# A $\sigma$-model for glassy dynamics

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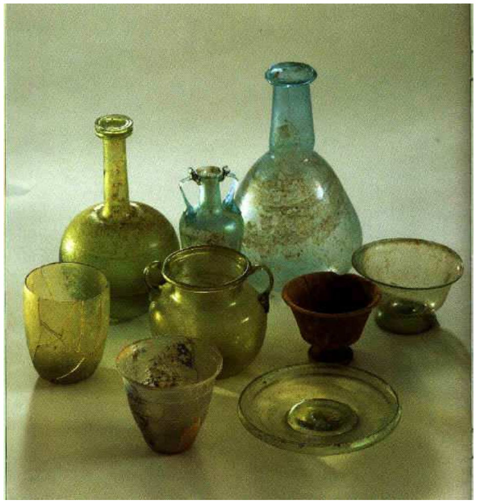
**Newton Institute, Cambridge, 30/05/2006**

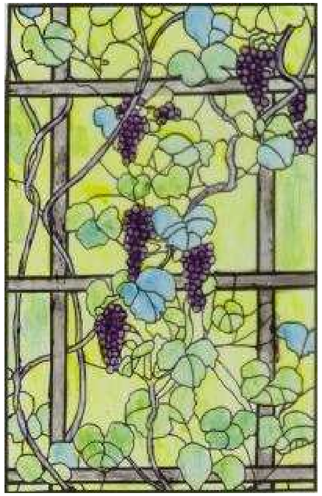
# Plan

- Recall : dynamics of mean-field models

  Fully connected disordered spin systems (& mode-coupling approx.)

- **Describing fluctuations**.





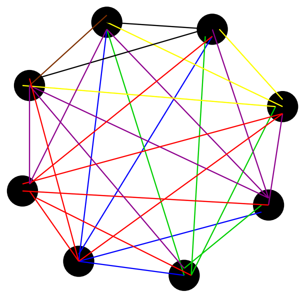
# Disordered spin models

The fully-connected $p$-spin spherical model

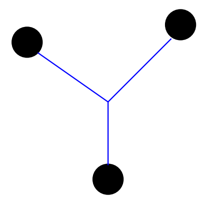
$$E_J(\vec{s}) = \sum_{i_1 \neq \cdots \neq i_p} J_{i_1 \ldots i_p} s_{i_1} \ldots s_{i_p} + \frac{z}{2} \left( \sum_i s_i^2 - N \right) ,$$

$z$ is a Lagrange multiplier,

$J_{i_1 \ldots i_p}$ are quenched i.i.d. Gaussian random variables with $[\, J_{i_1 \ldots i_p} \,] = 0$ and $[\, J^2_{i_1 \ldots i_p} \,] \propto N^{p-1}$.



***p=2***



***p=3, the unit***

# Stochastic dynamics

The system is coupled to its environment $\Rightarrow$ stochastic dynamics for $s_i$,

*e.g.* Langevin dynamics in the overdamped limit

$$\gamma \dot{s}_i(t) \;=\; -\frac{\delta E_J(\vec{s})}{s_i(t)} + \xi_i(t) \;,$$

with $\xi$ a Gaussian white noise :

$$\begin{aligned} \langle\, \xi_i(t) \,\rangle &= 0 \;, \\ \langle\, \xi_i(t)\xi_j(t') \,\rangle &= 2\gamma k_B T \delta_{ij} \delta(t-t') \;. \end{aligned}$$

$\gamma$ is the friction coefficient, $T$ is the temperature of the bath and $k_B$ is the Boltzmann constant ($k_B = \gamma = 1$ henceforth).

# Initial condition

- Rapid quench from high temperature mimicked by

random initial conditions, $s_i(0)$, taken, e.g. from a Gaussian pdf.

Note : the initial condition is uncorrelated with the quenched randomness ($J_{i_1 \dots i_p}$) $\Rightarrow$ easy average over disorder (no need to use replicas).

# Key quantities

In the $N \to \infty$ limit the relaxational dynamics can be described with

- the global correlation function

$$C(t, t_w) = N^{-1} \sum_{i=1}^{N} [\, \langle s_i(t) s_i(t_w) \rangle \,]$$

- its associated linear response function

$$R(t, t_w) = N^{-1} \sum_{i=1}^{N} \left[ \langle \frac{\delta s_i(t)}{\delta h_i(t_w)} \rangle \right] \Bigg|_{h=0}$$

or its integral over time $\chi(t, t_w) = \int_{t_w}^{t} dt' R(t, t')$ .

# Dynamic equations

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(\partial_t - z_t)C(t,t_w) = \int dt' \, [\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')] + 2TR(t_w,t) \; ,$$

$$(\partial_t - z_t)R(t,t_w) = \delta(t-t_w) + \int dt' \, \Sigma(t,t')R(t',t_w) \; ,$$

where the self-energy and vertex are functions of $C$ and $R$ :

$$D(t,t_w) = \frac{p}{2} C^{p-1}(t,t_w) \; , \quad \Sigma(t,t_w) = \frac{p(p-1)}{2} C^{p-2}(t,t_w) \, R(t,t_w) \; .$$

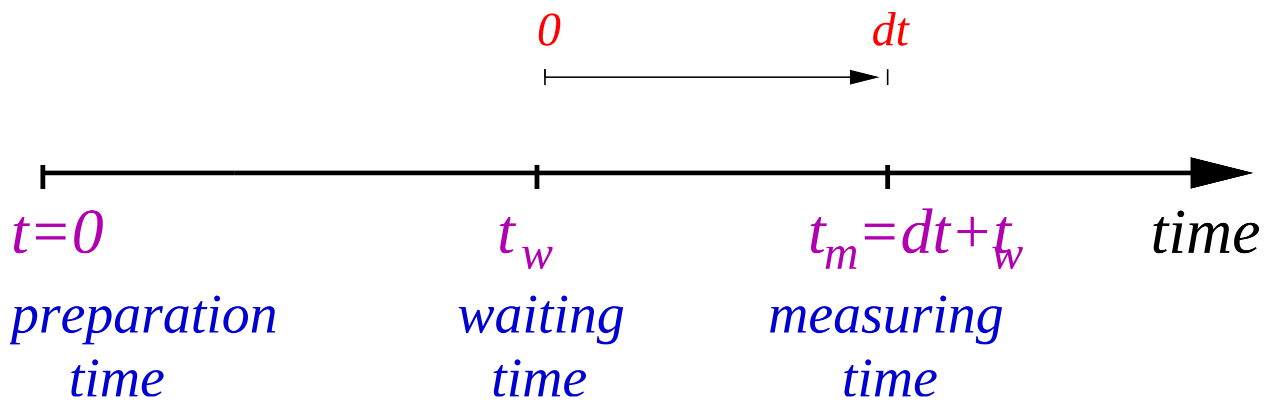
and the Lagrange multiplier $z_t$ is fixed by setting $C(t,t) = 1$.

# Glassy dynamics below $T_d$

**Out of equilibrium relaxation** $t, t_w \ll t_{eq}(N) \to \infty$

Two-time quantities age, *i.e.* the stationary relaxion is lost

separation of time-scales, rapid-slow, controlled by $t_w$.



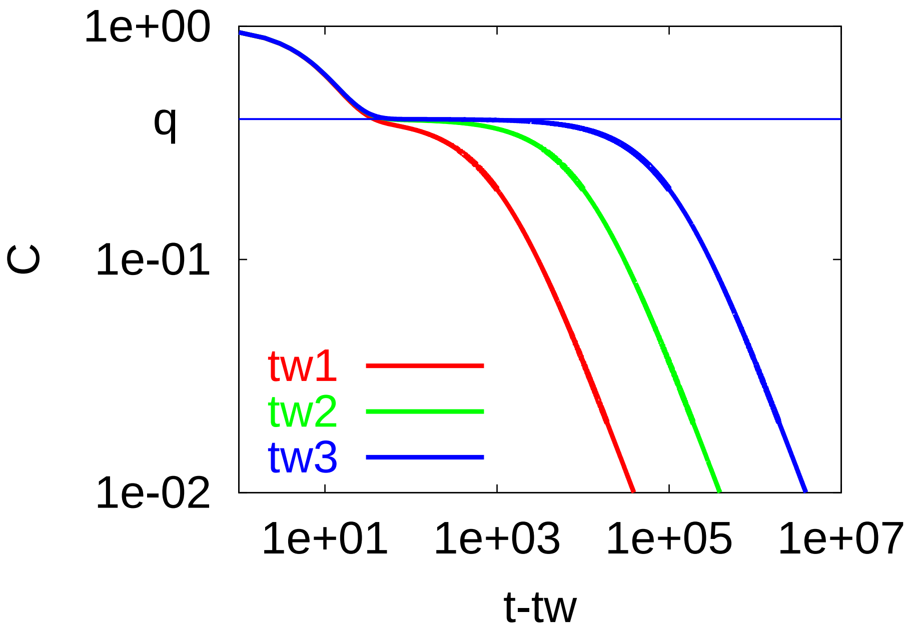

Numeric and analytic solution

LFC & J. Kurchan 93

# Separation of time-scales

The correlation in the long $t_w$ limit

Fast




Slow

$$C^s(t,t_w) \approx f_c\left(\frac{L(t)}{L(t_w)}\right)$$

$$\partial_t C^s(t,t_w) \ll C^s(t,t_w)$$

log-log scale !

Eqs. for the slow relaxation $C^s \equiv C < q$ :

Approx. asymptotic time-reparametization invariance $t \to h(t)$

# Time-reparametrization

Example : the equation $(\partial_t - z_t)R = \delta + \Sigma R$

- Take $t - t_w \gg t_w$ use $z_t \to z_\infty$, drop $\partial_t R$ separate the fast contributions to the integral $\Sigma R = \int_{t_w}^t dt' \; \Sigma(t,t')R(t',t_w)$ :

$$\tilde{z}_\infty R^s(t,t_w) \sim \int_{t_w}^{t} dt' \; D'[C^s(t,t')] \, R^s(t,t') \, R^s(t',t_w) \; . \qquad (1)$$

- The transformation

$$t \to h_t \equiv h(t) \; , \qquad \begin{cases} C^s(t,t_w) \to C^s(h_t,h_{t_w}) \; , \\ R^s(t,t_w) \to \frac{dh_{t_w}}{dt_w} \, R^s(h_t,h_{t_w}) \; . \end{cases}$$

with $h_t$ positive and monotonic leaves eq. (1) **invariant** :

$$\tilde{z}_\infty \, R^s(h_t,h_{t_w}) \sim \int_{h_w}^{h_t} dh_{t'} \; D'[C^s(h_t,h_{t'})] R^s(h_t,h_{t'}) \, R^s(h_{t'},h_{t_w}) \; .$$

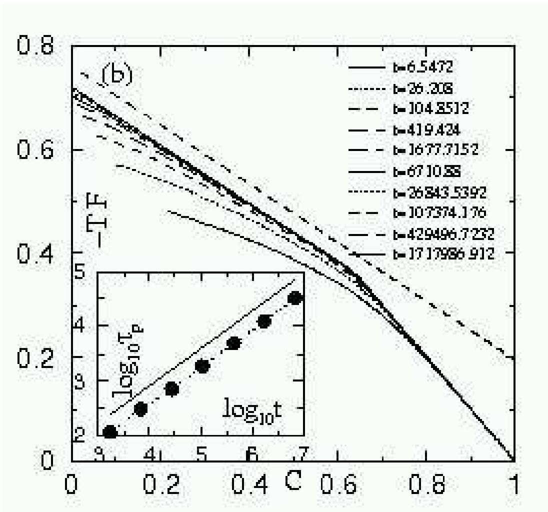
# Time reparametrization invariance

One can compute analytically $f_c$ and $\chi^s(C^s)$

$$C^s(t, t_w) \sim f_c\left(\frac{L(t)}{L(t_w)}\right) \; ,$$

$$\chi(t, t_w) \equiv \int_{t_w}^{t} dt' \, R(t, t') \sim \frac{1-q}{T} + \frac{1}{T_{eff}} \left[q - C^s(t, t_w)\right]$$

but not the 'clock' $L(t)$ .

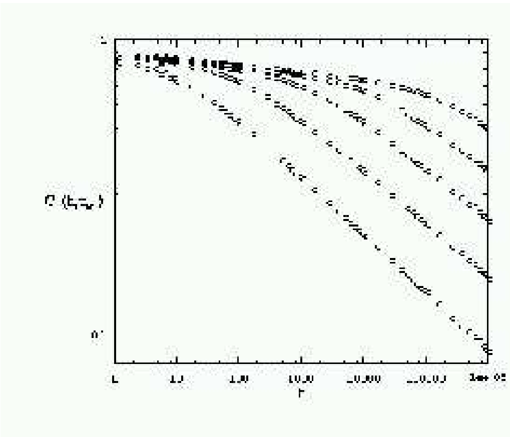


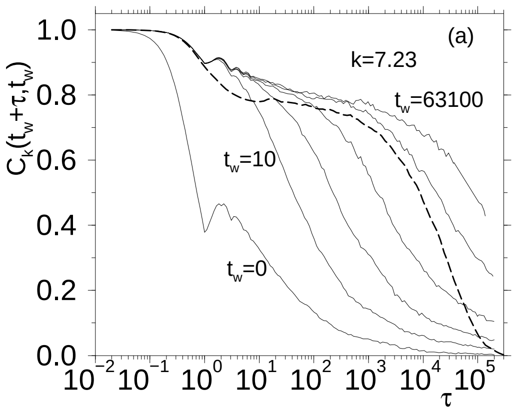

NOTE : the slow part of $\chi$ is finite ($\chi^s > 0; T_{eff} < +\infty$).

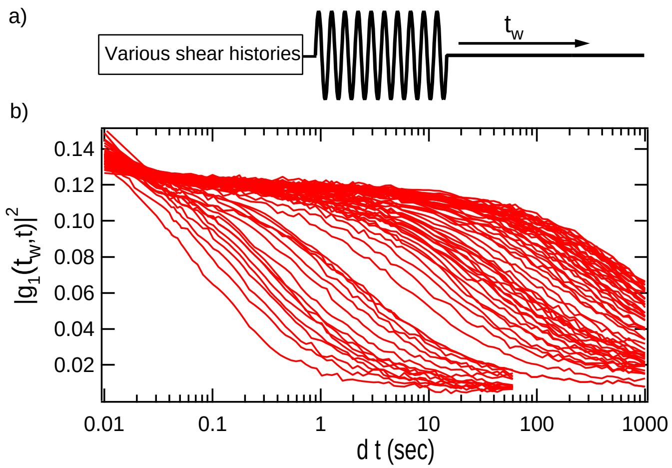
# Finite dimensions

- Slow dynamics : observed
- Separation of time-scales : observed, though less clear-cut.

Sim. 3d EA | Sim. L-J mixture | Exp. colloidal susp.






**H. Rieger (92)**

**J-L Barrat & W. Kob (99)**

**B. Viasnoff & F. Lequeux (03)**

# Finite dimensions

Dynamic action

$$S = S^s + S^f + S^{int}$$

- RG argument based on separation of time-scales allows one to show the approximate asymptotic invariance of $S^s$ under *global* time-reparametrizations

$$t \to h_t \equiv h(t) \; , \qquad \begin{cases} C_r^s(t,t_w) \to C_r^s(h_t, h_{t_w}) \; , \\ R_r^s(t,t_w) \to \frac{dh_{t_w}}{dt_w} \, R_r^s(h_t, h_{t_w}) \; . \end{cases}$$

Symmetry breaking terms become less important as $t_w$, $t - t_w \to \infty$.

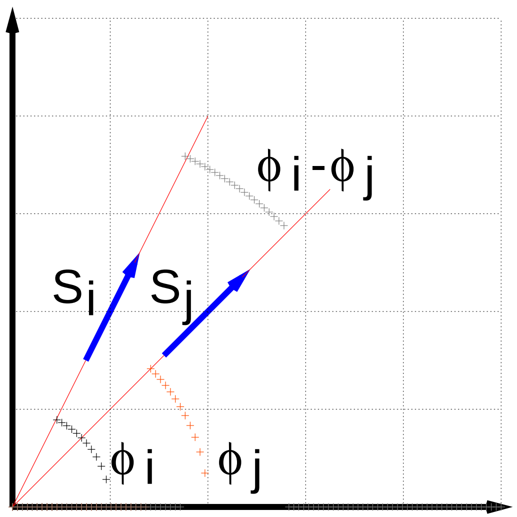
$3d$ Edwards-Anderson spin-glass

**Chamon, Kennett, Castillo & LFC (02).**

# Parenthesis : an analogy

# The Heisenberg ferromagnet

An analogy : a nuissance turned into a model




Landau free-energy

$$F = \int d^d r \left\{ [\nabla \vec{m}(\vec{r})]^2 + \lambda \left[ m^2(\vec{r}) - m_0^2 \right]^2 \right\} .$$

Invariant under the global rotation $m^a(\vec{r}) = R^{ab} m^b(\vec{r})$.

(`Global time-reparametrization invariance`)

# Statics of the Heisenberg ferro

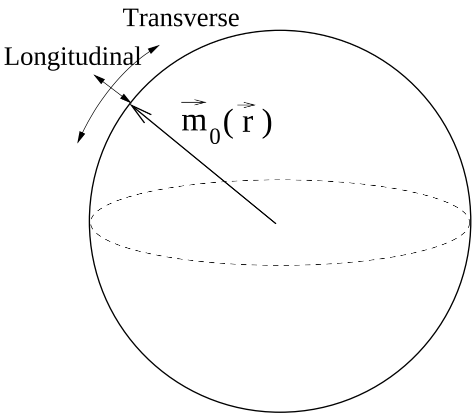
Ground state : $\vec{m}(\vec{r}) = \vec{m}_0$ for all $\vec{r}$.

Fluctuations : $\vec{m}(\vec{r}) = \vec{m}_0 + \delta\vec{m}(\vec{r})$.

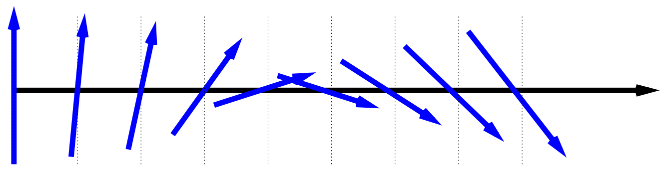
Longitudinal (easy) &

transverse (hard) fluctuations.

Spin-waves




Low energy excitation



(Time-reparametrization waves)

# Explicit symmetry breaking

A pinning field

Landau free-energy in a field

$$F = \int d^d r \left\{ [\nabla \vec{m}(\vec{r})]^2 + \lambda \left[ m^2(\vec{r}) - m_0^2 \right]^2 - \vec{h}\vec{m}(\vec{r}) \right\} .$$

No longer invariant under the global rotation

A particular direction is selected by the field : $\vec{m}(\vec{r}) = m_0 \hat{h}$.

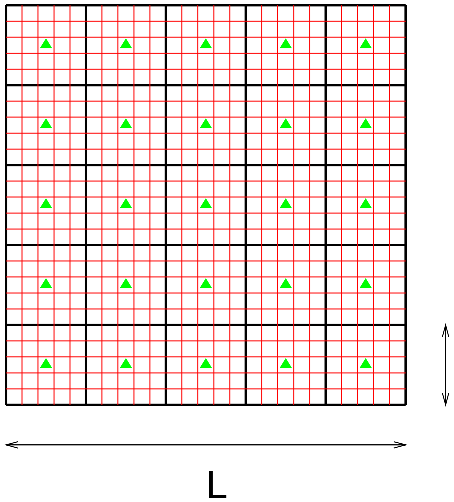
[$\partial_t C$ and $\partial_t R$ select the 'clock' $L(t)$
but they become less and less important as
the system ages & $t - t_w$ increases !]

# End of the parenthesis

# Next

Use the time-reparametrization symmetry

**to characterize the fluctuations**




$$C_r(t,t_w) \equiv \frac{1}{V_r}\sum_{i\in V_r} s_i(t)s_i(t_w) \; ,$$

$$\chi_r(t,t_w) \equiv \frac{1}{V_r}\sum_{i\in V_r}\int_{t_w}^{t} dt' \; \left.\frac{\delta s_i(t)}{\delta h_i(t')}\right|_{h=0} \; .$$

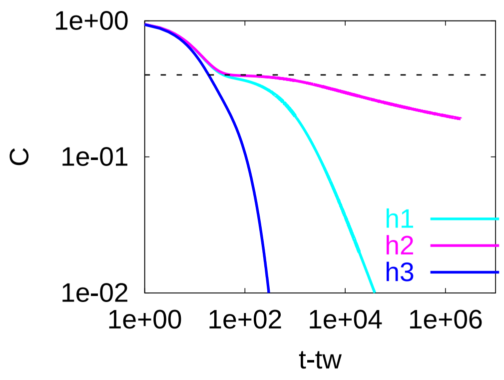
# Leading fluctuations

Scaling of the slow part of the global correlation

$$C^s(t,t_w) \approx f_c\left(\frac{L(t)}{L(t_w)}\right) \ .$$

The global time-reparametrization invariance $\Rightarrow$ $C^s_r(t,t_w) \approx f_c\left(\frac{h_r(t)}{h_r(t_w)}\right)$.

Ex. $h_{r_1} = \frac{t}{t_0}$, $h_{r_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{r_3} = e^{\ln^a\left(\frac{t}{t_0}\right)}$ on different regions




Same $t_w$, slower and faster decays

**Castillo, Chamon, LFC, Iguain, Kennett (02,03).**

# Turn it useful : $\sigma$ model

Easy fluctuations $t \longrightarrow h_r(t)$.

- Ideally : derive the action $S[h_r(t)]$.

  Doable in quasi-mean-field models, **C. Chamon, LFC, S. Franz, in progress.**

- In practice : propose the action $S[h_r(t)]$ ;

  predictions, *e.g.* $\rho[C_r^s; t, t_w]$ ; $\rho[R_r^s; t, t_w]$ ; $\rho[C_r^s, R_r^s; t, t_w]$

  that can be checked numerically & experimentally.

**P. Chamon, Charbonneau, LFC, D. Reichman & M. Sellitto (04).**

# $\sigma$-model

Slow decay in terms of $h_r(t) \equiv e^{-\varphi_r(t)}$

$$C_r^s(t,t_w) \approx f_c\left(\frac{h_r(t)}{h_r(t_w)}\right) = f_c\left(e^{-\int_{t_w}^{t} dt' \partial_{t'}\varphi_r(t')}\right)$$

The simplest

(i) global time-reparametrization invariant ;

(ii) local in space ;

(iii) positive definite ($\partial_t h_r(t) > 0 \Rightarrow \partial_t \varphi_r(t) > 0$) ;

(iv) invariant under $\varphi_r(t) \to \varphi_r(t) + \Phi(r)$ as $C_r^s$ effective action is

$$S = K \int d^d r \int dt \, \frac{[\nabla \partial_t \varphi_r(t)]^2}{\partial_t \varphi_r(t)}$$

Similar to a Gaussian surface after a redefinition of time and field.

# Some consequences

- Temporal scaling of the pdf of local correlations dictated by the global correlation $\rho(C_r; t, t_w) = \rho[C_r; C^s(t, t_w)]$ .

- Negatively-skewed, non-Gaussian $\rho(C_r; C^s)$ for $0 < C^s < q$.

- The two-time dependent correlation length $\xi(t, t_w)$,

$$\left[\sum_i C_i^s(t, t_w) C_j^s(t, t_w)\right]_c \approx e^{-|\vec{r}_i - \vec{r}_j|/\xi(t, t_w)} ,$$

should diverge with $t$ and $t_w$.

- Constant of motion. $\rho[C_r, \chi_r; t, t_w]$ should follow the global FDT rel. :

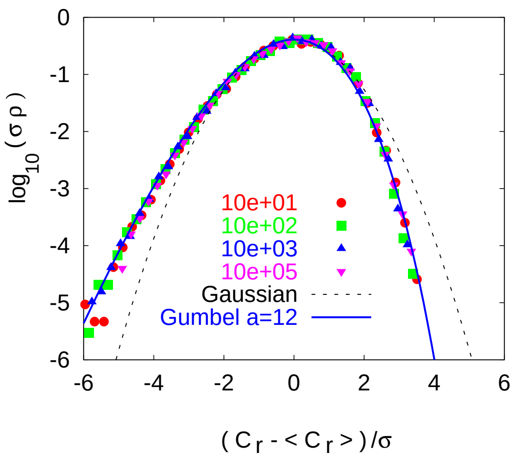
$$\lim_{t_w \to \infty; C(t, t_w) = C} \chi(t, t_w) = \chi(C) .$$

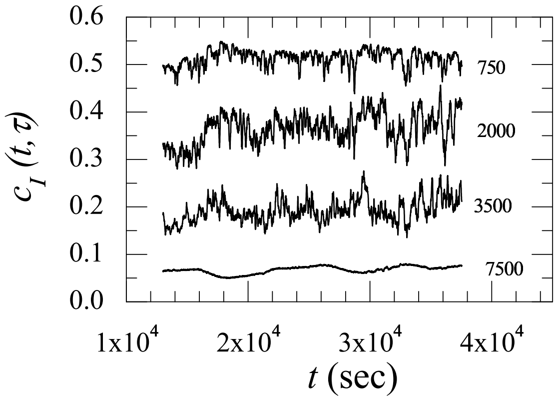
All can be tested with simulations & experiments.

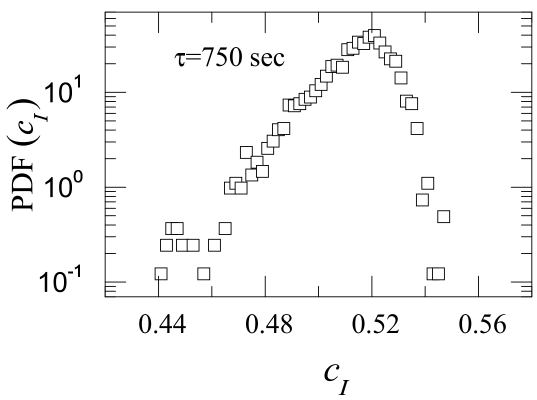
# e.g. pdf of local correlations

Kinetically constrained model ; four $(t, t_w)/\ C(t, t_w) = 0.8$.

Similar results for the $3d$ spin-glass.




**Chamon, Charbonneau, LFC, Reichman & Sellitto (04)**

a micellar polycrystal

**A. Duri, P. Ballesta, L. Cipelletti, H. Bissig, & V. Trappe (04)**

Other tests on joint pds of local correlations and responses

# How general is this ?

- Coarsening – domain growth

*e.g.* the $d$-dimensional $O(N)$ model in the large $N$ limit (continuous space limit of the Heisenberg ferro with $N \to \infty$)

$$\dot{\phi}_\alpha(\vec{r},t) = \nabla^2 \phi_\alpha(\vec{r},t) + \lambda |N^{-1}\phi^2(\vec{r},t) - 1|\, \phi_\alpha(\vec{r},t) + \xi_\alpha(\vec{r},t)$$

with $\phi(\vec{k},0)$ Gaussian distributed with variance $\Delta^2$. The solution is

$$\phi(\vec{k},t) = e^{-k^2 t - \int_0^t dt' z_{t'}}\ \phi(\vec{k},0)$$
$$+ \int_0^t dt'\ e^{-k^2(t-t') - \int_{t'}^t dt'\, z_{t'}}\ [\xi(\vec{k},t') + h(\vec{k},t')]\ ,$$

for any component $\alpha$, with $z_{t'} = -\lambda |N^{-1}\phi^2(\vec{r},t) - 1|$ for all $r$.

# The $O(N \to \infty)$ model

The global correlation, $C(t,t_w) = \int d^d r C(\vec{r},t,t_w)$ and linear response, $R(t,t_w) = \int d^d r R(\vec{r},t,t_w)$, are

$$C(t,t_w) \sim \frac{(tt_w)^{d/4}}{(t+t_w)^{d/2}} \left[ \Delta^2 + 2T \int_0^{t_w} dt'' \, \frac{e^{\,2\int_0^{t''} dt' \, z_{t'}}}{\left[1 - \frac{2t''}{(t+t_w)}\right]^{d/2}} \right] ,$$

$$R(t,t_w) \sim \left(\frac{t}{t_w}\right)^{d/4} (t - t_w + t_0)^{-d/2} ,$$

with $t_0 \equiv \Lambda^{-2}$ and $t + t_w \gg t_0$.

NOTE : the global correlation behaves as for the $p$-spin model but the slow integrated response vanishes as $\chi^s(t,t_w) \sim t^{1-d/2} f_\chi(t/t_w)$ for all $d > 2$.

# The $O(N \to \infty)$ model

**Invariance of the slow dynamic eqs. ?**

The full dynamic equation for $R(t,t') \equiv \int d^d r \; R(\vec{r},t,t')$ is

$$\frac{\partial R(t,t_w)}{\partial t} = -z_t R(t,t_w) + \sum_{n=0}^{\infty} A_n \int dt_n \int dt_{n-1} \dots \int dt_1 \; R(t,t_1) R(t_1,t_2) \dots R(t_n,t_w)$$

with $A_n$ fixed by the Fourier-mode density. After a separation of time-scales and $t - t_w \gg t_w$ one has

$$\frac{\partial R_s(t,t_w)}{\partial t} = -z_t R_s(t,t_w) + \sum_{n=0}^{\infty} B_n(t-t_w) \times \int dt_n \int dt_{n-1} \dots \int dt_1 \; R_s(t,t_1) R_s(t_1,t_2) \dots R_s(t_n,t_w)$$

# The $O(N \to \infty)$ model

**Invariance of the slow dynamic eqs. ?**

Knowing the exact $R$ one can plug in $R_s$ to find that, apart from a $g(t/t_w)$,

- the time-derivative $\partial_t R_s$ behaves as $t^{1-d/2}$ ;
- the Lagrange multiplier $z_t$ decays as $t^{-1}$ ; then $z_t \, R_s \sim t^{1-d/2}$ too ;
- the coefficients $B_n$ (stationary contributions) do not approach constants !

  INSTEAD $B_n(t-t_w) \sim (t-t_w)^{-[1+n(1-d/2)]}$.
- The integrals factors go as $I_n \sim t^{d/2-n(1-d/2)}$ in such a way that

  $B_n I_n \sim t^{1-d/2}$ as well.

No time-reparametrization invariance, just scale invariance $t \longrightarrow \zeta t$.

# The $O(N \to \infty)$ model

**$d > 2$ : a different mechanism**

due to the asymptotic absence of a slow response ($\chi_s \to 0$)

or the extreme violation of the fluctuation-dissipation equilibrium relation between correlations and responses ($T_{eff} \to \infty$) ;

and the 'bad' separation of time-scales.

**$d = d_l = 2$ : more like critical dynamics.**

**Chamon, LFC, Yoshino (06)**

Is it this way for **all coarsening systems** ?

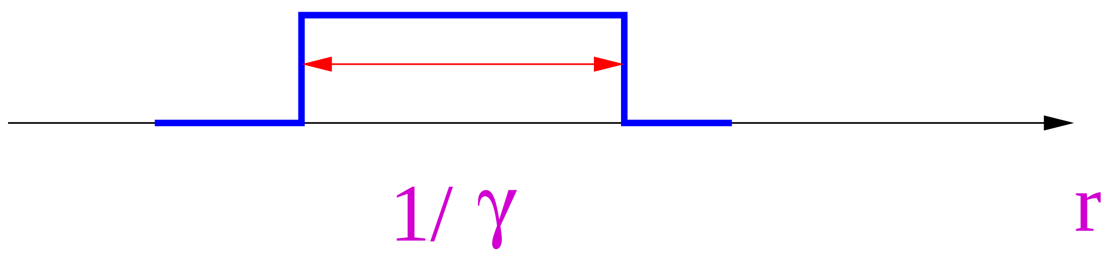
# A ‘doable’ non-trivial case ?

**The Kac $p$-spin spherical model**

$$E_J(\vec{s}) = \sum_{i_1 \neq \cdots \neq i_p} J_{i_1 \ldots i_p} s_{i_1} \ldots s_{i_p} + \frac{z_x}{2} \left( \sum_{i \in V_x} s_i^2 - N_x \right) ,$$

$z_x$ is a semi-local Lagrange multiplier,

$J_{i_1 \ldots i_p}$ are quenched i.i.d. Gaussian random variables such that interca-tions exist only when the sites are at distance $r \leq \gamma^{-1}$.




C. Chamon, LFC & S. Franz (in progress).

# A ‘doable’ non-trivial case ?

**The Kac $p$-spin spherical model**

Program : in order to compute $P(C_r, R_r)$ we have to solve :

$$(\partial_t + z_t)C = DR^\dagger + \Sigma C + 2TR^\dagger + 2T\zeta^{R\dagger}C \ ,$$

$$(\partial_t + z_t)R = \delta + DQ + \Sigma R + 2TQ + 2T\zeta^{R\dagger}R \ ,$$

$$(-\partial_t + z_t)R^\dagger = \delta + \Sigma^\dagger R^\dagger + \Delta C + (\zeta^C + 2T\zeta^R\zeta^{R\dagger})C + 2T\zeta^{R\dagger}R^\dagger \ ,$$

$$(-\partial_t + z_t)Q = \Sigma^\dagger Q + \Delta R + (\zeta^C + 2T\zeta^R\zeta^{R\dagger})R + 2T\zeta^{R\dagger}Q \ ,$$

$$R = R^* - C\zeta^R \ , \qquad D = \hat\psi f'(\hat\psi C) \ , \qquad \Sigma = \hat\psi[f''(\hat\psi C) \bullet \hat\psi R] \ ,$$

$$\Delta = \hat\psi[f''(\hat\psi C) \bullet \hat\psi Q + f'''(\hat\psi C) \bullet \hat\psi R \bullet \hat\psi R^\dagger] \ ,$$

$$2f(x) = x^p \ .$$

$C, R, R^*, R^\dagger, Q, \zeta^C, \zeta^R, z$ are functions of space $r$ and two-times. The products involve $\int_0^\infty dt' \ \cdots$ ; $\hat\psi$ has the spatial information ; $\zeta^C, \zeta^R$ are Lagrange multipliers that enforce $C_r, R_r$. **Causality is lost !**

# Summary

Theory for the nonequilibrium dynamics in the glassy phase.

dictated by (the assumption of)

Global time reparametrization invariance

*cfr.* Spin-waves in Heisenberg ferromagnets.

Predictions for the behaviour of local correlations and responses,

in rather good agreeement with simulations in **disordered spin models**

and **kinetically constrained models** ;

experiments on **colloidal systems** on their way

# Summary

Classification of non-equilibrium systems ?

The theory suggests a strong link between $T_{eff}$ and the fluctuations.

- Structural and spin glasses – aging, $T_{eff} < +\infty$.
- Domain growth – aging in the correlations but ‘no memory’, $T_{eff} \to \infty$
- Critical dynamics – interrupted aging, no asymptotic $T_{eff}$.

with different properties of the fluctuations.

to be confirmed !

# $\sigma$-model

Using the 'proper' time $\tau(t) \equiv \ln L(t)$

with $L(t)$ the "growth" law in the global corr.

& the change of variables $\psi_r^2(\tau) \equiv \partial_\tau \varphi_r(\tau)$

$$C_r^s(t,t_w) \approx f_c\left(e^{-\int_{\ln L(t_w)}^{\ln L(t)} d\tau' \psi_r^2(\tau')}\right)$$

$$\mathcal{A} = K \int d^d r \int d\tau \, [\nabla \psi_r(\tau)]^2$$

**Chamon, Charbonneau, LFC, Reichman & Sellitto (04)**

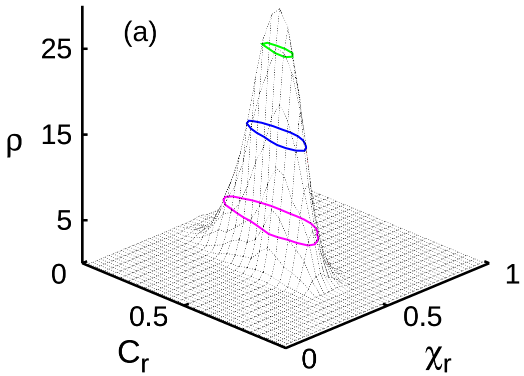
**cfr. Bramwell, Holdsworth & Pinton (98)** xy-model – spin waves ;

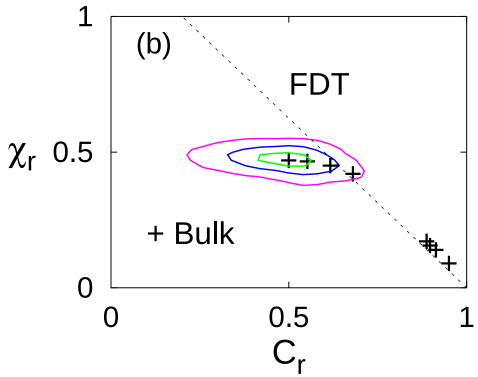
**Antal & Rácz (94)** Edwards-Wilkinson manifold.

# pdf of correlations & responses

$3d$ Edwards-Anderson spin-glass.

$$C_r(t,t_w) \equiv \frac{1}{V_r} \sum_{i \in V_r} s_i(t) s_i(t_w) \; , \quad \chi_r(t,t_w) \equiv \frac{1}{V_r} \sum_{i \in V_r} \int_{t_w}^{t} dt' \; \left. \frac{\delta s_i(t)}{\delta h_i(t')} \right|_{h=0}$$




+ Bulk : Parametric plot $\chi(t,t_w)$ vs $C(t,t_w)$ for $t_w$ fixed and 7 $t\ (> t_w)$.

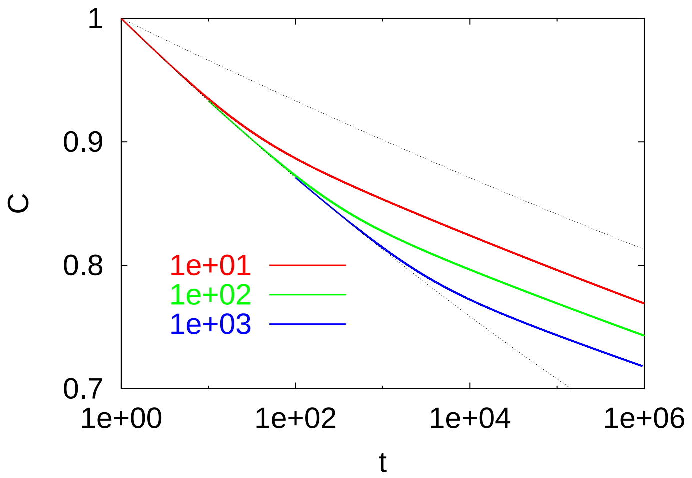
$\rho$ corresponds to the maximum $t$ yielding the smallest $C$ (left-most +).

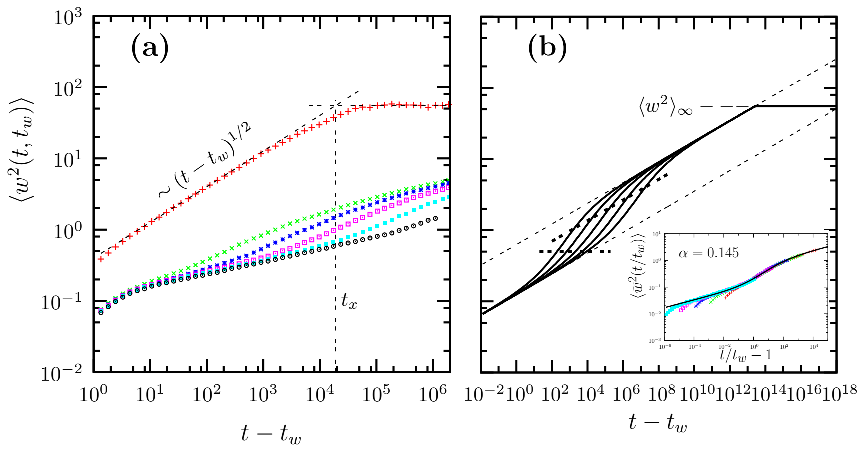
**Castillo, Chamon, LFC, Kennett (02).**

# How general is this ?

- Critical dynamics

*e.g.* the $2d$ xy model, an elastic line in a random environment




**Berthier, Holdsworth, Sellitto (03)**

**Yoshino (96), Bustingorry, Iguain, LFC, Chamon, Domínguez (06)**

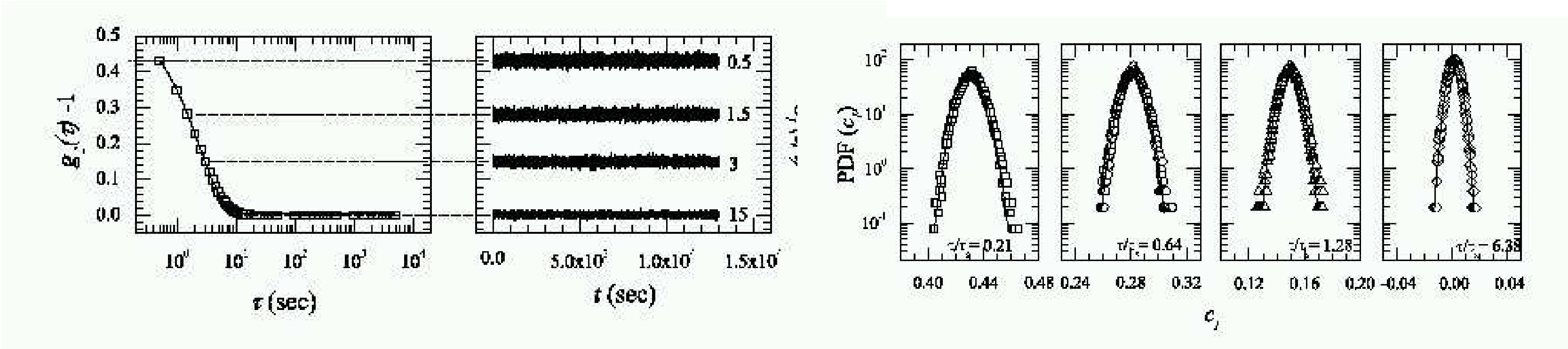
Multiplicative scaling

$$C(t,t_w) \approx t_w^{-\alpha} f_c\left(\frac{L(t)}{L(t_w)}\right)$$

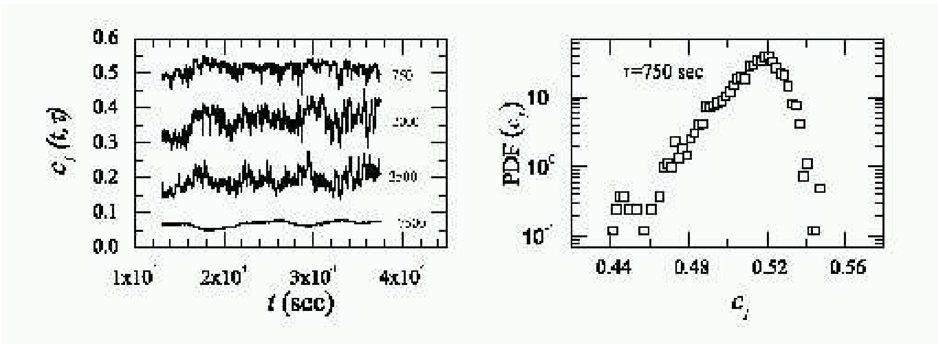
Take care of $t_w^{-\alpha}$ & saturation !

# Experiments

## Time fluctuations in Brownian particles




## a micellar polycrystal




**A. Duri, P. Ballesta, L. Cipelletti, H. Bissig, & V. Trappe (04)**

Spatial fluctuations in polymer glasses (cantilevers) **K. Sinnathamby, H. Oukris, N. Israeloff (06)**

colloidal suspensions (confocal microscopy) **P. Wang, C. Song, H. Makse et al, in progress**