RETURN TIMES FOR GAUSSIAN PROCESSES WITH POWER-LAW SCALING

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EVIDENCE FOR STRETCHED EXPONENTIAL SCALING OF OCCURRENCE TIME PDF'S

- Prob[no zero crossing in time τ] $\sim$ exp$(-\alpha\tau^s)$

  [Newell & Rosenblatt, Ann. Math. Statis. **33**, 1306 (1962)]

  - Rigorous bounds

- Prob[Return time (to $y > q$) $> \tau$] $\sim$ exp$(-\alpha\tau^s)$

  [Bunde et Al., Physica A **330**, 1 (2003)]

  - Numerical results.

The same occurs with the distribution of permanence times S_q during which $y(t) > 1$. Will see if there is time.

PROBLEM SETTING:

- $y(t)$ stationary stochastic process
- Zero mean, unit variance
- Gaussian statistics
- Power-law correlation:

$$C(t) = \langle y(t)y(0)\rangle = A|t|^{-s}, \qquad |t| > \tau_0$$

- Slow decay: $0 < s < 1$.

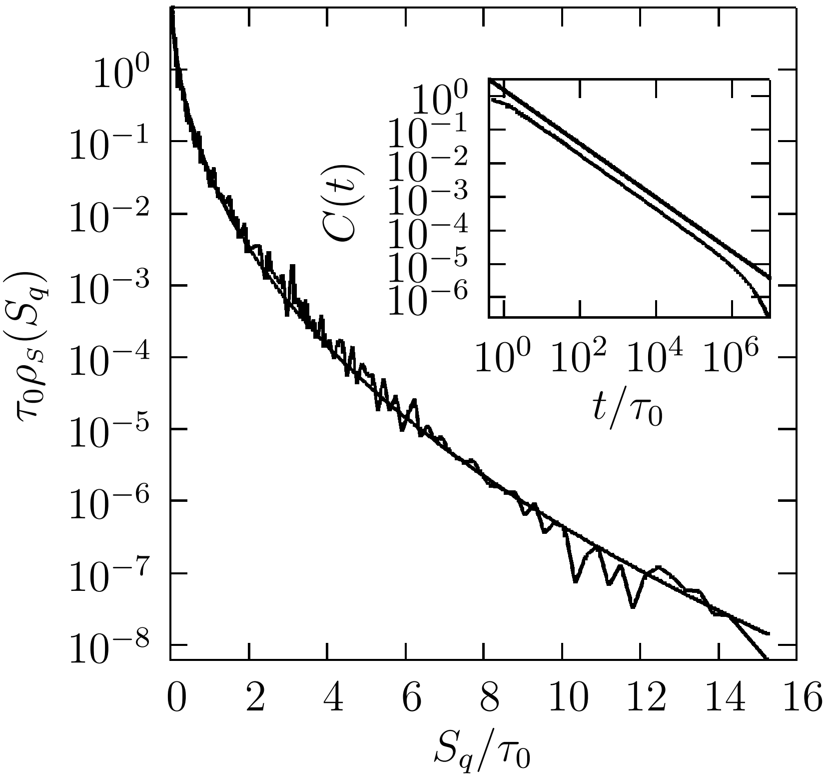
EXTREME EVENT:

$y > q \gg 1$; q fixed.

KAC THEOREM:

$$\frac{\text{Time}(y > q)}{\text{Time}(y < q)} = \frac{\langle \text{Permanence time} \rangle}{\langle \text{Return time} \rangle} \simeq P(y > q)$$

PERMANENCE TIME DISTRIBUTION:



Permanence time PDF from numerical simulation for s= 0.4 and q=3, and the fit $\exp(-7.5 S_q^{0.4})$

From Olla (2006) COND−MAT/0606323

WHAT IS HAPPENING?

$$P(S_q > \tau)P(y > q)$$

$$= P\{y(t) > q, t \in [T, T+\tau]\} < P(y_\tau > q);$$

$$y_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau} y(t')\mathrm{d}t'.$$

But, using the Wiener-Khinchin formula:

$$\langle y_\tau^2 \rangle \sim \int_{|\omega\tau|<1} C_\omega \mathrm{d}\omega \sim \int_{|\omega\tau|<1} \omega^{s-1}\mathrm{d}\omega \sim \tau^{-s}$$

HENCE:

$$P(y_\tau > q) \sim \exp\left(-\frac{q^2}{2\langle y_\tau^2 \rangle}\right) \sim \exp(-\alpha\tau^s)$$

That was easy...

...but what about the return times R_q?

Consider simpler (but relevant) case of discrete time sampling:

$$t \to t_n = n\Delta, \quad y(t) \to y_n = y(t_n); \quad \Delta > \tau_0.$$

We can then write:

$$P(R_q > t_n) = \prod_{k=1}^{n} (1 - p_k),$$

$$p_k = \text{Prob[exit at } t_k|\text{first exit; } y_0 = q]$$

and the main role will be played by PDF's in the form:

$$\rho_k(n) = \rho(y_n|y_i < q, i = 1, 2, ...k - 1; y_0 = 1).$$

In particular:

$$p_n = \int_q^\infty \rho_n(n) \mathrm{d}y_n$$

WE CAN ESTABLISH A RECURSION RELATION

$$\rho_{n+1} = \rho_n(n+1|y_n < q) = \frac{\rho_n(n+1; y_n < q)}{1 - p_n}$$

$$= \frac{1}{1-p_n}\Big[\rho_n(n+1) - p_n \rho_n(n+1|y_n > q)]$$

$$\simeq \rho_n(n+1) + p_n[\rho_n(n+1) - \rho_n(n+1|y_n = q)],$$

which we can iterate:

$$\rho_n(n) \simeq \rho_1(n) + \sum_{k=1}^{n-1} p_k[\rho_k(n) - \rho_k(n|k)],$$

where $\rho_k(n|k)$ is the PDF for y_n conditioned to a first exit at t_k.

HIERARCHICAL STRUCTURE:

$$\rho_n(n) \simeq \rho_1(n) + \sum_{k=1}^{n-1} p_k[\rho_k(n) - \rho_k(n|k)],$$

$$\rho_n(n|k) \simeq \rho_1(n|k) + \sum_{l=1}^{k-1} p_l(k)[\rho_l(n|k) - \rho_l(n|k,l)],$$

...

PERTURBATIVE APPROACH:

To lowest order: $\rho_n(n) = \rho_1(n)$.

For $n \to \infty$: $\rho_n(n) \to \text{const.}\,\exp(-y^2/2)$.

$p_n \to \bar{p}_\infty = \text{const.} \int_q^\infty \exp(-y^2/2)\mathrm{d}y; \qquad \bar{p}_\infty > 0$

Hence, for $n \to \infty$:

$$P(R_q > t_n) \to (1 - \bar{p}_\infty)^n$$

EXPONENTIAL SCALING!

NEXT ORDER:

$$\rho_n(n) = \rho_1(jn) + \sum_{k=1}^{n-1} p_k[\rho_1(n) - \rho_1(n|k)]$$

Take $\int_q^\infty \mathrm{d}y_n$ at both sides to obtain an equation for p_n; for $n \to \infty$:

$$p_n = \bar{p}_\infty + \sum_{k=1}^{n-1} p_k[\rho_1(n) - \int_q^\infty \mathrm{d}y_n \rho(y_n|y_k = q)]$$

$$= \bar{p}_\infty - \text{const.} \sum_{k=1}^{n-1} p_k t_{n-k}^{-s}$$

BARE PERTURBATION THEORY, i.e. $p_k = \bar{p}_\infty$:

$$p_n = \bar{p}_\infty \left[1 - \text{const.} \sum_{k=1}^{n-1} t_{n-k}^{-s} \right]$$

DIVERGENCE FOR $s < 1$!

TRY ϵ-EXPANSION TO RENORMALIZE THEORY, WITH $\epsilon = 1 - s$.

$$p_n = \bar{p}_\infty - c \sum_{k=1}^{n-1} p_k t_{n-k}^{-1+\epsilon}$$

$$= \bar{p}_\infty - \frac{c p_k}{\epsilon \Delta^{1-\epsilon}} (n^\epsilon - 1) + O(\epsilon^0)$$

HENCE TO LEADING ORDER IN ϵ:

$$p_n \simeq \frac{\bar{p}_\infty}{1 + \frac{c}{\epsilon \Delta^{1-\epsilon}} (n^\epsilon - 1)} \to n^{-\epsilon}$$

THIS WILL PRODUCE STRETCHED EXPONENTIAL SCALING OF PDF FOR R_q:

$$P(R_q > t_n) = \prod_{k=1}^{n} (1 - p_k) \simeq \prod_{k=1}^{n} (1 - a k^{-\epsilon})$$

$$\simeq \exp(- \sum_{k=1}^{n} a k^{-\epsilon}) \simeq \exp(-b n^s)$$

Then, stretched exponential can be associated with divergenct contributions from the far past to the exit probability p_n.

IMPORTANT POINT:

The stretched exponential scaling of $P(R_q > t)$ is not due to details in the tails of $\rho(y)$. It is produced by the scaling of the conditional mean:

$$\langle y(t_n)|y(t_k) = q\rangle = qC(t_{n-k}) = qA|t_{n-k}|^{-s}$$

We can repeat the calculations:

$$\rho_n(n) \simeq \rho_1(n) + \sum_{k=1}^{n-1} p_k[\rho_k(n) - \rho_k(n|k)]$$

take the mean:

$$\mu_n(n) \simeq \mu_1(n) + \sum_{k=1}^{n-1} p_k[\mu_k(n) - \mu_k(n|k)]$$

and the last term scales like t_{n-k}^{-s}. It is the same behavior obtained for the exit probability.

Good reason to proceed again by ϵ-expansion:

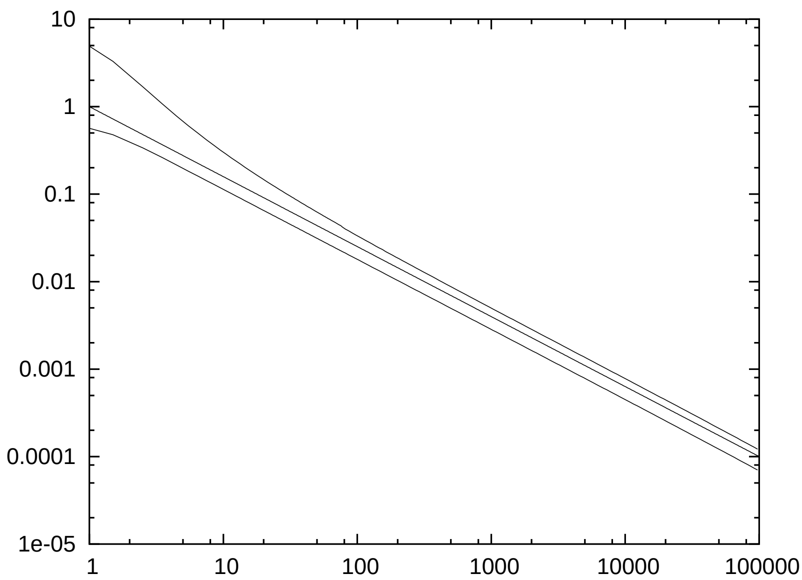
$$\mu_n(n) = \mu_1(n) - \frac{p_{n-1}}{\epsilon\Delta^{1-\epsilon}}(n^\epsilon - 1)$$

If $p_n \to n^{-\epsilon}$, $\mu_n(n) \to \bar{\mu}_\infty < 0$: a negative constant.

RENORMALIZED PERTURBATION CALCULATION FOR EXIT STATISTICS:

It is possible to prove that only μ requires renormalization

The scaling of $\mu_k(n|k) \simeq \langle y_n|y_k = q\rangle$ (lower curve) is cleaner than that for $P(y_n > q|y_k = q) - \bar{p}_\infty$ (upper curve):



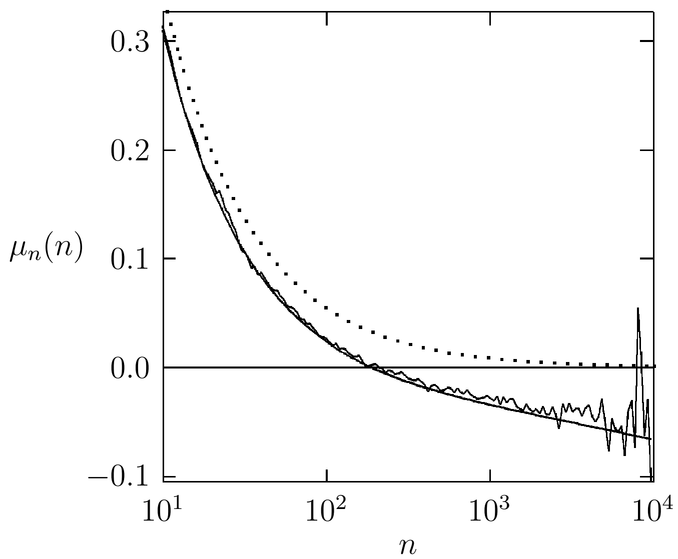
Calculating p_n from the renormalized expression for μ may be therefore a convenient strategy

In fact, we calculate $p_n = \int_q^\infty \rho_n(n)$ from

$$\rho_n(n) = (2\pi)^{-1/2} \exp(-(1/2)[y - \mu_n(n)]^2$$

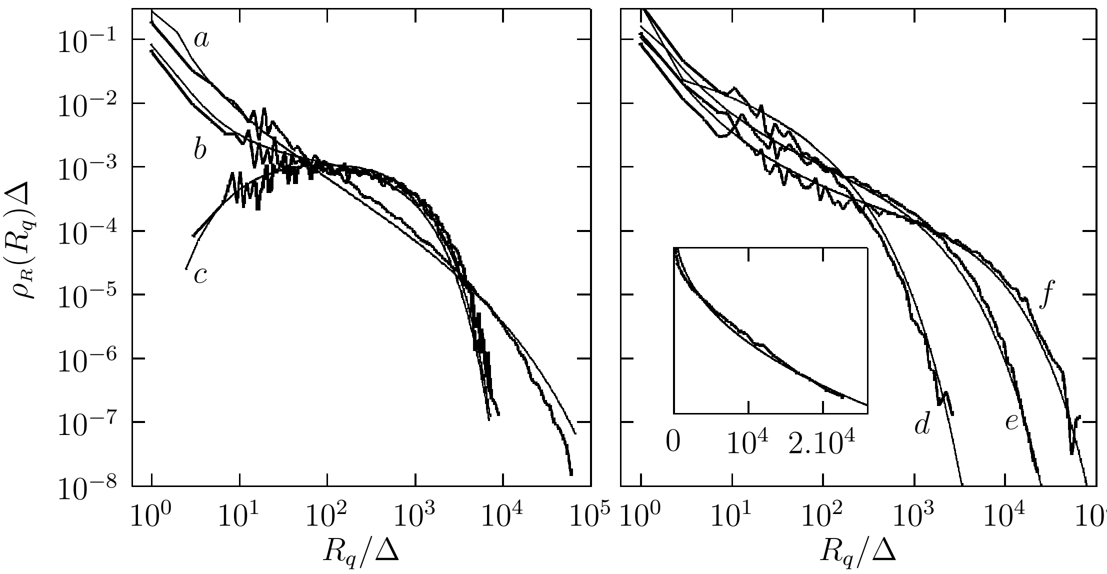
with the renormalized expression $\mu_n(n) = \mu_1(n) - \frac{p_{n-1}}{\epsilon \Delta^{1-\epsilon}}(n^\epsilon - 1)$.

This is the result, for $q = 3$, $s = 0.4$



Dotted line bare theory, continuous renormalized, from Olla (2006) cond-mat/0606323

This is the result for the return time PDF's. Theory vs. numerics:



From Olla (2006) cond-mat/0606323.

On the left $q = 3$; a: $s = 0.2$, b: $s = 0.8$; c: $s = 0.8$ with exit at $-q$. On the right: $s = 0.4$; d: $q = 2.5$, e: $q = 3$, f: $q = 3.5$. Insert: stretched exponential fit for e.

CONCLUSIONS

- Stretched exponential scaling is a characteristic of both the return and permanence time distribution.

- Stretched exponential scaling of the return time distribution is associated with bulk properties of the statistics (conditional mean).

- The identity of the exponent in the stretched exponential and the correlation $C(t)$ is associated with the fact that for a Gaussian: $\langle y(t)|y(0) = q\rangle = qC(t)/\sigma_y^2$.

- Analytical calculations of return time PDF's for finite values of the argument can be carried on as an expansion in $1 - s$ and compare in excellent way with the result of numerical simulations.