Social Balance on Networks: The Dynamics of Friendship and Hatred

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Basic question:

How do social networks evolve when both friendly and unfriendly relationships exist?

Partial answer: *(Heider 1944, Wasserman & Faust 1994)*

Social balance; on a complete graph socially-balanced states must be either utopia or bipolar.

This talk:

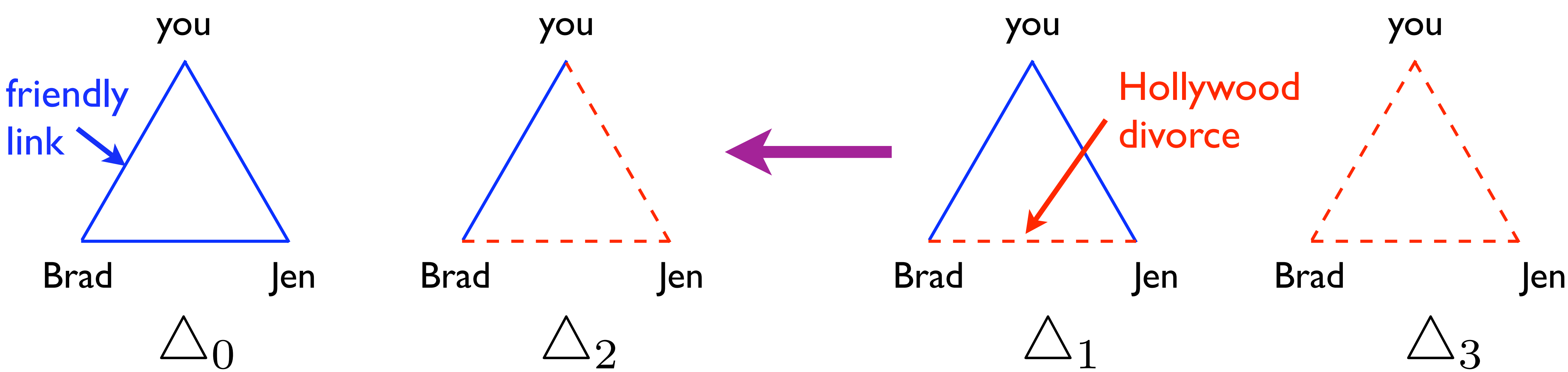
Endow a network with the simple social dynamics and investigate the evolution of relationships.

related work: Kulakowsi et al.

Main result:

Dynamical phase transition between bipolarity and utopia.

Socially Balanced States



unfrustrated/balanced

frustrated/imbalanced

Social Balance Defined

a friend of my friend
an enemy of my enemy } *is my friend;*

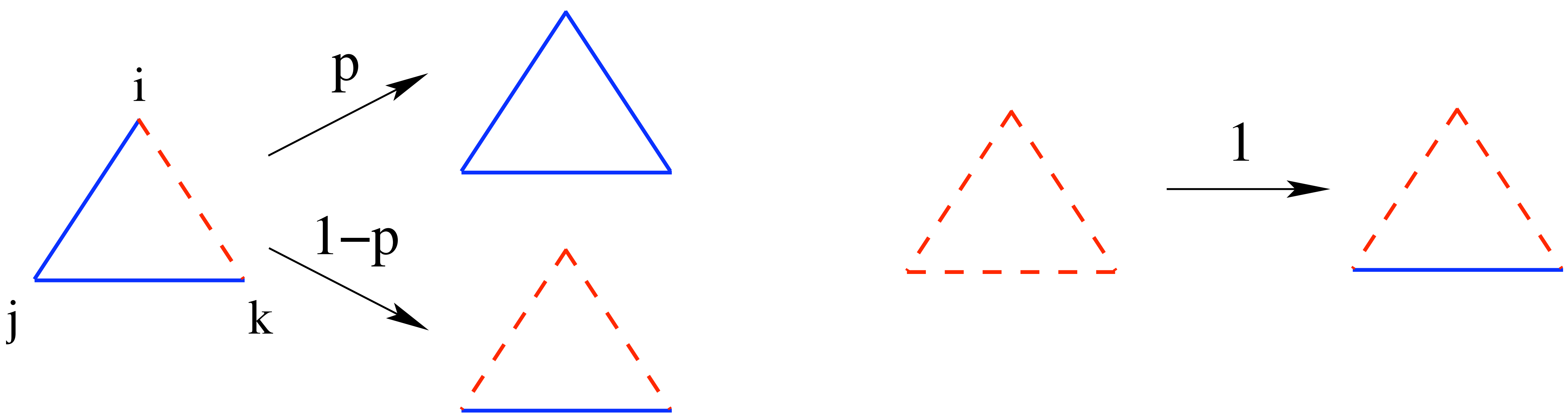
a friend of my enemy
an enemy of my friend } *is my enemy.*

Local Triad Dynamics on Arbitrary Networks

(social graces of the clueless)

1. Pick a random imbalanced (frustrated) triad
2. Reverse a single link so that the triad becomes balanced

free parameter: *probability p: unfriendly → friendly;*
probability 1-p: friendly → unfriendly



Triad Evolution on the Complete Graph

Basic graph characteristics:

N nodes

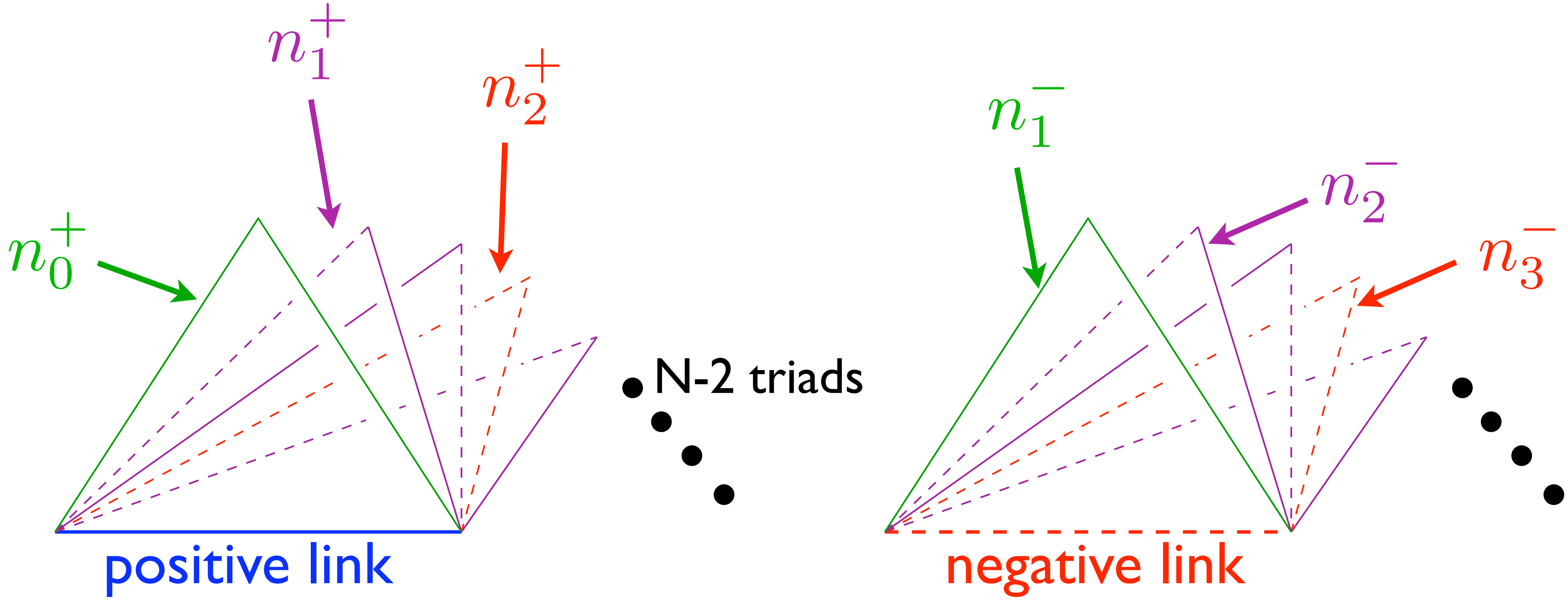
$\frac{1}{2}N(N-1)$ links

$\frac{1}{6}N(N-1)(N-2)$ triads

$\rho =$ friendly link density

$n_k =$ density of triads of type k

$n_k^{\pm} =$ density of triads of type k attached to a $\pm$ link



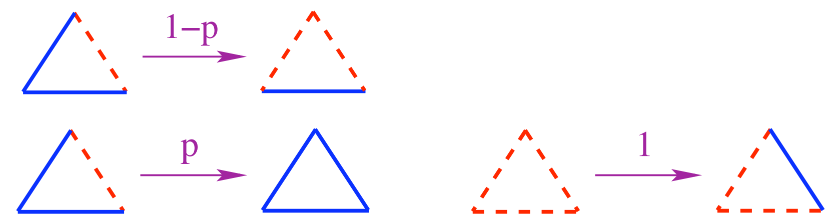
Triad Evolution on the Complete Graph

n_k = density of triads of type k

$n_k^{\pm}$ = density of triads of type k attached to a $\pm$ link

$\pi^{+} = (1-p)\, n_1$ flip rate $+ \;\rightarrow\; -$

$\pi^{-} = p\, n_1 + n_3$ flip rate $- \;\rightarrow\; +$



Master equations:

$\triangle_1 \rightarrow \triangle_0$ $\quad$ $\triangle_0 \rightarrow \triangle_1$

$$
\begin{aligned}
\frac{dn_0}{dt} &= \pi^{-} n_1^{-} - \pi^{+} n_0^{+} \,, \\
\frac{dn_1}{dt} &= \pi^{+} n_0^{+} + \pi^{-} n_2^{-} - \pi^{-} n_1^{-} - \pi^{+} n_1^{+} \,, \\
\frac{dn_2}{dt} &= \pi^{+} n_1^{+} + \pi^{-} n_3^{-} - \pi^{-} n_2^{-} - \pi^{+} n_2^{+} \,, \\
\frac{dn_3}{dt} &= \pi^{+} n_2^{+} - \pi^{-} n_3^{-} \,.
\end{aligned}
$$

Steady State Solution

$$\frac{dn_0}{dt} = \pi^- n_1^- - \pi^+ n_0^+ \,,$$
$$\frac{dn_1}{dt} = \pi^+ n_0^+ + \pi^- n_2^- - \pi^- n_1^- - \pi^+ n_1^+ \,,$$
$$\frac{dn_2}{dt} = \pi^+ n_1^+ + \pi^- n_3^- - \pi^- n_2^- - \pi^+ n_2^+ \,,$$
$$\frac{dn_3}{dt} = \pi^+ n_2^+ - \pi^- n_3^- \,.$$

impose $\dot{n}_i$ and $\pi^+ = \pi^-$

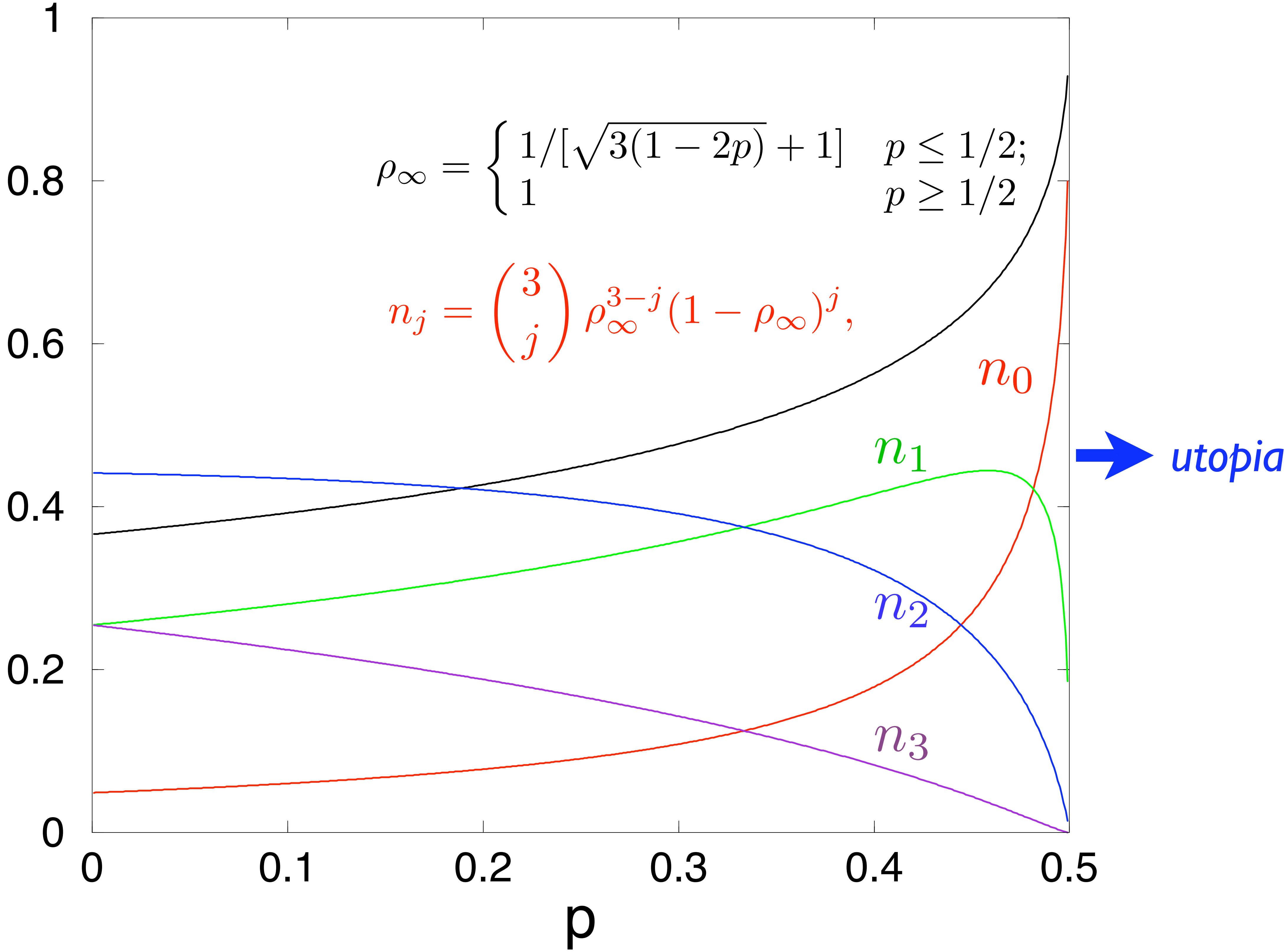
gives $n_k^+ = n_{k+1}^-$

then use $n_k^\pm = \begin{cases} \frac{(3-k)n_k}{3n_0+2n_1+n_2} \\ \frac{kn_k}{n_1+2n_2+3n_3} \end{cases}$ to give

$$n_j = \binom{3}{j} \rho_\infty^{3-j} (1-\rho_\infty)^j, \quad \rho_\infty = \begin{cases} 1/[\sqrt{3(1-2p)}+1] & p \le 1/2; \\ 1 & p \ge 1/2 \end{cases}$$

Steady State Triad Densities

p≤½, steady state;   p>½, utopia



The Evolving State

rate equation for the friendly link density:

$- \to +$ in $\triangle_1$ $\quad + \to -$ in $\triangle_1$ $\quad - \to +$ in $\triangle_3$

$$\frac{d\rho}{dt} = 3\rho^2(1-\rho)[p-(1-p)] + (1-\rho)^3$$

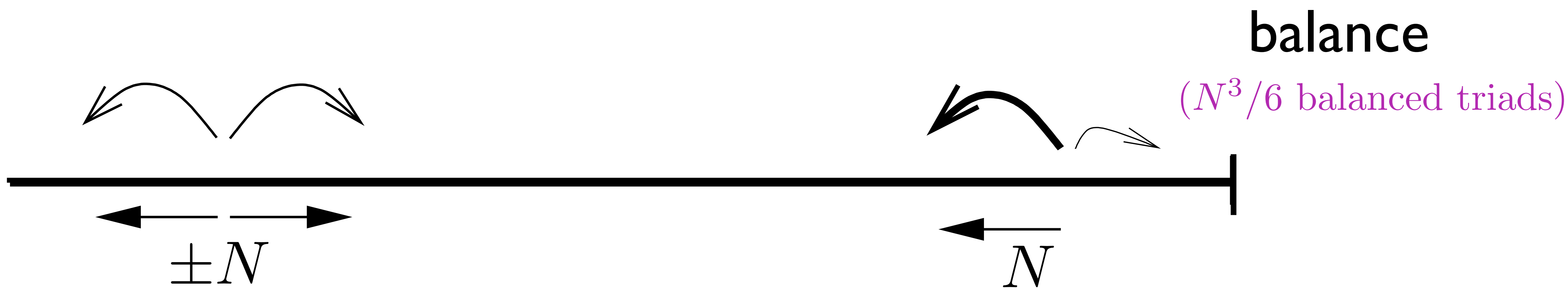
Solution:

$$\rho(t) \sim \begin{cases} \rho_\infty + Ae^{-Ct} & p < 1/2; \\ 1 - \dfrac{1-\rho_0}{\sqrt{1+2(1-\rho_0)^2 t}} & p = 1/2; \\ 1 - e^{-3(2p-1)t} & p > 1/2. \end{cases}$$

- $p < 1/2$: quick frustration
- $p = 1/2$: slow relaxation to utopia
- $p > 1/2$: rapid attainment of utopia

Fate of a Finite Society

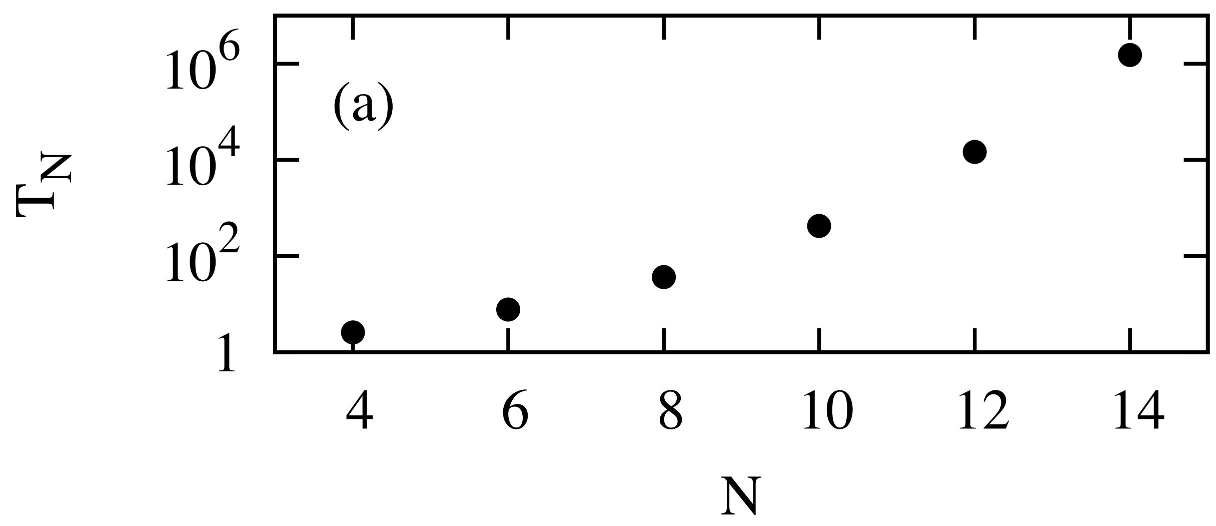
p<1/2: effective random walk picture



$D \propto N^2$ $\qquad$ $v \propto N$

$$\rightarrow T_N \sim e^{v\mathcal{L}_N/D} \sim e^{N^2}$$

Fate of a Finite Society

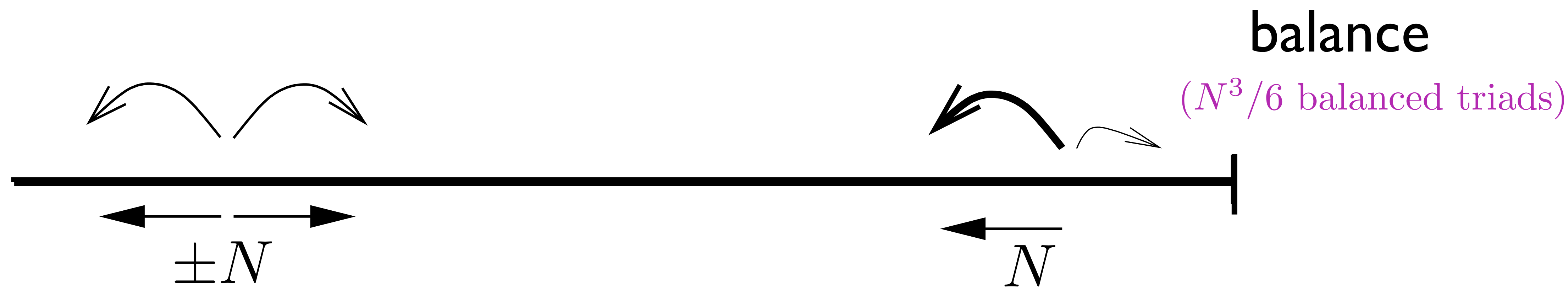


$$p < \frac{1}{2}, \qquad T_N \sim e^{N^2}$$

$$\left(\text{from } T_N = e^{N \cdot N^3/N^2}\right)$$

Fate of a Finite Society

p<1/2: effective random walk picture



$D \propto N^2$ $\quad$ $v \propto N$

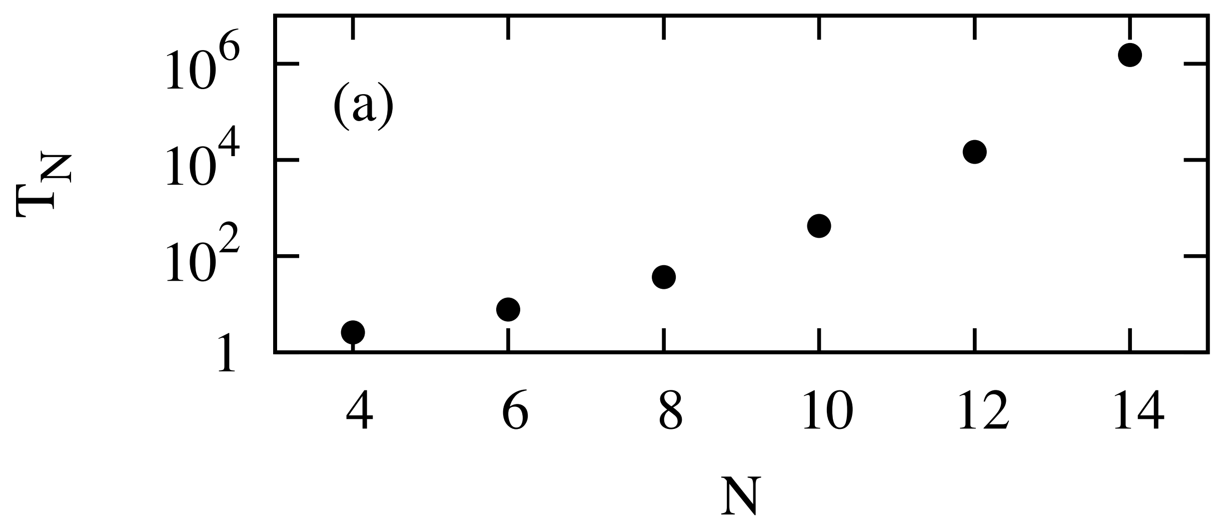
$$\rightarrow T_N \sim e^{v\mathcal{L}_N/D} \sim e^{N^2}$$

p>1/2: inversion of the rate equation

$$u \sim e^{-3(2p-1)t} \approx N^{-2} \quad \rightarrow \quad T_N \sim \frac{\ln N}{2p-1}$$

$u \equiv 1-\rho = \text{unfriendly link density}$

Fate of a Finite Society



$$p < \frac{1}{2}, \qquad T_N \sim e^{N^2}$$

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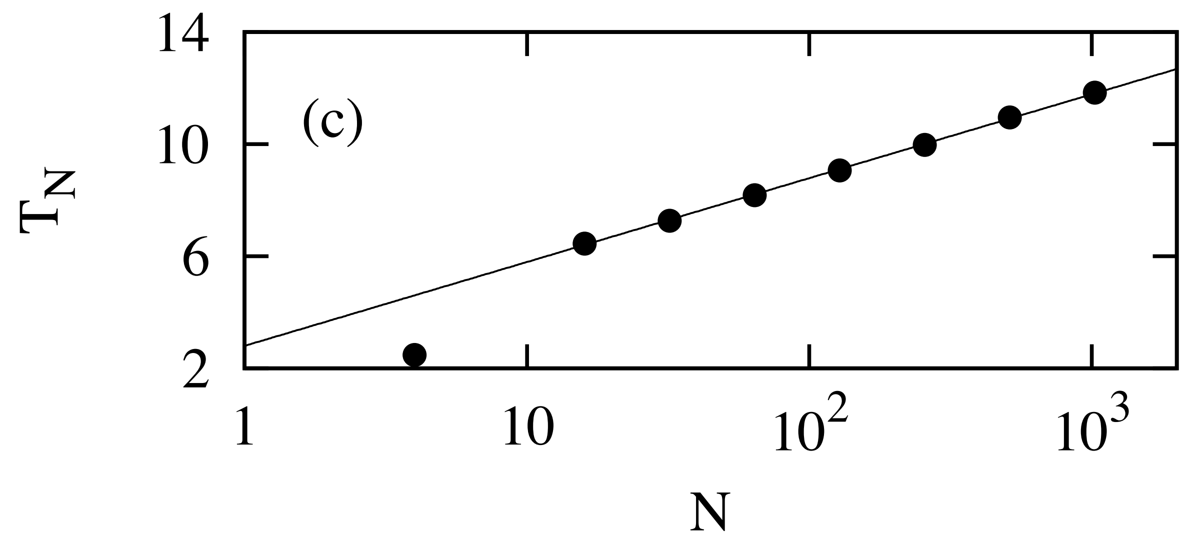


$$p > \frac{1}{2}, \qquad T_N \sim \frac{\ln N}{2p-1}$$

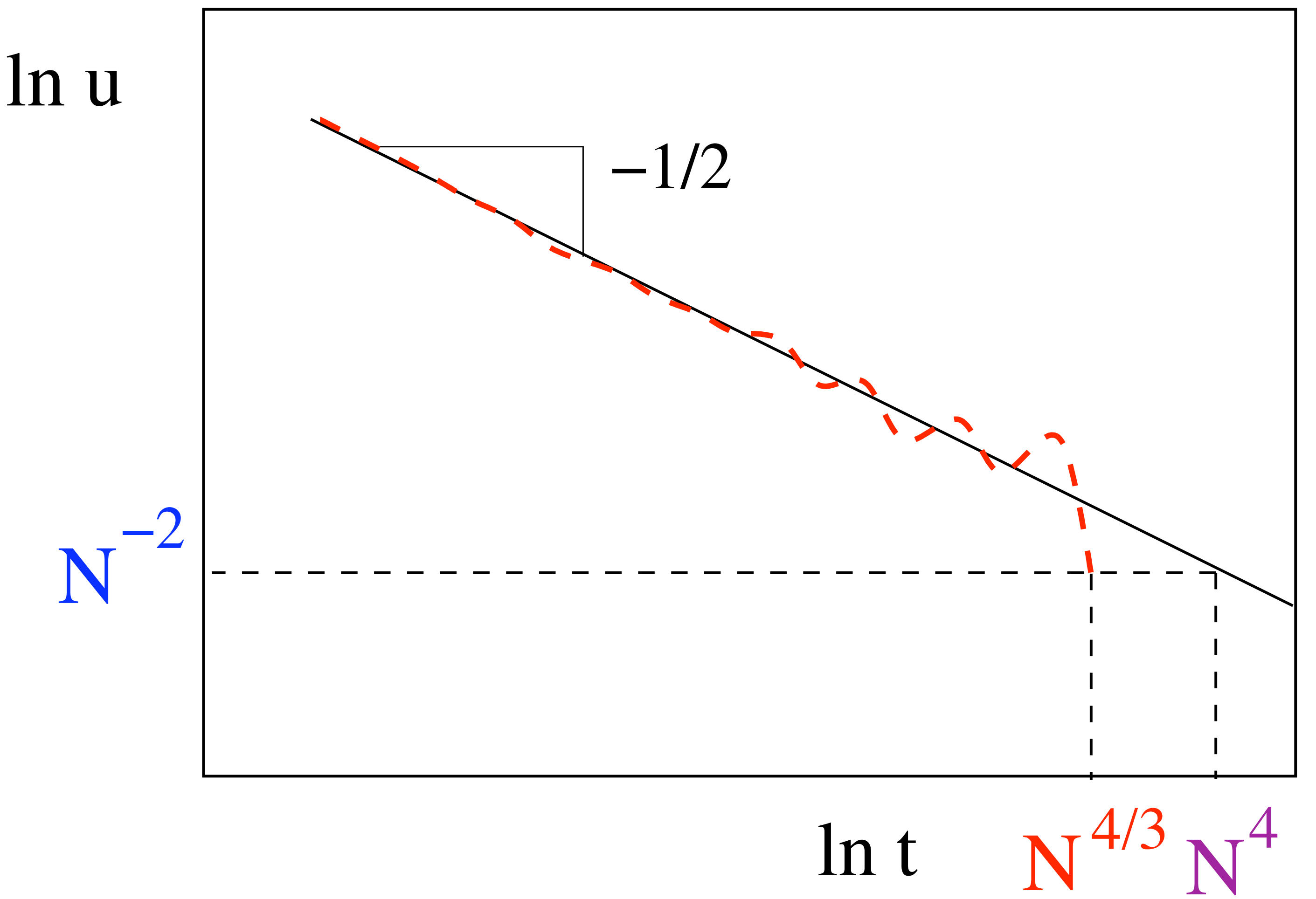
$$\left(\text{from } 1-\rho \sim e^{-3(2p-1)t} = \tfrac{1}{N^2}\right)$$

p=1/2

naive rate equation estimate:

$$u \equiv 1 - \rho \propto t^{-1/2} \approx N^{-2} \qquad \rightarrow \; T_N \sim N^4$$

incorporating fluctuations as balance is approached:

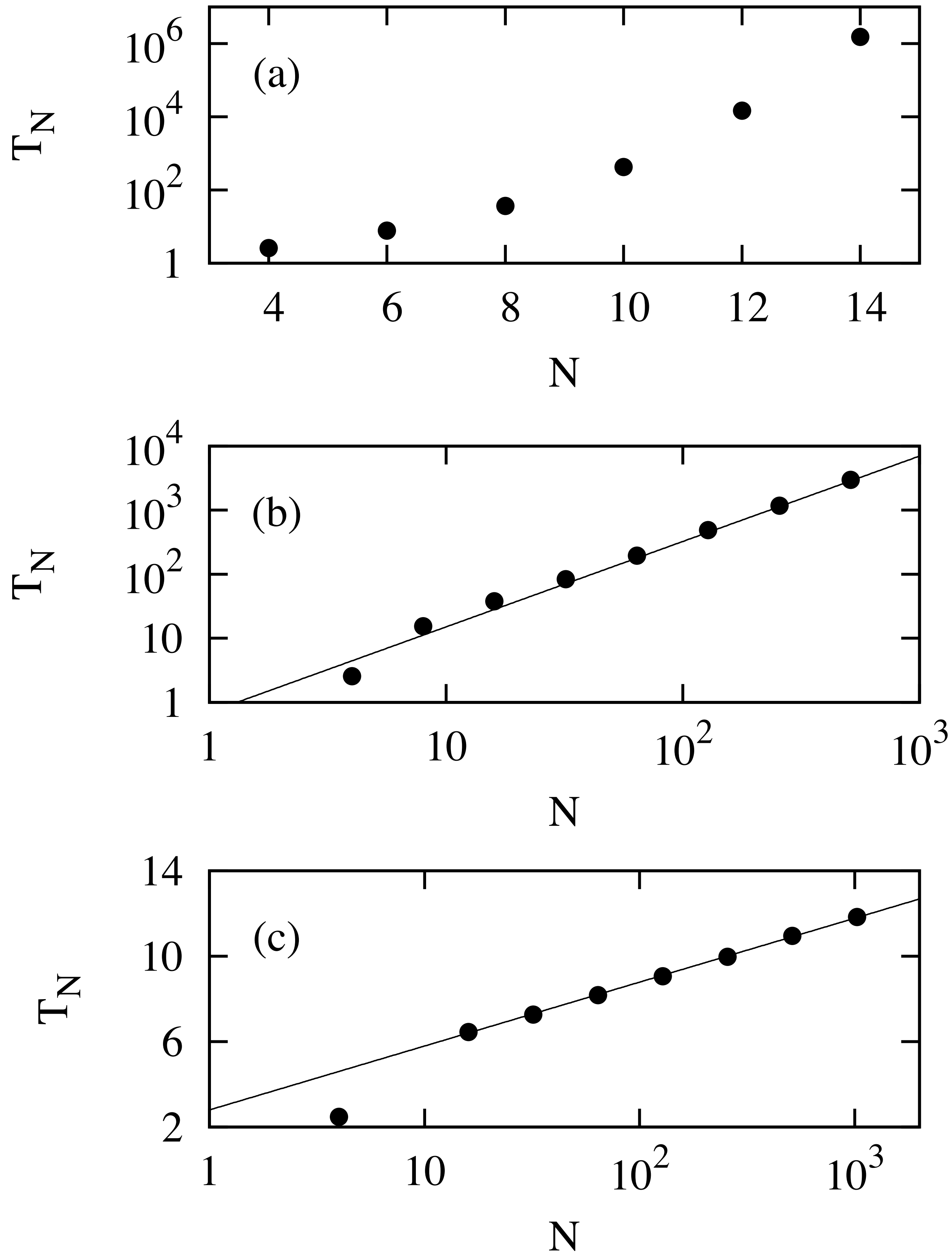


$$\begin{aligned} U &= Lu + \sqrt{L}\,\eta \\ &\sim \frac{L}{\sqrt{t}} + \sqrt{L}\,t^{1/4} \end{aligned}$$

equating the 2 terms in U:

$$T_N \sim L^{2/3} \sim N^{4/3}$$

Fate of a Finite Society



$$p < \frac{1}{2}, \qquad T_N \sim e^{N^2}$$

(from $T_N = e^{N \cdot N^3/N^2}$)

$$p = \frac{1}{2}, \qquad T_N \sim N^{4/3}$$

(from $1 - \rho \sim \frac{1}{\sqrt{t}} = \frac{1}{N^2}$

$+$ fluctuations)

$$p > \frac{1}{2}, \qquad T_N \sim \frac{\ln N}{2p-1}$$

(from $1 - \rho \sim e^{-3(2p-1)t} = \frac{1}{N^2}$)

Constrained (Socially Aware) Triad Dynamics

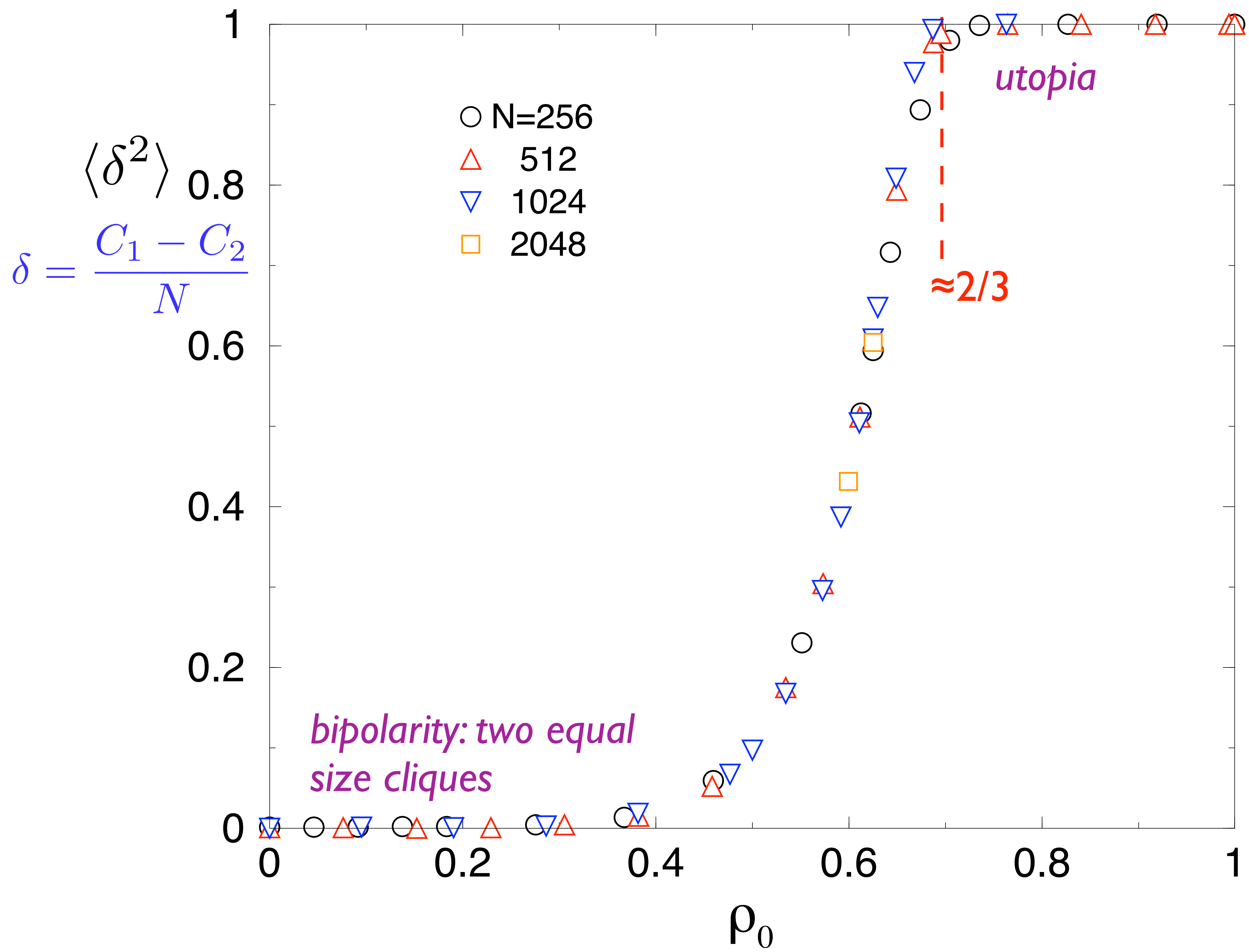
1. Pick a random imbalanced (frustrated) triad.

2. Reverse a random link ($p=1/3$) to eliminate a frustrated triad
only if the total number of frustrated triads does not increase.

only free parameter:
initial density of friendly links ρ_0

Outcome: Quick approach to a final static state.
Typically: $T_N \sim \ln N$

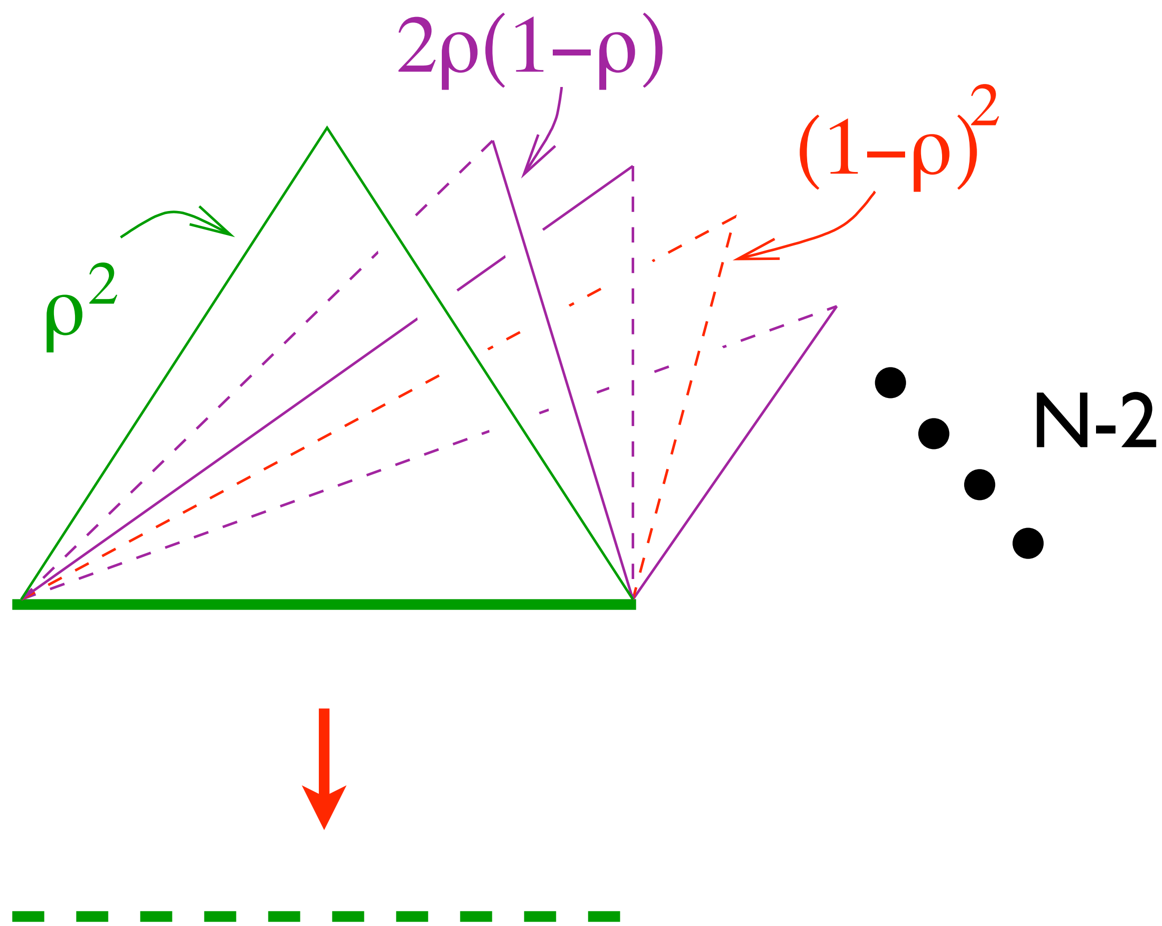
Final Clique Sizes



Bipolarity/Utopia Transition

First consider evolution of an uncorrelated network:

for + → −

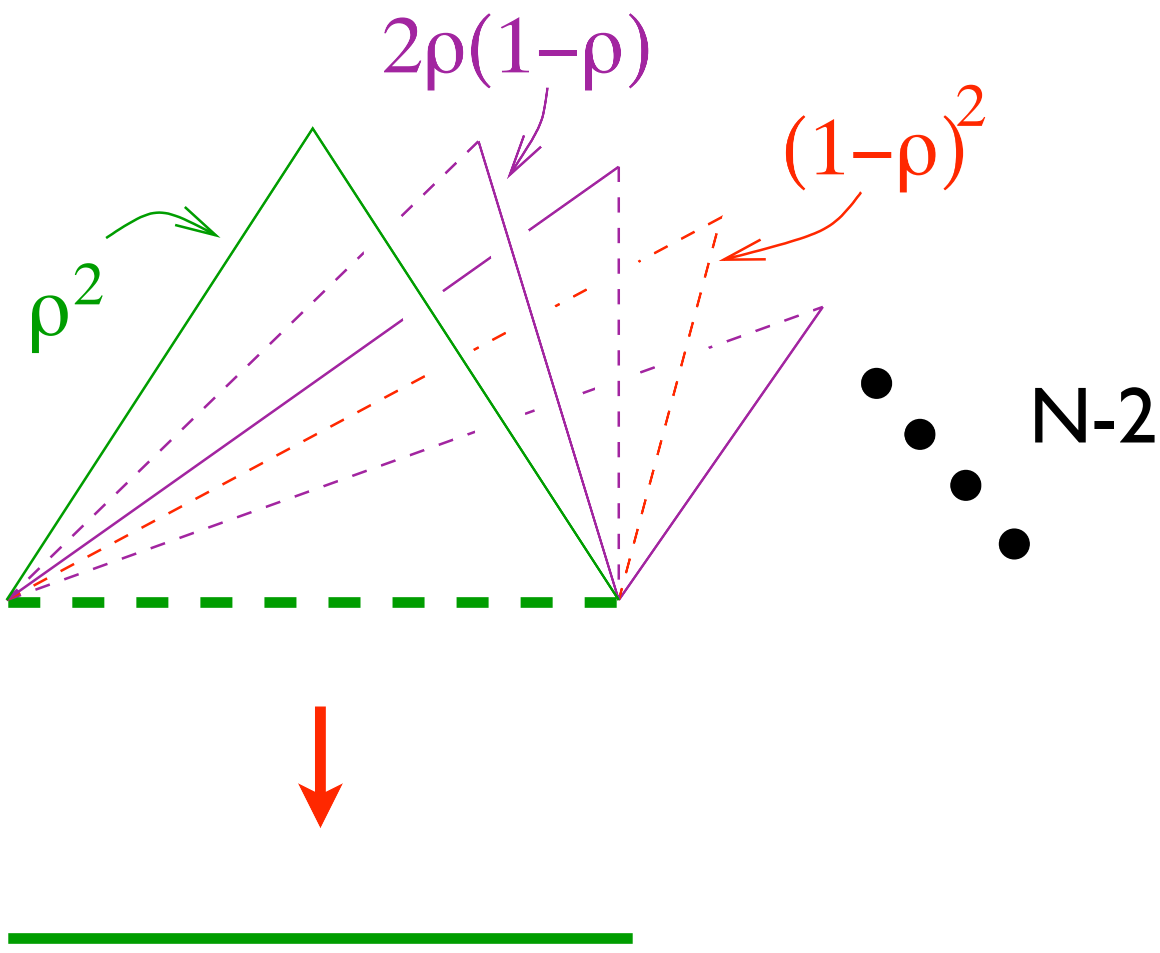


we need:

$$\underbrace{n_1^+ + n_3^+}_{\text{frustrated}} > \underbrace{n_0^+ + n_2^+}_{\text{unfrustrated}}, \ \text{with } \vec{n}_+ = [\rho^2, 2\rho(1-\rho), (1-\rho)^2, 0]$$

$$\to 1 - 4\rho(1-\rho) < 0, \ \text{impossible, so } + \text{ links never flip}$$

for $- \rightarrow +$



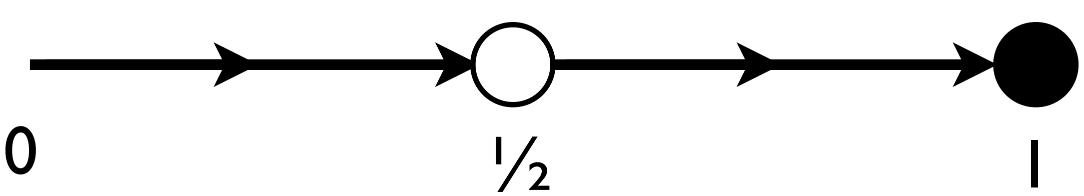
we need:

$$\underbrace{n_1^- + n_3^-}_{\text{frustrated}} > \underbrace{n_0^- + n_2^-}_{\text{unfrustrated}}, \text{ with } \vec{n}_- = [0, \rho^2, 2\rho(1{-}\rho), (1{-}\rho)^2]$$

$$\rightarrow 1 - 4\rho(1-\rho) > 0, \text{ valid when } \rho \neq 1/2$$

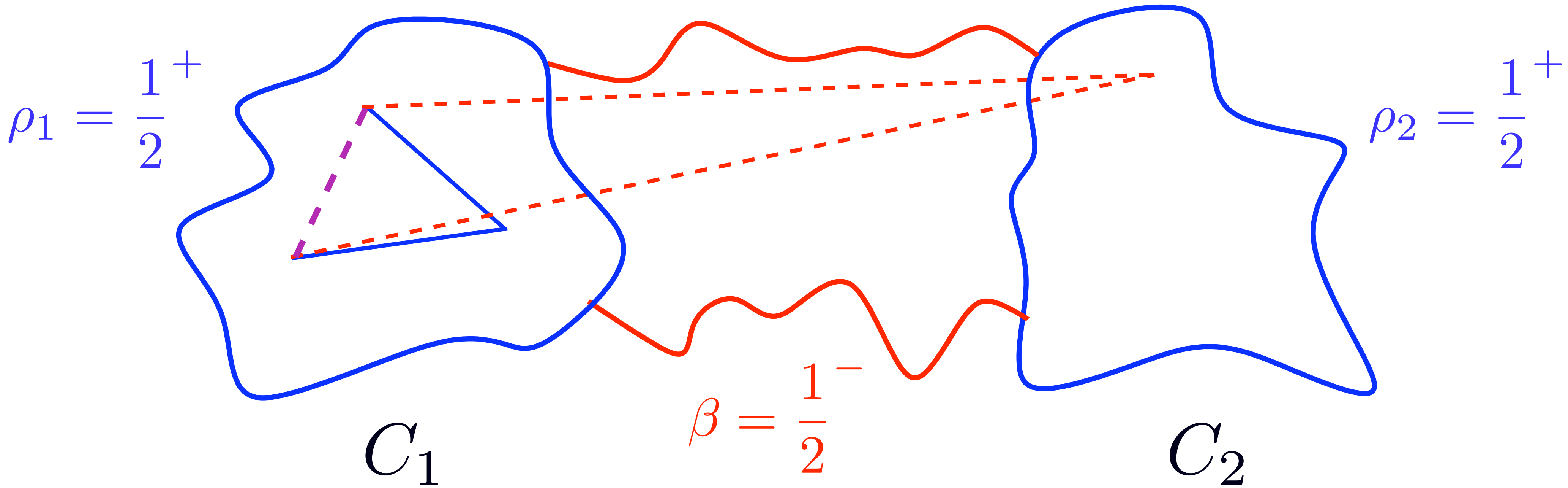
Only negative links flip, except when ρ→½

flow diagram for ρ:



Instability near ρ=½

intraclique relationship evolution



for $-\rightarrow +$ within cliques, we need:

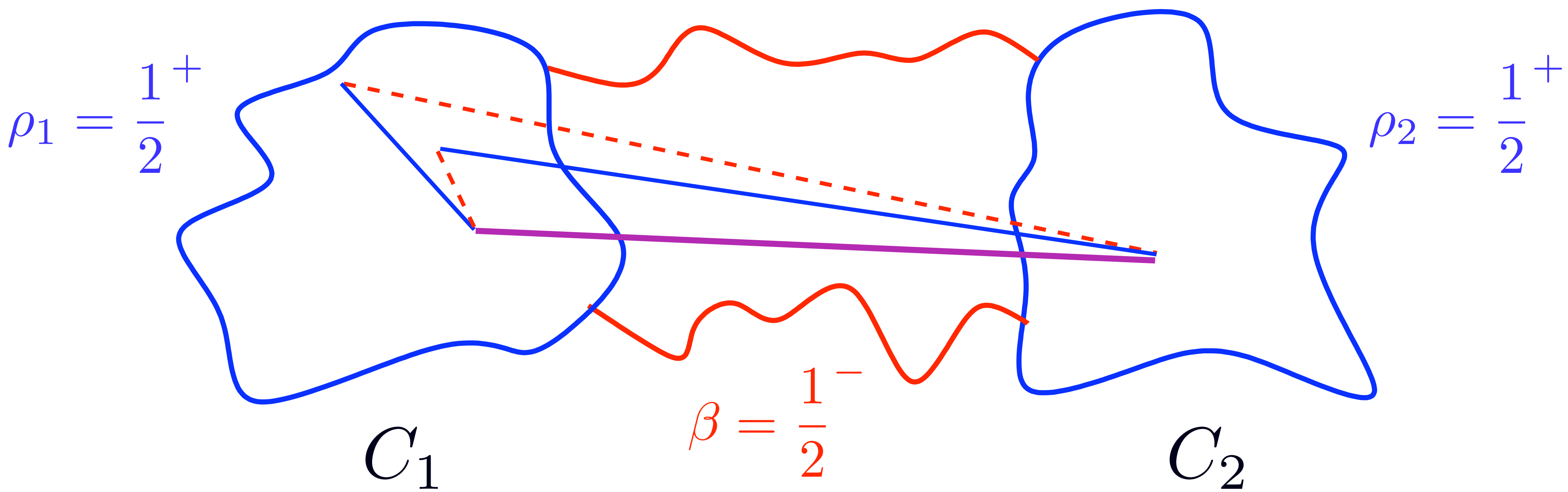
$$\underbrace{n_1^- + n_3^-}_{\text{frustrated}} > \underbrace{n_0^- + n_2^-}_{\text{unfrustrated}}, \text{ with } \vec{n}_- = \begin{cases} [0, \rho_i^2, 2\rho_i(1-\rho_i), (1-\rho_i)^2] & \text{intraclique} \\ [0, \beta^2, 2\beta(1-\beta), (1-\beta)^2] & \text{interclique} \end{cases}$$

$$\rightarrow C_1[1-4\rho_i(1-\rho_i)] + C_2[1-4\beta(1-\beta)] > 0, \text{ always true}$$

negative intraclique links disappear

increased cohesiveness within cliques

interclique relationship evolution



for + → − between cliques, we need:

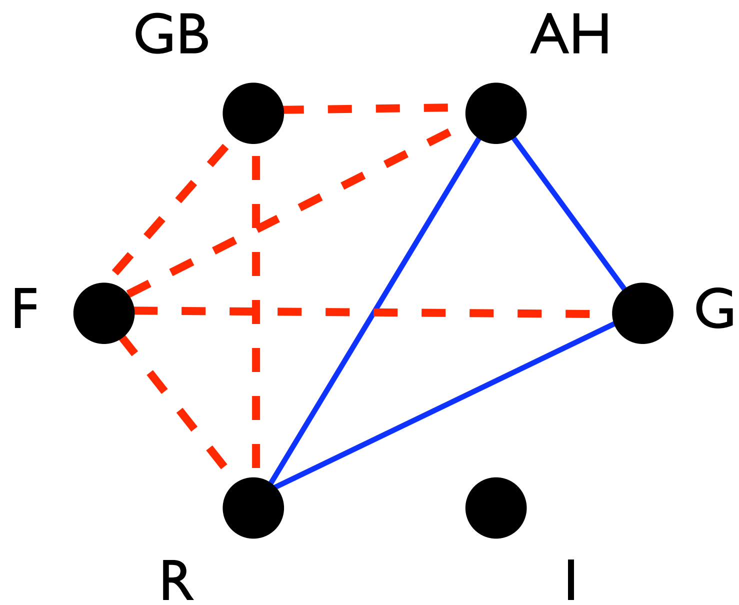
$$\underbrace{n_1^+ + n_3^+}_{\text{frustrated}} > \underbrace{n_0^+ + n_2^+}_{\text{unfrustrated}}, \text{ with } \vec{n}_+ = [\beta\rho_i, \beta(1-\rho_i) + \rho_i(1-\beta), (1-\beta)(1-\rho_i), 0]$$

$$\rightarrow [C_1(2\rho_1 - 1) + C_2(2\rho_2 - 1)](1-2\beta) > 0, \quad \text{true if } \rho_1, \rho_2 > 1/2, \beta < 1/2$$

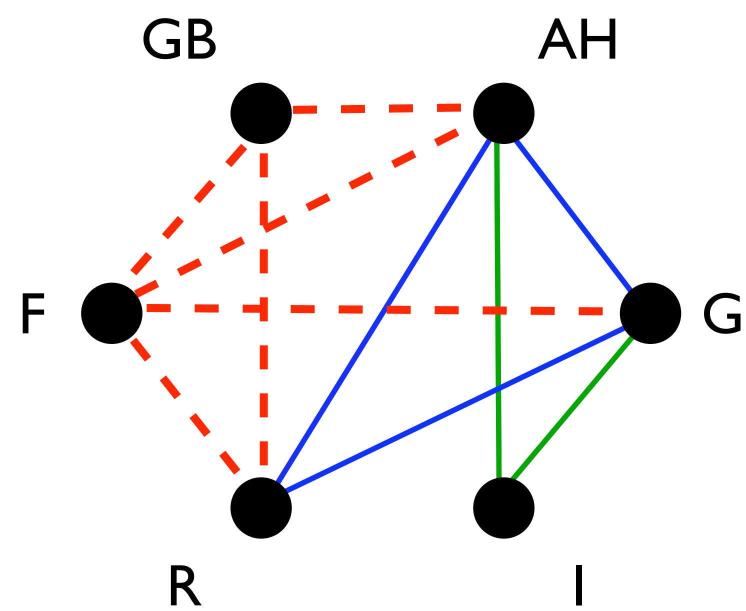
positive interclique links disappear

increased emnity between cliques

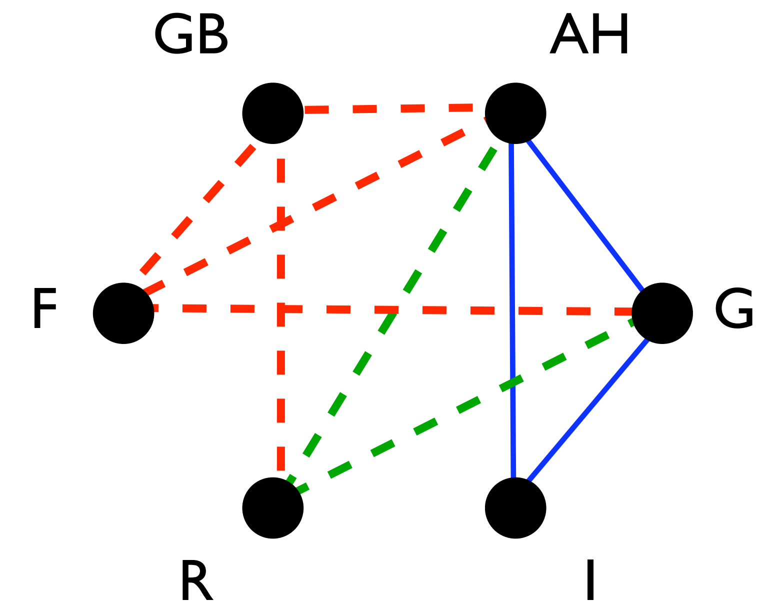
A Historical Lesson



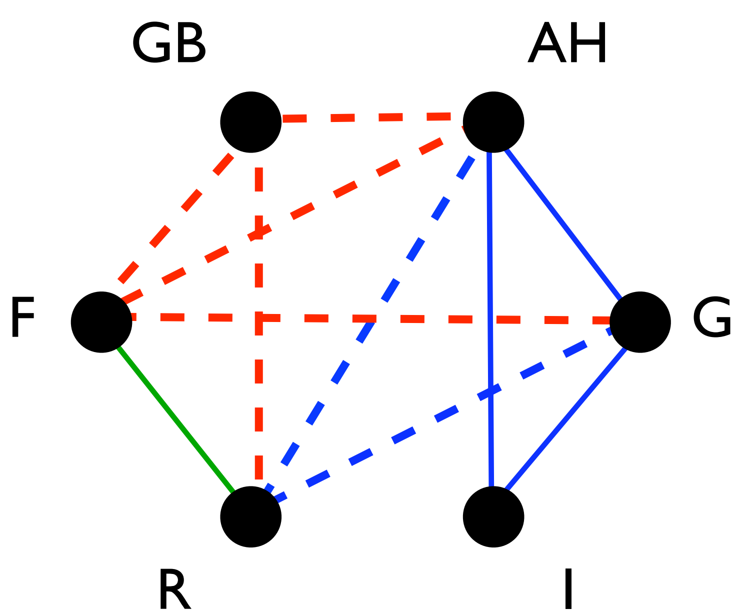
3 Emperor's League 1872-81



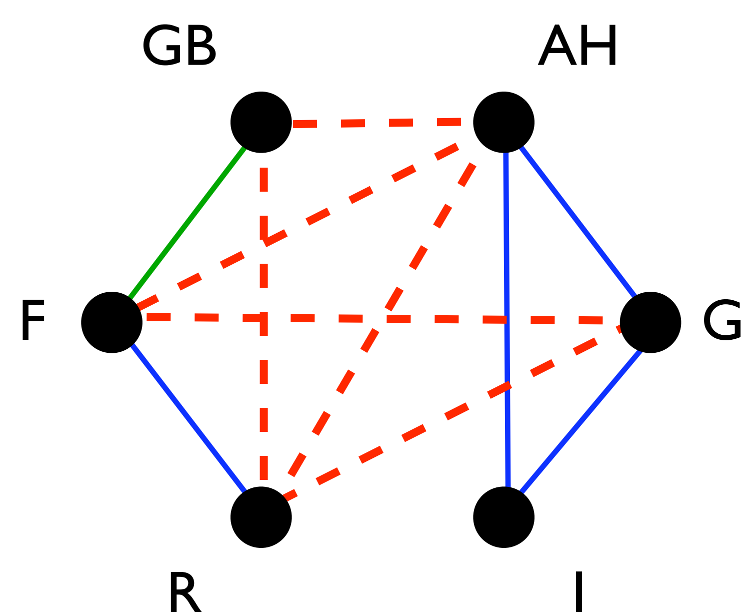
Triple Alliance 1882



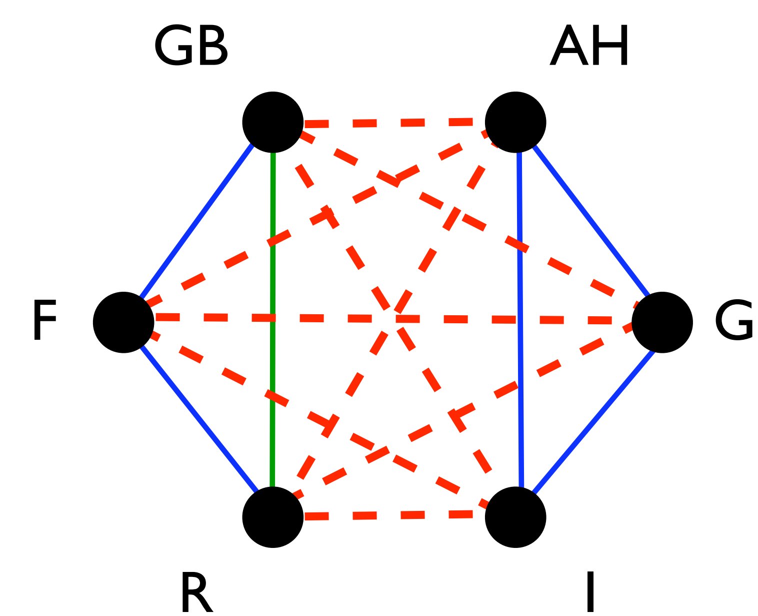
German-Russian Lapse 1890



French-Russian Alliance 1891-94



Entente Cordiale 1904



British-Russian Alliance 1907

Summary & Outlook

If we can't all love each other → *social balance*

Local triad dynamics:

finite network: social balance is achieved, with the time until balance strongly dependent on p

infinite network: phase transition between utopia and bipolarity at p=½

Global triad dynamics (p=⅓):

jammed states possible but never occur

infinite network: two cliques emerge, with utopia when $\rho_0 \cong 2/3$ (rough argument gives $\rho_0 = ½$)

Open questions:

incomplete graphs/indifference, continuous interactions

allow  → Machiavellian society

asymmetric relations