First-Passage Problems in Systems with Many Degrees of Freedom

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- You are swimming in a pool of alligators. What is the probability density, $P(t)$, to be eaten ("for the first time") at time t? What is the probability, $Q(t)$, to survive until time t? Note that these are basically the same question, since $P(t) = -dQ/dt$.

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What is the probability, $Q(x,t)$, that a walker *starting* at $x > 0$ has not yet reached zero at time t?

$$\frac{\partial Q}{\partial t} = D\frac{\partial^2 Q}{\partial x^2} \quad \text{(BFPE)}$$

Boundary conditions: $Q(0,t) = 0$, $Q(\infty,t) = 1$, all t
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Solution:

$$Q(x, t) = \mathrm{erf}\left(\frac{x}{\sqrt{4Dt}}\right) \propto t^{-1/2}, \qquad t \gg x^2/D$$

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$Q(x, v, t)$ is the probability that a particle *starting* at $x > 0$ with velocity v has not yet reached the absorbing boundary at $x = 0$ at time t.

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Boundary Condition: $Q(0, v, t) = 0$ for all $v < 0$ and all t.

$$Q(x, v, t) = F\left(\frac{x^{2/3}}{D^{1/3}t}, \frac{v^3}{Dx}\right) \sim_{t\to\infty} \left(\frac{x^{2/3}}{D^{1/3}t}\right)^{\theta} G\left(\frac{v^3}{Dx}\right)$$

Obtain second-order ordinary differential equation (Kümmer's equation) for $G(z)$ in which θ appears as a parameter. The boundary conditions fix the value of θ:

$$\theta = \frac{1}{4}$$

(McKean, Sinai, Burkhardt)

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n	θ
1	1/2
2	1/4
3	0.2202
4	0.2096
5	0.2041
6	0.2008
∞	0.1875

The $n \to \infty$ limit is related to an entirely different problem – diffusion in two dimensions (see later).

All these processes are Gaussian, so everything, including the exponent θ, in determined by the two-time correlator, $C(t_1, t_2) = \langle x(t_1)x(t_2)\rangle$.

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$$\theta = (2-d)/2, \qquad d < 2$$

For dimensions $d > 2$ there is a non-zero probability that the walker never hits the absorbing region (borderline dimension signalling recurrence-transience transition of a random walker). The process is *recurrent* for $d < 2$, and *transient* for $d > 2$.

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- Random Acceleration: $\ddot{\mathbf{x}} = \eta(t)$

Is there a borderline dimension? If so, what is it?

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The following class of *non-Gaussian* processes is also exactly soluble:

$$\dot{y} = \eta(t)$$

$$\dot{x} = \begin{cases} v_+ \, y^\alpha, & y > 0 \\ -v_- \, (-y)^\alpha, & y < 0 \end{cases}$$

which corresponds to a particle diffusing (random walk) in the y direction under a deterministic (and asymmetric) "shear flow" in the x-direction. There is an absorbing boundary at $x = 0$.

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Using the same method (after Burkhardt) as for the random acceleration problem (which corresponds to $\alpha = 1$ and $v_+ = v_-$) gives

$$\theta = \frac{1}{4} - \frac{1}{2\pi\beta} \tan^{-1}\left[\frac{\gamma - 1}{\gamma + 1} \tan\left(\frac{\pi\beta}{2}\right)\right] \qquad (\mathrm{AB} + \mathrm{Gonos(2004)})$$

where $\gamma = (v_+/v_-)^\beta$ and $\beta = 1/(2+\alpha)$.

For $v_+ = v_-$, the "universal" result $\theta = 1/4$ is obtained, independent of α.

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We consider the probability $Q(p,t)$ for the process to survive to time t *given that it survives each zero-crossing with probability* p. It has the form

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\begin{aligned}
Q(p,t) &= \sum_{n=0}^{\infty} p^n P_n(t) \sim t^{-\theta(p)}, \qquad t \to \infty \\
\Rightarrow \ln Q(p,t) &= \sum_{r=1}^{\infty} \frac{(\ln p)^r}{r!} \langle n^r \rangle_c \sim -\theta(p) \ln t, \qquad (1)
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$$\theta(p) = \frac{1}{4} - \frac{1}{2\pi\beta} \sin^{-1}\left(\sqrt{A}\sin\left(\frac{\pi\beta}{2}\right)\right), \text{ where}$$

$$A = \frac{2p^2\cos^2\left(\frac{\pi\beta}{2}\right) + 2\sinh^2\left(\frac{1}{2}\ln\gamma\right)}{\cos(\pi\beta) + \cosh(\ln\gamma)},$$

and $\gamma = (v_+/v_-)^\beta$, $\beta = 1/(2+\alpha)$. (Majumdar and AB, 2006, in preparation)

"Persistence" in Spatially Extended Systems

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"Simple" example: The Diffusion Equation

$$\frac{\partial \phi}{\partial t} = D\nabla^2\phi$$

with random (white noise) initial conditions,

$$\langle \phi(\mathbf{x},0)\rangle = 0; \quad \langle \phi(\mathbf{x},0)\phi(\mathbf{x}',0)\rangle = \Delta\delta^d(\mathbf{x}-\mathbf{x}')$$

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The normalized one-point, two-time correlator is

$$C(t_1,t_2) = \frac{\langle \phi(\mathbf{x},t_1)\phi(\mathbf{x},t_2)\rangle}{[\langle \phi^2(\mathbf{x},t_1)\rangle \langle \phi^2(\mathbf{x},t_2)\rangle]^{1/2}} = \left(\frac{4t_1 t_2}{(t_1+t_2)^2}\right)^{d/4} \quad \text{(scaling form)}$$

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Introduce a new time variable $T = \ln t$. Then $C(t_1,t_2) \to C(T)$, where $T = T_2 - T_1$ and

$$C(T) = \mathrm{sech}^{d/2}(T/2) \quad \text{(Gaussian Stationary Process)}$$

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- This distribution can be determined from the correlator of the *clipped* process, $\mathrm{sgn}\, X(T)$, with correlator

$$A(T) = \langle \mathrm{sgn}\, X(T)\, \mathrm{sgn}\, X(0)\rangle = \frac{2}{\pi} \sin^{-1} C(T) \quad \text{(true for } \textit{any} \text{ GSP)}$$

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- The "persistence", $Q(t) \sim t^{-\theta}$, is the fraction of space in which the field has not changed sign up to time t.
- This approach has recently been generalised to non-crossing of an arbitrary level (Sire, cond-mat/0606145).

Diffusive Persistence: Results

d	θ_{IIA}	θ^*
1	0.1203	0.1205
2	0.1862	0.1875
3	0.2358	0.2382

*From the “correlator expansion”

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Coarsening Dynamics of other Scalar Fields

- Two-dimensional Ising model at zero temperature ($\theta \approx 0.22$).
- Two-dimensional Ginzburg-Landau equation: $\partial_t \phi = \nabla^2 \phi + \phi - \phi^3$ ($\theta \approx 0.18$).
- One-dimensional Potts model (Derrida, Hakim, Pasquier, 1995):

$$\theta = -\frac{1}{8} + \frac{2}{\pi^2}\left[\cos^{-1}\left(\frac{2-q}{\sqrt{2}q}\right)\right]^2 \qquad (q \;=\; \text{number of Potts states})$$

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The probability that two points separated by r both belong to the persistent set has the form

$$P(r,t) = t^{-2\theta} f(r/t^{1/2})$$

where $f(x) \to \mathrm{const.}$ for $x \gg 1$, and $f(x) \sim x^{-2\theta}$ for $x \ll 1$. In the latter regime

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The probability for the second point to be in the set *given that the first point is in the set* is therefore

$$R(r,t) \sim r^{-2\theta} \quad (r \ll t^{1/2})$$

The set therefore has a fractal structure, on scales smaller that $t^{1/2}$, with fractal dimension

$$d_f = d - 2\theta$$

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Define $Q(t)$ to be the fraction of A particles that survive till time t, which is also the probability that a randomly chosen A-particle survives till time t.

Rate equation approach:

$$\frac{dQ}{dt} = -\lambda\rho_B Q \Rightarrow Q(t) = \exp(-\lambda\rho_B t)$$

However, spatial fluctuations are important for $d \leq 2$ (reactions are diffusion-limited).

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Exact asymptotic forms (Bramson and Lebowitz 1988):

$$Q(t) \sim \begin{cases} \exp(-\lambda_d \; t^{d/2}), & d < 2 \\ \exp(-\lambda_2 t/\ln t), & d = 2 \\ \exp(-\lambda_d t), & d > 2 \end{cases}$$

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$$Q(t) \leq Q_U(t) = Q_{TAP}(t) = exp\left(-\frac{4}{\sqrt{\pi}}\,\rho_B(D_B t)^{1/2}\right)$$

where $Q_{TAP}(t)$ is the result for the "Target Annihilation Problem", corresponding to $D_A = 0$ (static A-particle with mobile traps).

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For general dimensions $d \leq 2$ the results have the Bramson-Lebowitz forms

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with

$$\lambda_d = \begin{cases} \frac{2}{\pi d}\sin\left(\frac{\pi d}{2}\right)(4\pi D_B)^{d/2}\rho_B, & d < 2 \\ 4\pi D_B \rho_B, & d = 2 \end{cases}$$

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(Pascal, Pensées, fragment 139 (1670)).

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"*All the misfortune of man comes from the fact that he does not stay peacefully in his room.*"

The Target Annihilation Problem (d=1)

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A B-particle (trap), starting at x, has not yet reached the target (located at $x = 0$) at time t with probability

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The probability that *none* of N traps has reached the target is

$$Q_{TAP}(t) = \left(1 - \frac{1}{L}\frac{4}{\sqrt{\pi}}\sqrt{D_B t}\right)^N \Rightarrow \exp\left(-\frac{4}{\sqrt{\pi}}\rho_B\sqrt{D_B t}\right)$$

(in the limit $N \to \infty$, $L \to \infty$ with $\rho_B = N/L$ fixed).

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Create a fictitious box with edges at $x = \pm\ell/2$, and consider the subset of trajectories in which

1. There are no B-particles initially inside the box.
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3. No B-particles enter the box up to time t.

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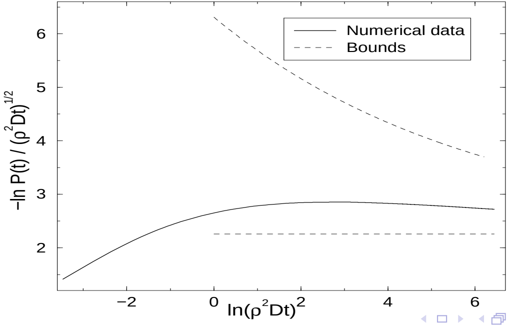
These trajectories are a subset of all trajectories for which the A-particle survives till time t, so

$$\begin{aligned} Q(t) \geq Q_L(t) &\sim \max_{\ell} \exp(-\rho_B \ell - \pi^2 D_A t/\ell^2)\, Q_{TAP}(t) \\ &\sim \exp[-\mathrm{const.}(\rho_B^2 D_A t)^{1/3}]\, Q_{TAP}(t) \end{aligned}$$

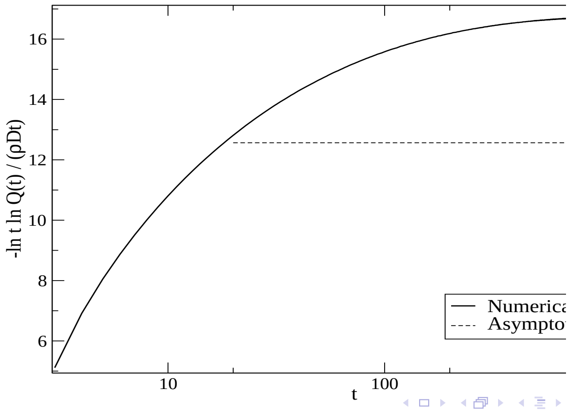
The prefactor is *subdominant* for $t \to \infty$:

$$Q_L(t) \sim \exp\left(-\frac{4}{\sqrt{\pi}}\rho_B (D_B t)^{1/2} - \mathrm{const.}(\rho_B^2 D_A t)^{1/3}\right)$$

Compare with data from Mehra and Grassberger (2002):



In two dimensions (Mehra and Grassberger (2002)):



ELIMINATING THE B-PARTICLES (AB, Majumdar, Blythe (2003))

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Consider the A and B particles to be *non-interacting* and consider events in which different B-particles hit the A-particle *for the first time*. These events are described by a *Poisson distribution* – the probability that the A-particle, *with a given trajectory* $z(t)$, has been hit by n different B-particles up to time t is

$$p_n = \frac{\mu^n}{n!} \exp(-\mu)$$

where $\mu = \mu[z(\tau)]$, $0 \leq \tau \leq t$, is a *functional* of the A-particle trajectory $z(t)$.

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The probability of *no hits up to time t* is

$$Q(t) = \langle p_0(t) \rangle_z = \langle \exp(-\mu[z]) \rangle_z$$

where the average $\langle \ldots \rangle_z$ is taken over all A-particle trajectories weighted with the usual Wiener measure.

To determine $\mu[z]$ we calculate, in two ways, the probability density for a B-particle to reach $z(t)$ at time t (in a *noninteracting* system):

$$\rho_B = \int_0^t dt' \, \dot{\mu}(t') \, G_B \left(z(t), t | z(t'), t'\right)$$

where

$$G_B \left(z(t), t | z(t'), t'\right) = \frac{1}{[4\pi D_B(t - t')]^{1/2}} \exp\left(-\frac{[z(t) - z(t')]^2}{4D_B(t - t')}\right)$$

is the B-particle diffusion propagator. For the target problem ($z(t) = 0$ for all t) this gives

$$\mu(t) = \mu_0(t) = \frac{4}{\sqrt{\pi}} \rho_B (D_B t)^{1/2}$$

as before.

PROOF OF THE PASCAL PRINCIPLE (AB, Majumdar, Blythe (2003); Moreau et al. (2003))

In general we can write $\mu[z] = \mu_0 + \mu_1[z]$. Then $\mu_1[z]$ satisfies the equation:

$$\mu_1[z] = \frac{1}{\pi} \int_0^t \frac{dt_1}{\sqrt{t - t_1}} \int_0^{t_1} \frac{dt_2}{\sqrt{t_1 - t_2}} \, \dot{\mu}(t_2) \, K(t_1, t_2)$$

where

$$K(t_1, t_2) = 1 - \exp\left(\frac{-[z(t_1) - z(t_2)]^2}{4D_B(t_1 - t_2)} \right)$$

The obvious inequalities $K(t_1, t_2) \geq 0$ and $\dot{\mu}[z] \geq 0$ prove that

$$\mu_1[z] \geq 0 \Rightarrow \mu[z] \geq \mu_0$$

i.e. the "Pascal principle" is proved.

EXTENSIONS OF THE BASIC RESULT

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- Diffusion on a fractal space (Oshanin et al., 2003)
- Calculation of subleading (preasymptotic) terms in the trapping reaction for $D_A \ll D_B$ (Anton + AB, 2004)
- Trapping reaction with subdiffusive traps and particles (Yuste and Lindenberg, 2006)
- Some exact results in certain limits for the double trapping reaction, $A + B \to B$, $B + C \to C$ (AB + Smith, 2006): See Richard Smith's poster.

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   - The random walk (N=1); the random acceleration process (N=2); the "windy cliff" and its generalizations (N=2).
   - The one-dimensional Ising model with Glauber Dynamics.
   - The asymptotics of the trapping reaction, $A + B \to B$, for $d \leq 2$.
   - The one-dimensional TDGL equation (AB, Derrida, Godreche, 1994).

EPILOGUE

I am sorry this talk has been so long, but I didn't have time to write a shorter one.

(After Pascal)