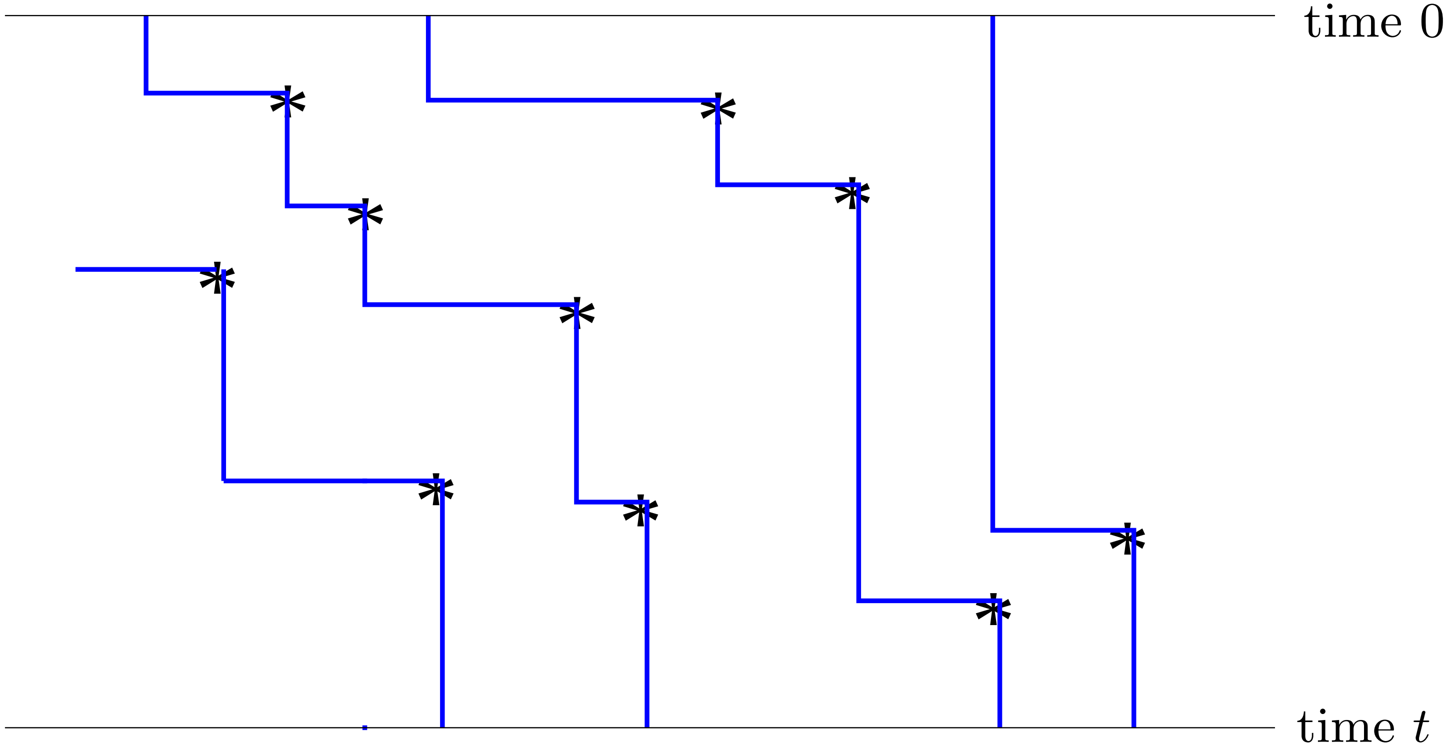
Let $\omega^n = \omega$, and

$$\omega^k = \Delta_{\rho^{k+1}}(\omega^{k+1})$$

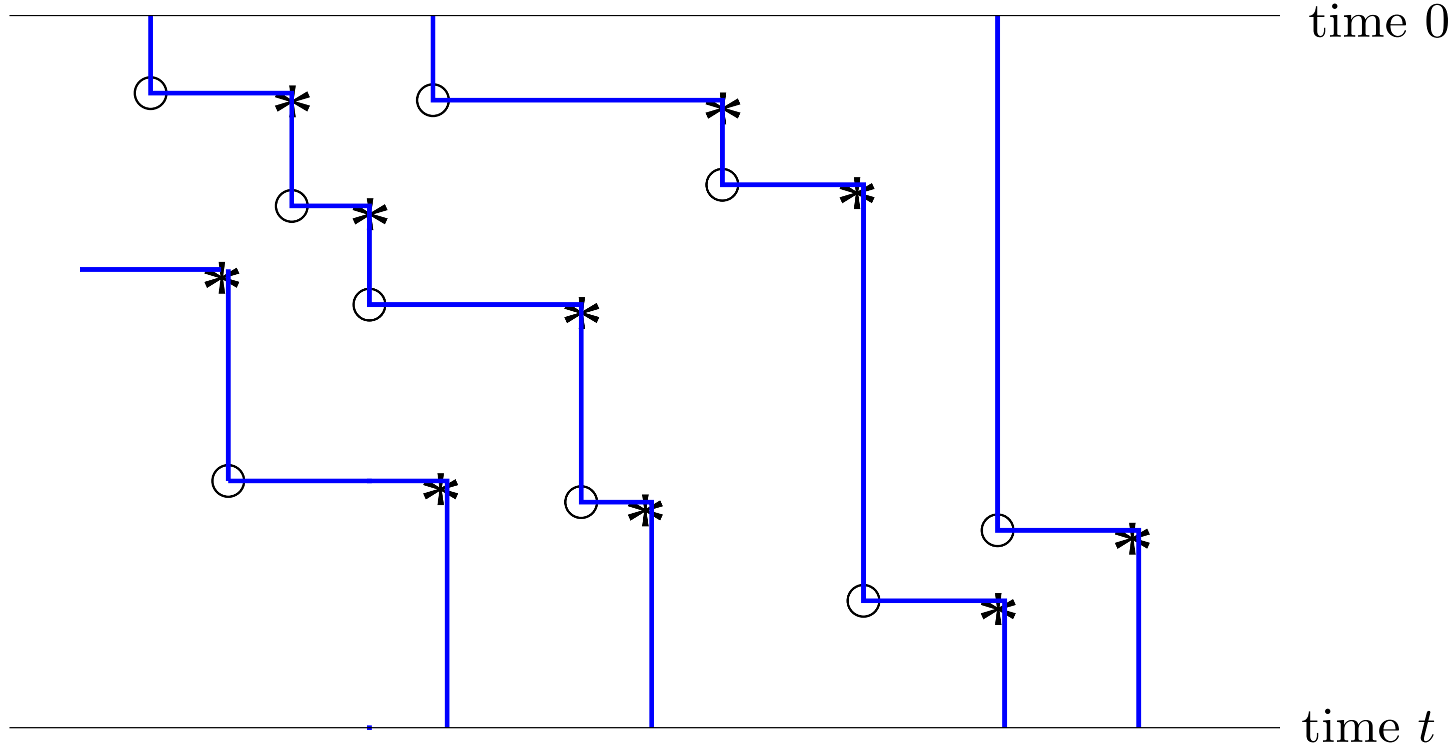
is Poisson on $\mathbb{R}^2$.

The "kth line" of $(\alpha_t^k, t \in \mathbb{R})$ is HAD trajectory with density ρ^k governed by ω^k

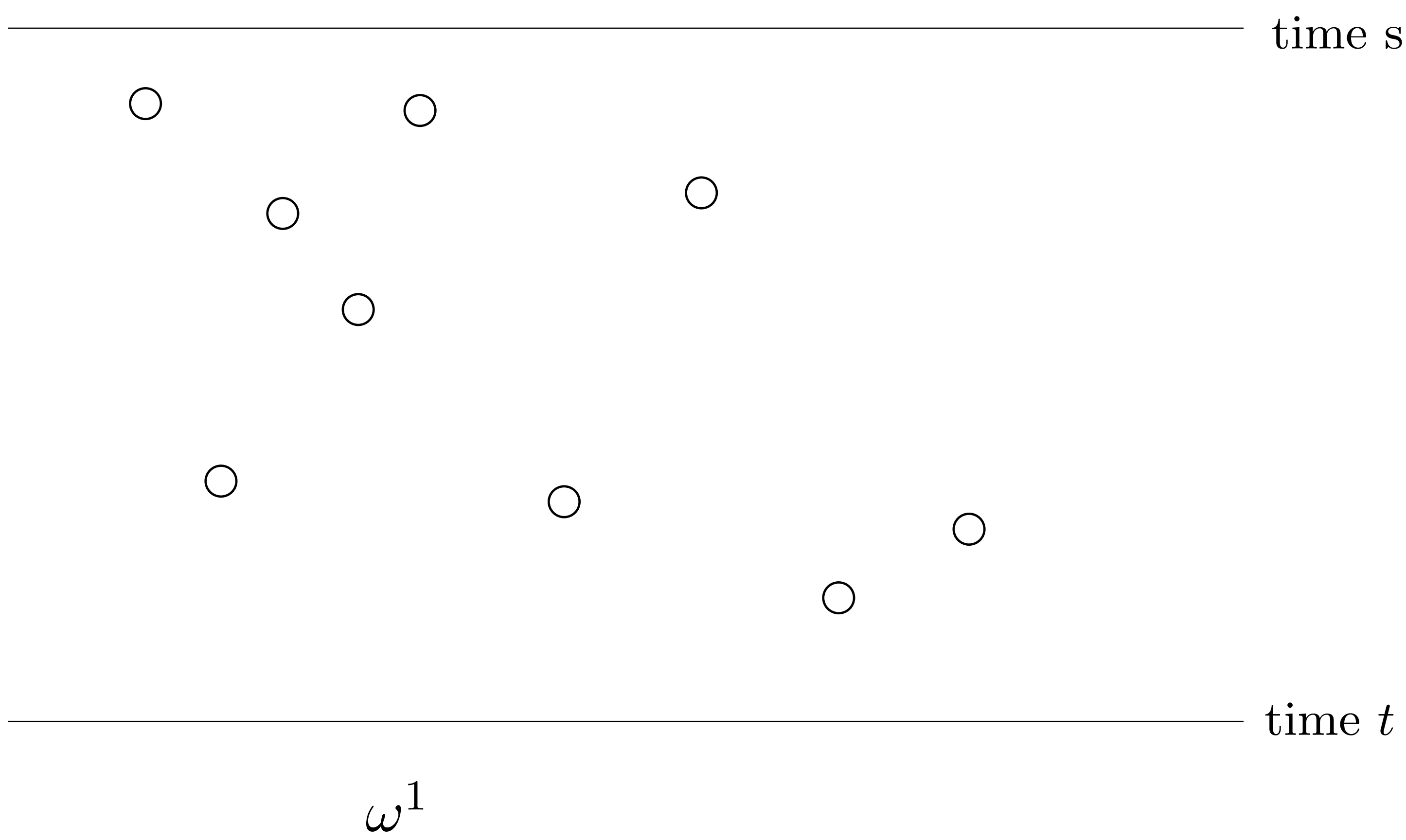
Each line is a HAD trajectory governed by the dual points produced from the line below.

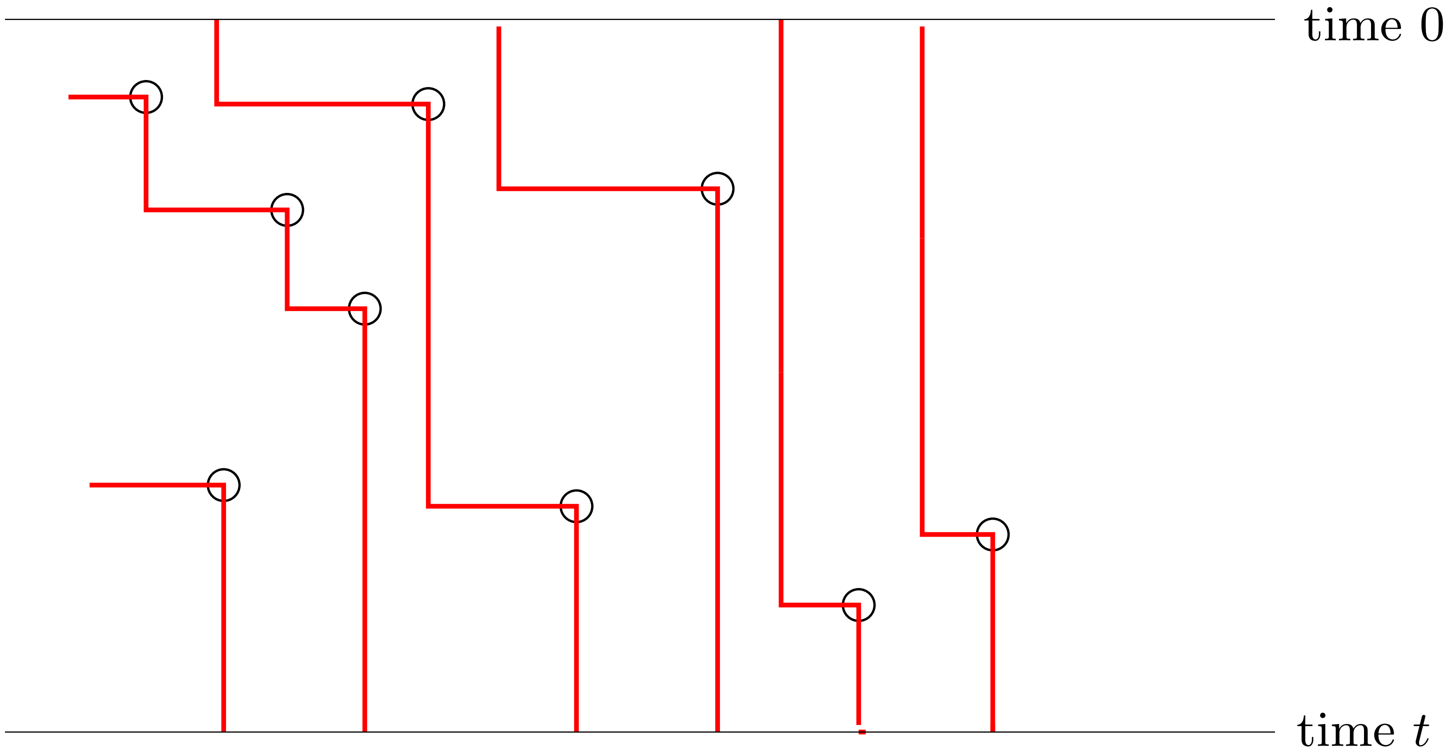


ω and α_t^1



ω, α_t^1 and ω^1


time s
time t
ω^1



ω^1 and α_t^1

Product of Poisson is invariant for multiline process

Proposition 2 *The multi-line HAD process* $(\alpha_t, t \in \mathbb{R})$ *is stationary, and the distribution of* α_t *for each* t *is the product measure* $\nu = \nu^{\rho^1} \times \ldots \times \nu^{\rho^n}$.

Proof By construction, process is stationary and the marginal distribution of α_t^k is ν^{ρ^k} for any k and t.

Need to show $\alpha_t^1, \alpha_t^2, \ldots, \alpha_t^n$ are independent.

Let $2 \leq k \leq n$.

α_t^k is independent of the set of dual points (x, s) in $\Delta_{\rho^k}(\omega^k)$ such that $s < t$.

But $(\alpha_s^{k-1}, s \leq t)$ is a function of precisely this set of dual points.

Recursively, also the processes $(\alpha_s^j, s \leq t)$ for each $1 \leq j \leq i$. $\square$

Local description of multiline process and its time reversal

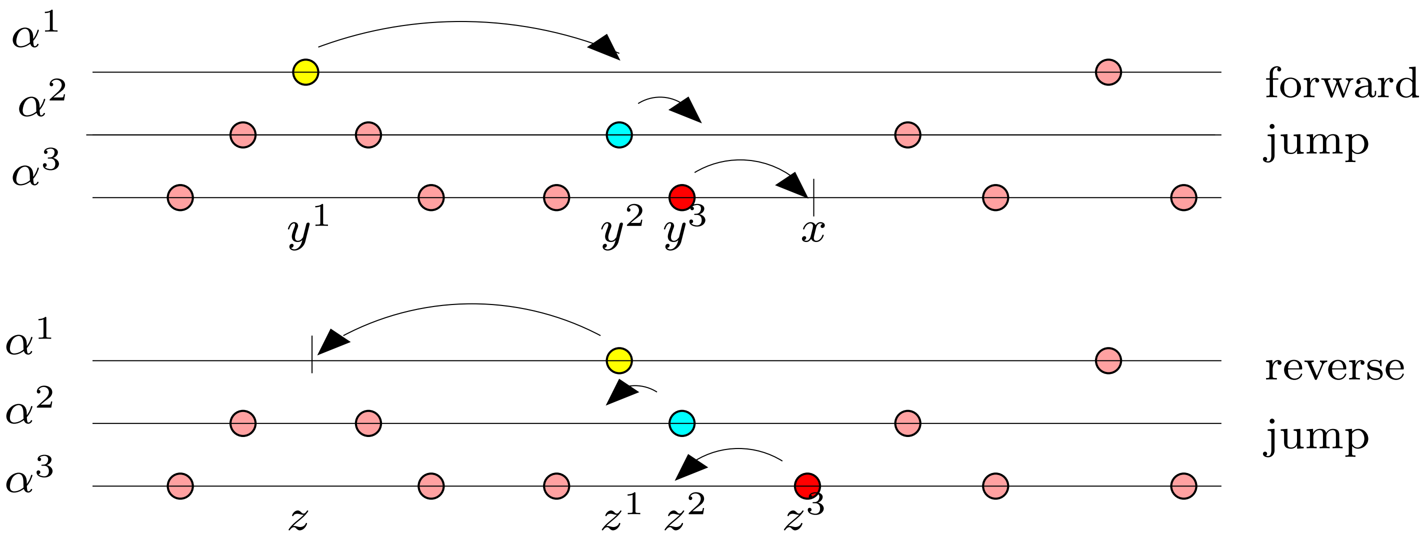


Image of multi-line process under map T is coupled process

Proposition 3

Let $0 < \rho^1 < \ldots < \rho^n < 1$*, and let* $(\alpha_t, t \in \mathbb{R})$ *multiline HAD governed by* ω

Let $\eta_t = T\alpha_t \in \mathcal{X}^{n\uparrow}$*.*

Then $(\eta_t^k, t \in \mathbb{R})$ *is the HAD trajectory governed by* ω*.*

Sketch of Proof From the definition of T:

$$\eta_t^k = D^{(n-k+1)}\left(\alpha_t^k, \ldots, \alpha_t^n\right). \tag{6}$$

We know that η_t^k has distribution ν^{ρ^k}.

Need to show that RHS of (6) is a HAD trajectory governed by ω.

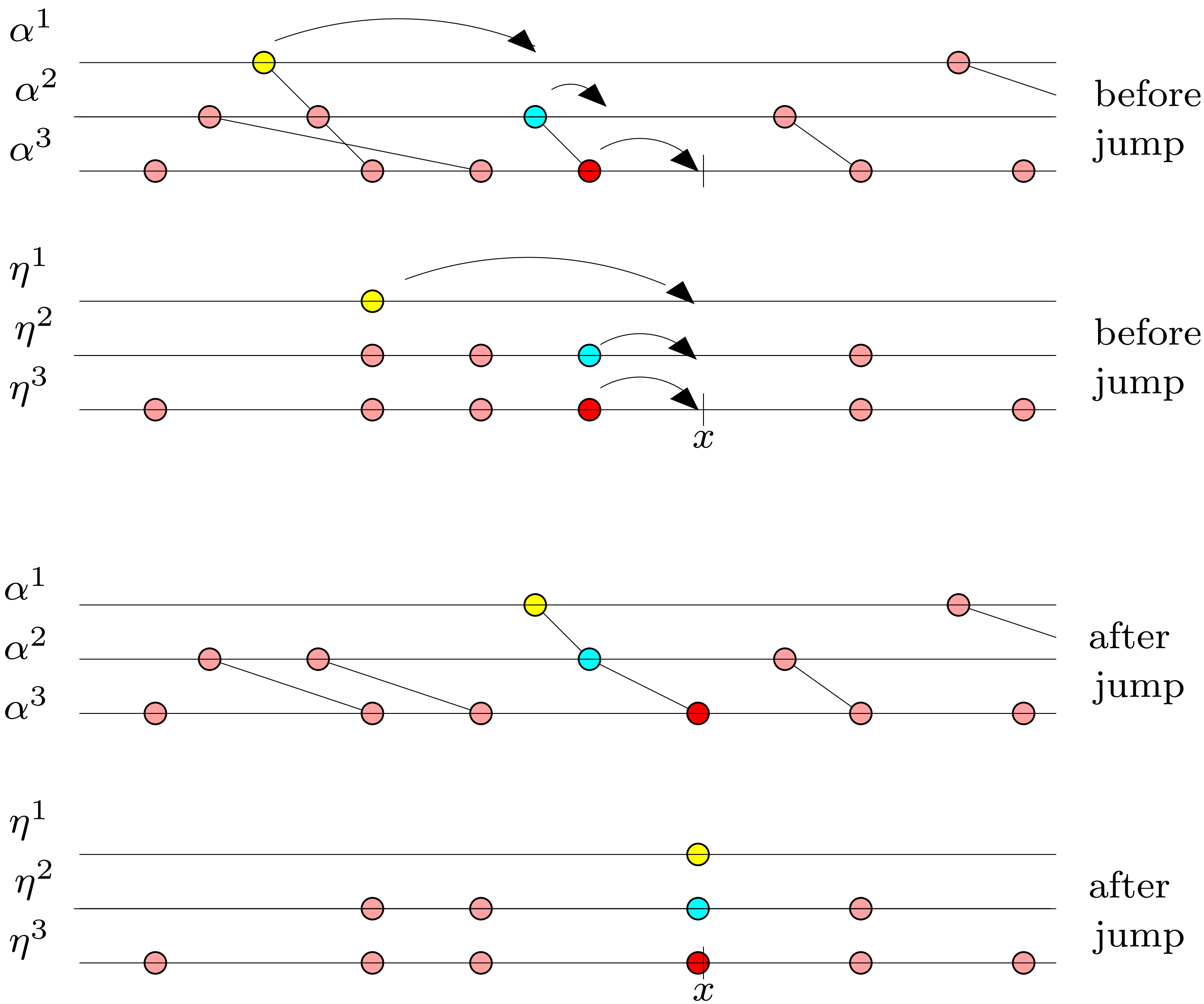
Since $(\alpha^k, \ldots, \alpha^n)$ is multi-line governed by ω, enough to show that $D^{(n)}(\alpha_t^1, \ldots, \alpha_t^n)$ is HAD governed by ω.

$D^{(n)} :=$ departures of first class customers at nth queue. Induction:

$$D^{(n)}(\alpha_t^1, \ldots, \alpha_t^n) = D^{(2)}\left(D^{(n-1)}(\alpha_t^1, \ldots, \alpha_t^{n-1}), \alpha_t^n\right)$$

and the fact that $(\alpha_t^1, \ldots, \alpha_t^{n-1})$ is an $(n-1)$-line multiline process governed by ω^{n-1}.

For $n = 2$ need to show $D(J(\alpha^1, y), J(\alpha^2, x)) = J(D(\alpha^1, \alpha^2), x)$. Check a small number of cases. □



α^1
α^2
α^3
before jump
η^1
η^2
η^3
before jump
x
α^1
α^2
α^3
after jump
η^1
η^2
η^3
after jump
x

Properties of invariant measure

Fix n and $\rho^1, \ldots, \rho^n$, and let $m < n$.

• Let $\xi^{(n)}$ be distributed according to $\mu^{(n)}_{\rho^1,\ldots,\rho^n}$ (multiclass with n classes),

• $\xi^{(m)}$ according to $\mu^{(m)}_{\rho^1,\ldots,\rho^m}$.

Then

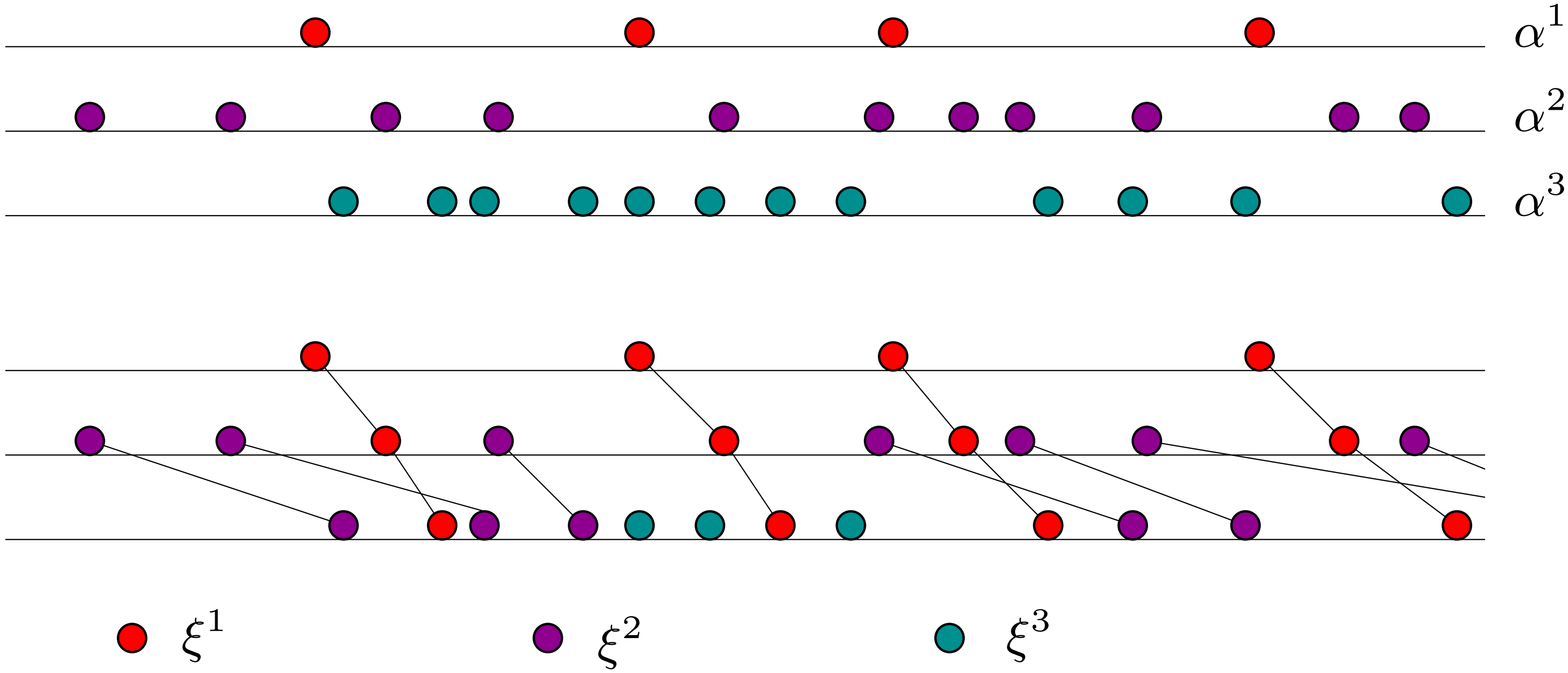
$$\xi^{(m)} \text{ has the same distribution as } [\xi^{(n)}]^m, \tag{7}$$

where $[\xi^{(n)}]^m$ is the truncated configuration defined by

$$[\xi^{(n)}]^m(i) = \min\left\{\xi^{(n)}(i), m+1\right\}.$$

• m-class input process to queue m has the same law as the m-class departure process from the same queue.

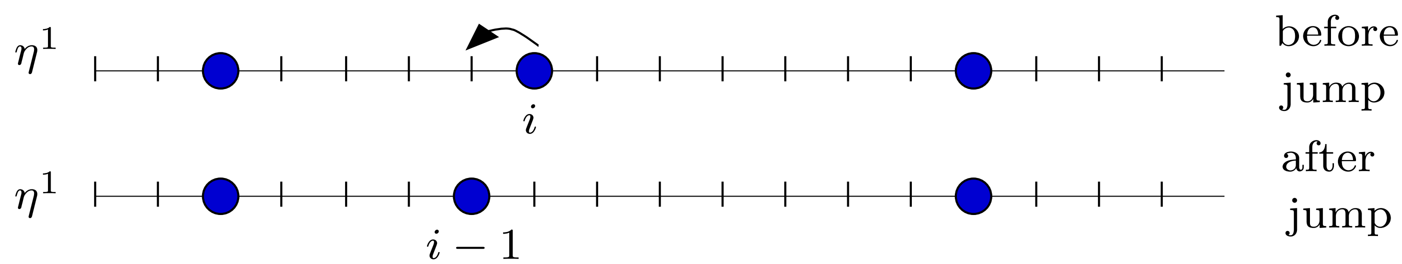
Multiclass Burke.



Law of (ξ^1, ξ^2) in second line is the same as law of (ξ^1, ξ^2) in third line.

TASEP

The totally asymmetric simple exclusion process.



Generator:

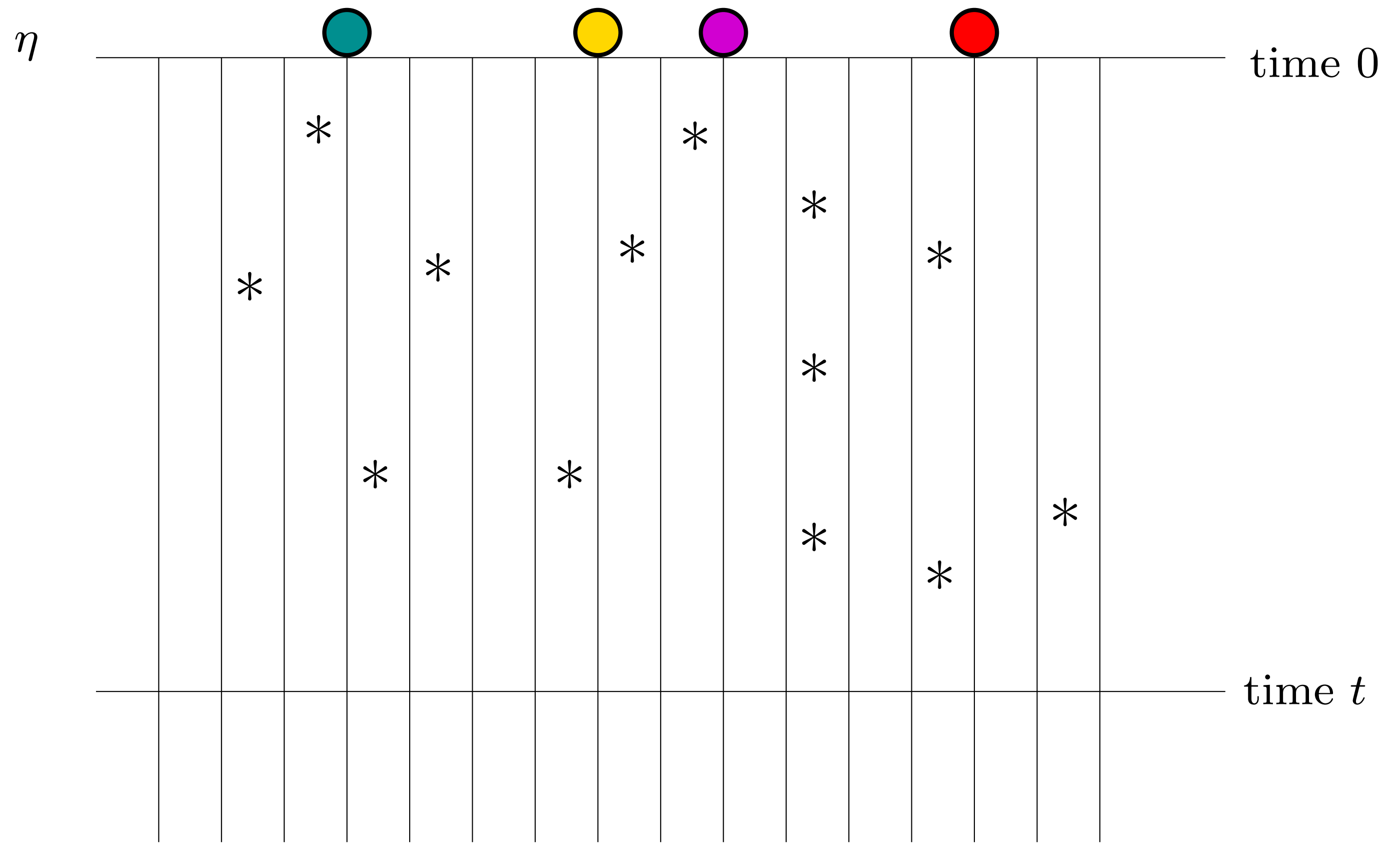
$$Lf(\eta) = \sum_j [f(A_j\eta) - f(\eta)] \tag{8}$$

where

$$(A_j\eta)(k) = \begin{cases} \eta(k) & \text{if } k \notin \{j-1, j\} \\ \max\{\eta(j-1), \eta(j)\} & \text{if } k = j-1 \\ \min\{\eta(j-1), \eta(j)\} & \text{if } k = j. \end{cases} \tag{9}$$

Graphical construction

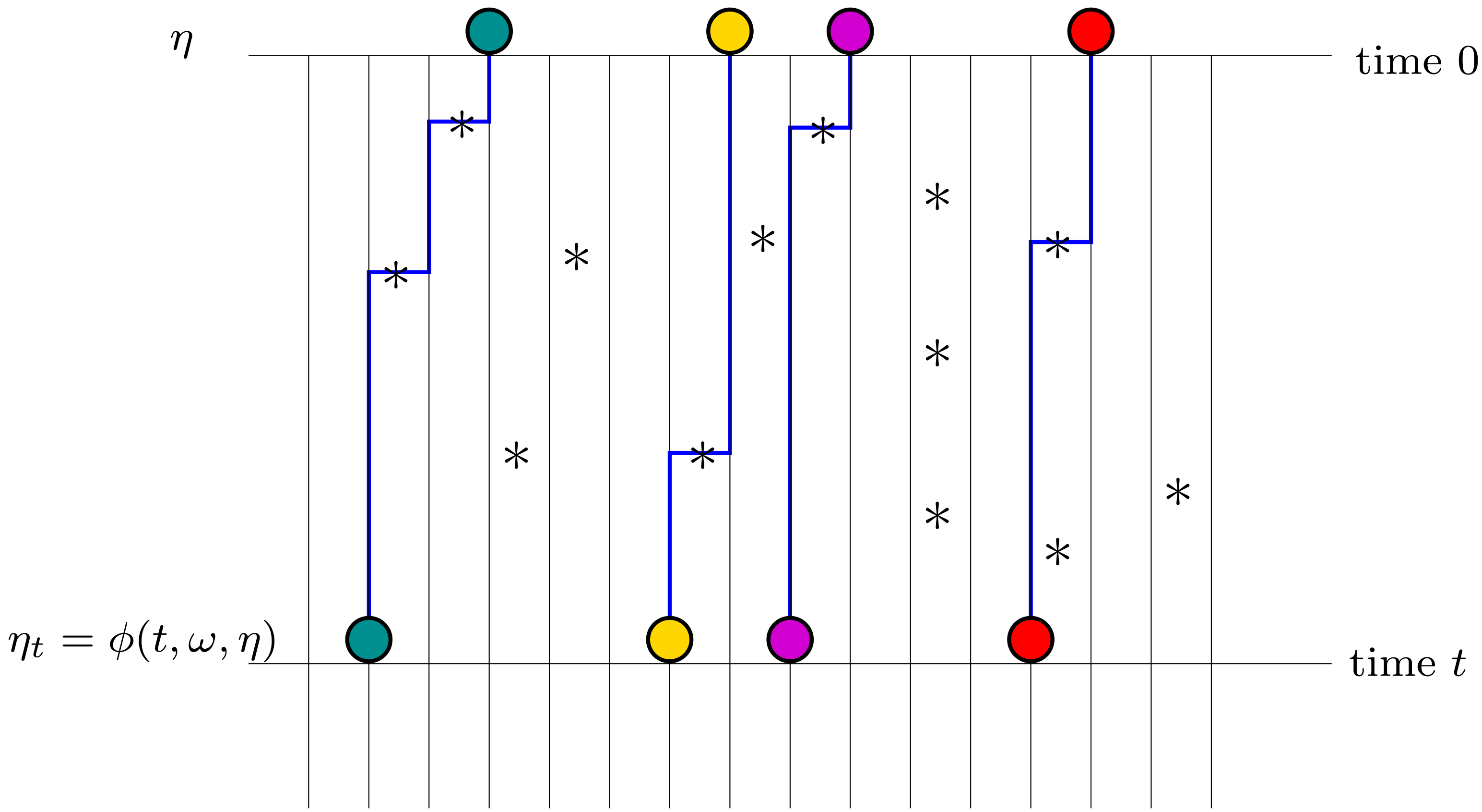
Poisson points or marks, on ω on $\mathbb{R} \times (\mathbb{Z} + \frac{1}{2})$.



The * represent the events of the Poisson process ω.

Graphical construction

Poisson points or marks, on ω on $\mathbb{R} \times (\mathbb{Z} + \frac{1}{2})$.



The * represent the events of the Poisson process ω.

The Bernoulli measures ν^ρ (and mixtures of them) are again invariant for the TASEP. (Also *blocking measures*).

Again stationary version governed by ω with marginal law ν^ρ:

$$\eta_t = \phi(t - s, \tau_s \omega, \eta_s) \tag{10}$$

for all $0 \leq s < t < \infty$.

Coupled TASEP

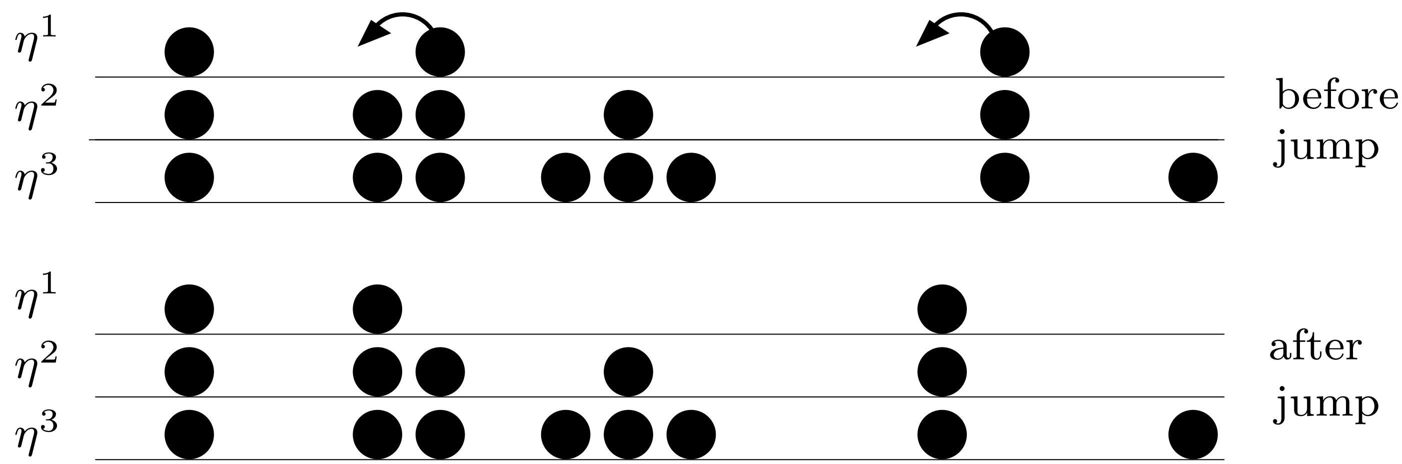


Figure 1: Coupling in TASEP

The basic coupling with initial configurations $\eta = (\eta_0^1, \ldots, \eta_0^n)$:

$\eta_t = (\eta_t^1, \ldots, \eta_t^n) = \phi^{(n)}(t, \omega, \eta_0)$, where

$(\phi^{(n)}(t, \omega, \eta_0))^k = \phi(t, \omega, \eta_0^k)$.

Multiclass process $\xi_t = R\eta_t$.

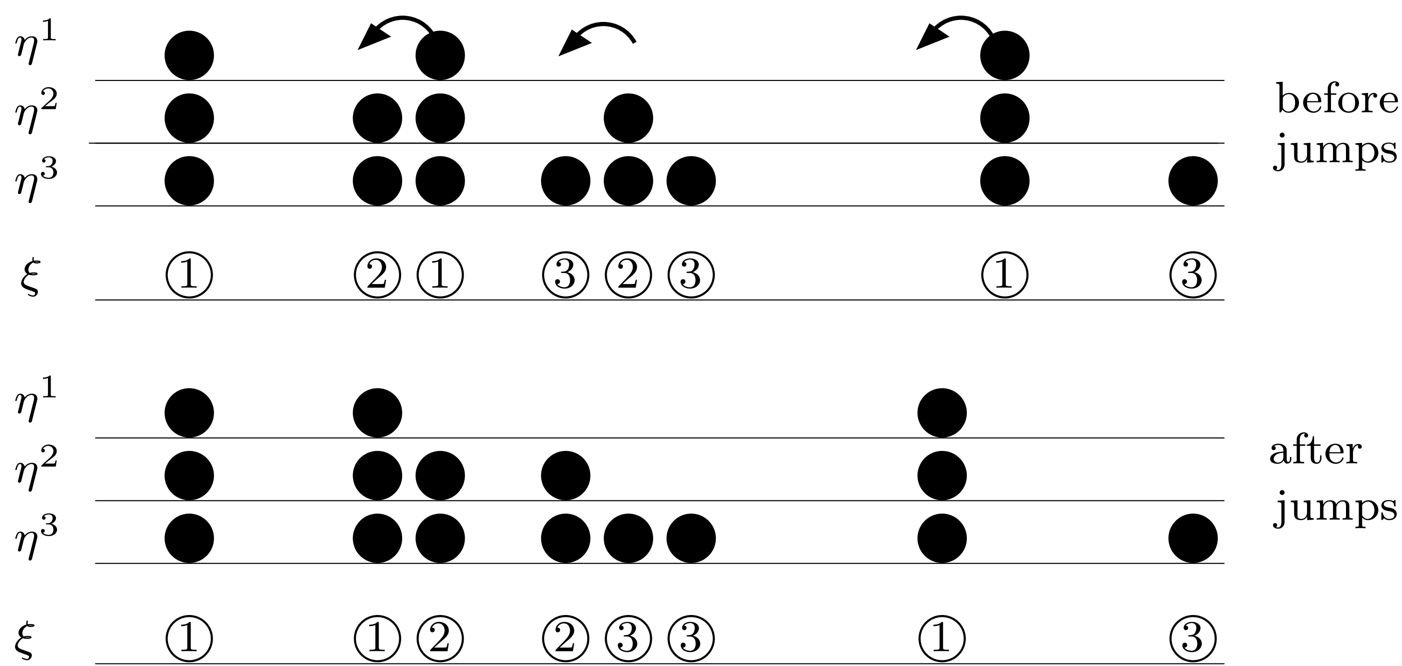


Figure 2: Coupled and multiclass TASEP

Invariant measures for coupled and multiclass TASEP

Theorem 4 *Under the conditions of Theorem 1,*

- π*, the distribution of* $\eta = T\alpha$ *is invariant for coupled TASEP* (η_t)
- μ*, the law of* $M\alpha$*, is invariant for the multiclass TASEP* (ξ_t)*.*

The strategy is the same as for HAD.

The differences come in the definitions of dual points and of the multi-line process.

Dual points in TASEP

Density ρ fixed

ω be a Poisson process on $\mathbb{R} \times (\mathbb{Z} + \frac{1}{2})$.

Let $(\eta_t, t \in \mathbb{R})$ be the TASEP trajectory governed by ω

Define

$$\begin{aligned} \Delta_\rho(\omega) &= \{(x,t) \in \omega : \eta_{t-}(x + \frac{1}{2}) = 1\} \\ &\qquad \cup \{(x+1,t) : (x,t) \in \omega \text{ and } \eta_{t-}(x + \frac{1}{2}) = 0\}. \end{aligned}$$

See Figure 3.

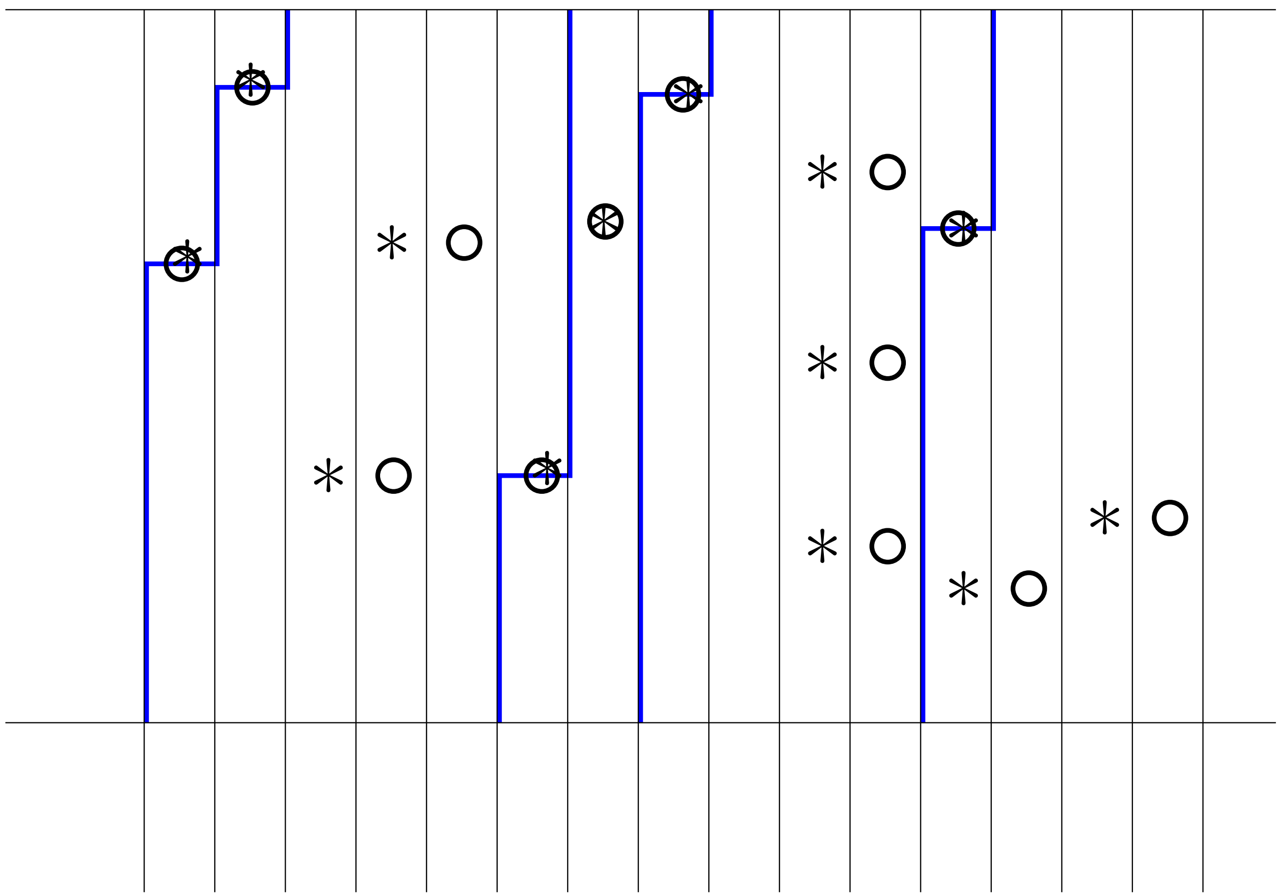


Figure 3: Circles represent the dual points of the TASEP trajectory. Turning the picture upside-down exchanges the roles of circles and stars.

Proposition 5 *ω Poisson process in $\mathbb{R} \times \mathbb{Z}$.*

$\Delta_\rho(\omega)$ dual points for the TASEP trajectory with density ρ governed by ω.

Then $\Delta_\rho(\omega)$ is also a Poisson process in $\mathbb{R} \times \mathbb{Z}$.

Furthermore $\{(x, s) \in \Delta_\rho(\omega) : s < t\}$, the set of dual points earlier than t, is independent of the configuration η_t.

Proof Same proof but need two spin flip dynamics to mark dual points missed by the trajectory.

Now Multi-line TASEP is defined in the same way and the results proven following the same strategy. Only difference is case $n = 2$ where the argument is model specific.

Other dynamics

The long range exclusion process

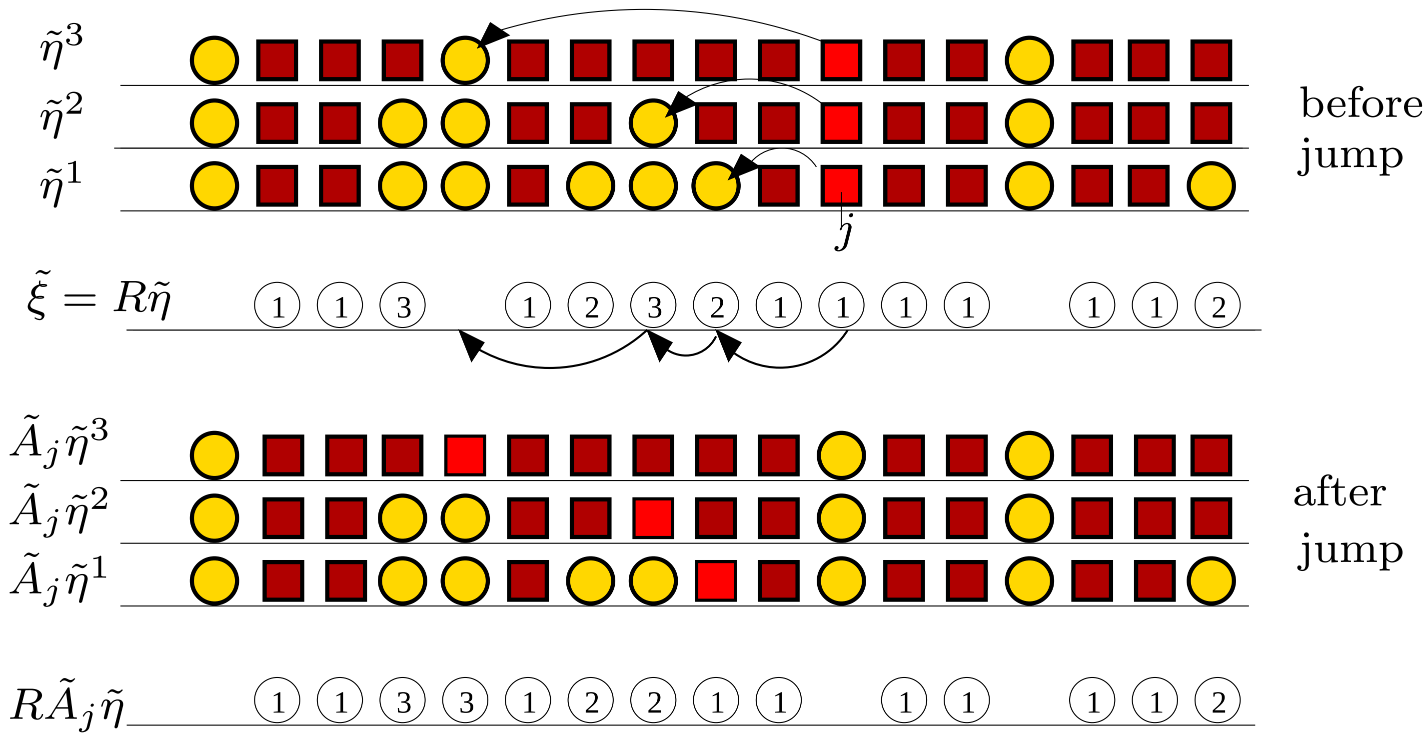


Figure 4: Jumps in coupled and multiclass LREP due to bell at j. LREP particles are represented by squares and empty sites by balls.

Sequential TASEP .

• discrete-time $\mathcal{X}$.

• At each time step, each particle tries to jump left with probability p, succeeding if the site to its left is empty.

• Updates are carried out sequentially from left to right.

• governing points ω have Bernoulli product measure on $\mathbb{Z} \times \mathbb{Z}$

• μ is invariant for the multiclass process.

Examples where the method does not work

Parallel TASEP .

- All sites are updated simultaneously;
- jumps are only allowed at sites containing a particle with a hole to its left before any other update occurred.
- basic coupling does not even preserve ordering of configurations,
- product measure ν^ρ is no longer invariant for the parallel TASEP.

ASEP

- each particle tries to jump left at rate p and right at rate $1 - p$.
- Product measure ν^ρ is invariant for the process;
- μ is no longer invariant for the multiclass process.

The End