Fluctuation-regularized Front Propagation Dynamics in a Reaction-Diffusion System

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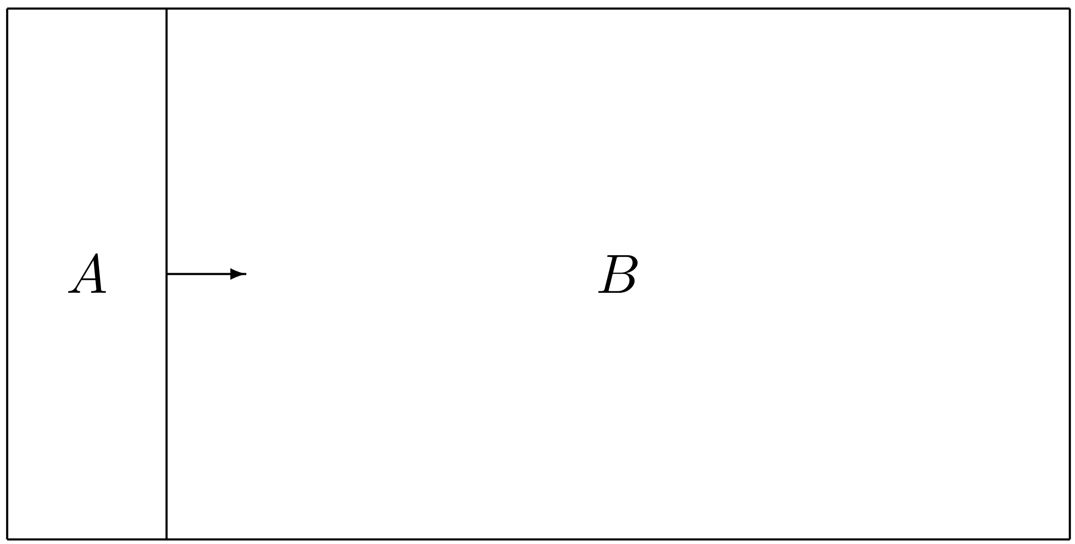
The Standard Problem: $A + B \to 2A$

Start with a system full of B's

Introduce some A's at left hand edge

Nonlinear "wave" of A's propagates to right, consuming B's



Propagation into an Unstable Phase

The Standard Approach: Reaction-Diffusion Equation

If $D_A = D_B$, get Fisher (*aka* Kolmogorov-Petrovsky-Piscounov) Equation (1937)

$$\dot{A} = D_A \nabla^2 A + rA(N - A)$$

Has Positive Traveling wave Solutions for all $v > v_F \equiv 2\sqrt{D_A N r}$

First Sign of Something Funny

All Initial Conditions with Compact Support Lead to Wave with $v = v_F$! (Aronson-Weinberger, 1978)

$v \to v_F$ very slowly: $v(t) = v_F(1 - \frac{3}{4Nrt})$ (Bramson, 1983)

Why are things funny?

Fisher (1937):

The equation ''must omit some essential element of the problem, and it is indeed clear that while a coefficient of diffusion may represent the biological conditions adequately in places where large numbers of individuals of both types are available, it cannot do so at the extreme front and back of the advancing wave, where the numbers ... are small, and where their distribution must be largely sporadic.''

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Can't add Regular Noise! Would Kill Wave Due to Instability!

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What have we gained??!!

For recent progress on velocity bounds, see (Conlon and Doering, 2005)

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Let us kill off the reaction by hand! (Brener, Levine and Tu (1991), Kepler and Perelson (1993), Kessler, Levine and Tsimring (1996), Brunet and Derrida (1997))

$$r = r_0 \theta(A - A_c)$$

No reaction if number of A's at a site $< A_c$, an $O(1)$ threshold.

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$$v = v_F \left(1 - \frac{\pi^2}{\ln^2(N/A_c)}\right) \qquad N \to \infty$$

Huge Correction to Velocity. In fact, for small N, $v \propto N$, not $\sqrt{N}$!

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In Two Dimensions, Cutoff gives rise to instability for small enough D_A. (Kessler and Levine, 2001)

Verified in Particle Simulations.

So much for simple things...

What Happens if Reaction Rate r Not Constant?

Not at all far-fetched

- Imposed Temperature Gradient
- Natural Inhomogeneities - Propagation of Roll State in Convection with Nonparallel Top and Bottom Walls
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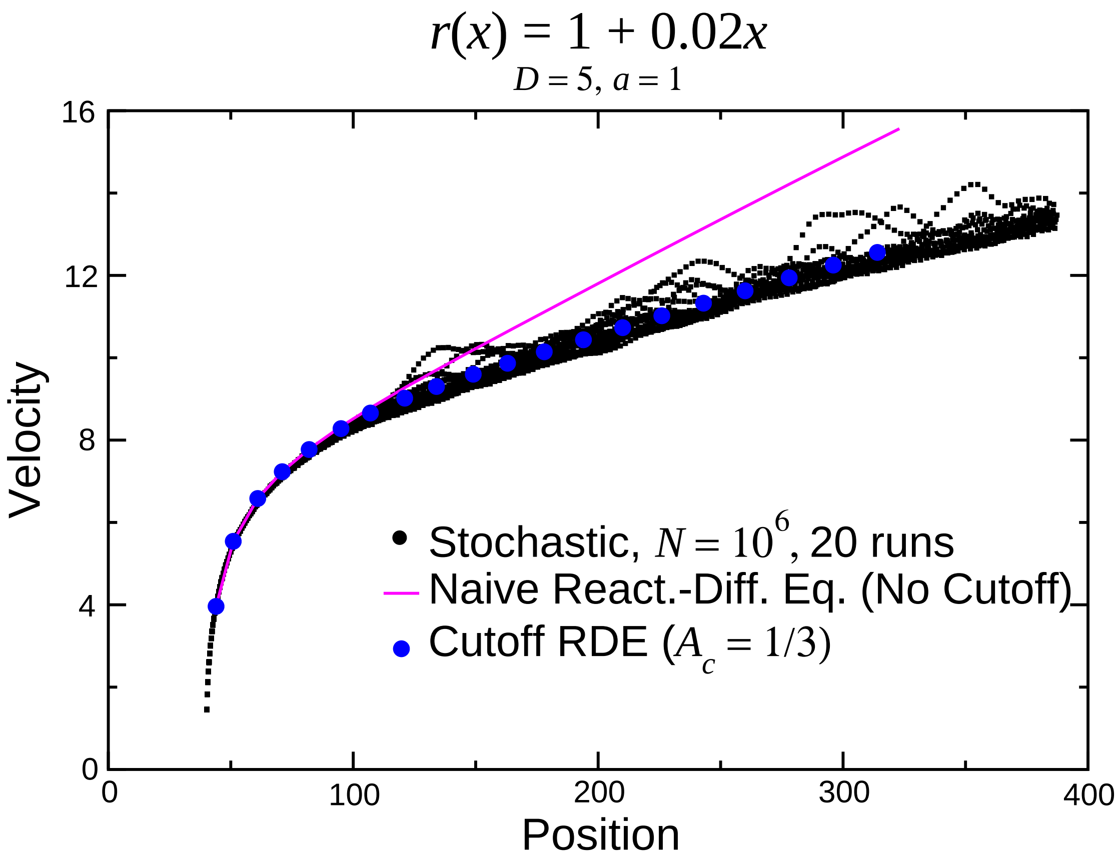
BUT --- Width of front depends on N, Width $\to \infty$ as $N \to \infty$

Conclusion: Does not work for large N
2nd Conclusion: N effects even more interesting than before!

Again: Study N Dependence via Cutoff



Cutoff works quite well!

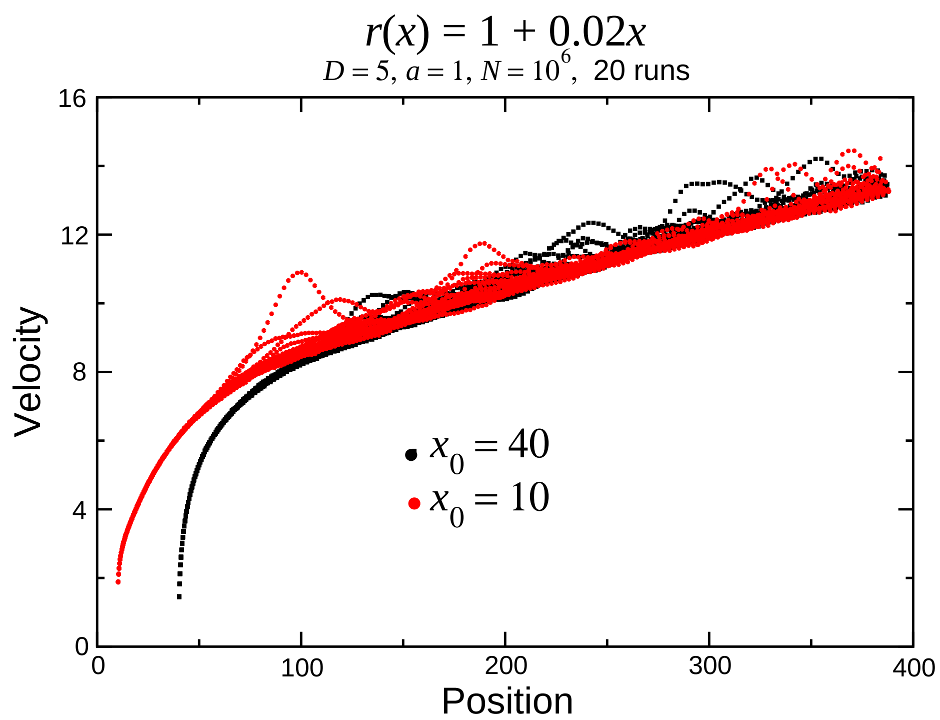
$$\dot{A} = \frac{D}{a^2}\left(A(x+a) - 2A(x) + A(x-a)\right) + r(x)\theta(A - A_c)A(N - A)$$
$$r(x) = r_0(1 + \alpha x)$$

The Cutoff and the Attractor

Problem: The Front Accelerates --- What can one say about the velocity?

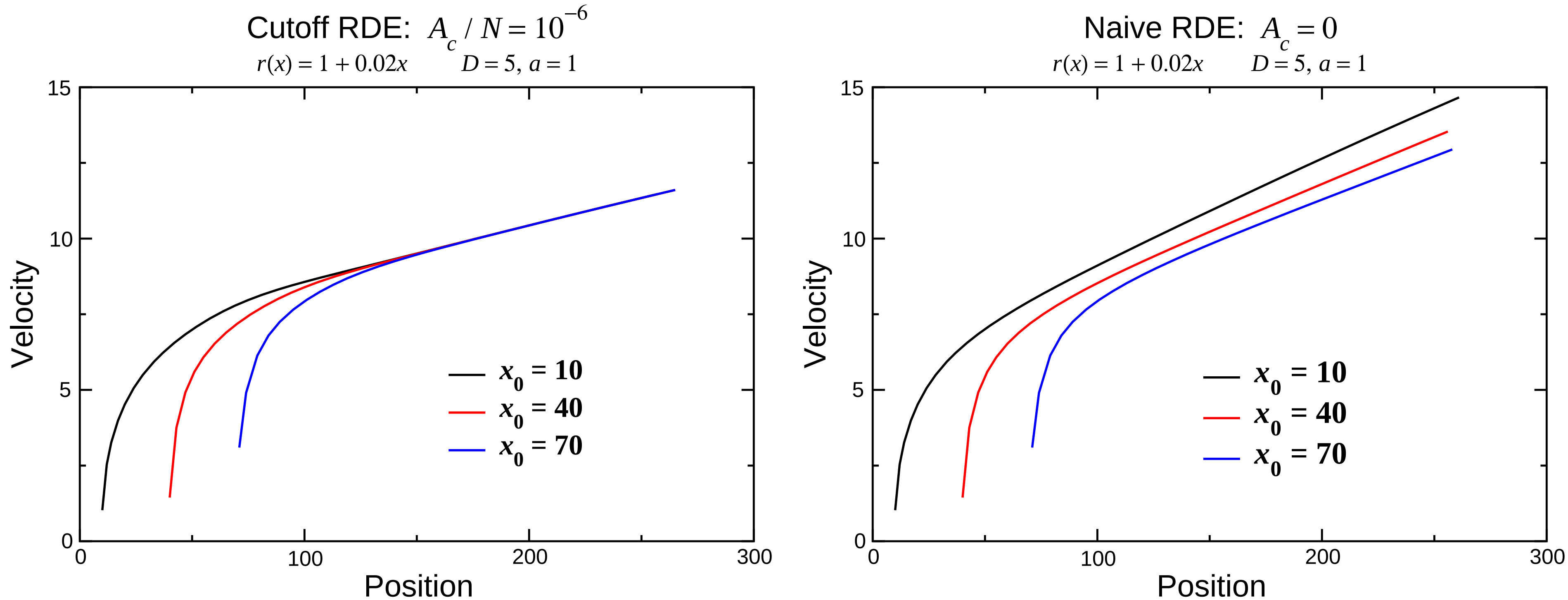
- Want to make predictions independent of details of the initial conditions
- "Normally", there is an asymptotic steady-state velocity (and front profile) indep. of IC's
- Here, no such solution exists

Solution: v asymptotically only depends on front location, not IC's



The Cutoff and the Attractor, cont.

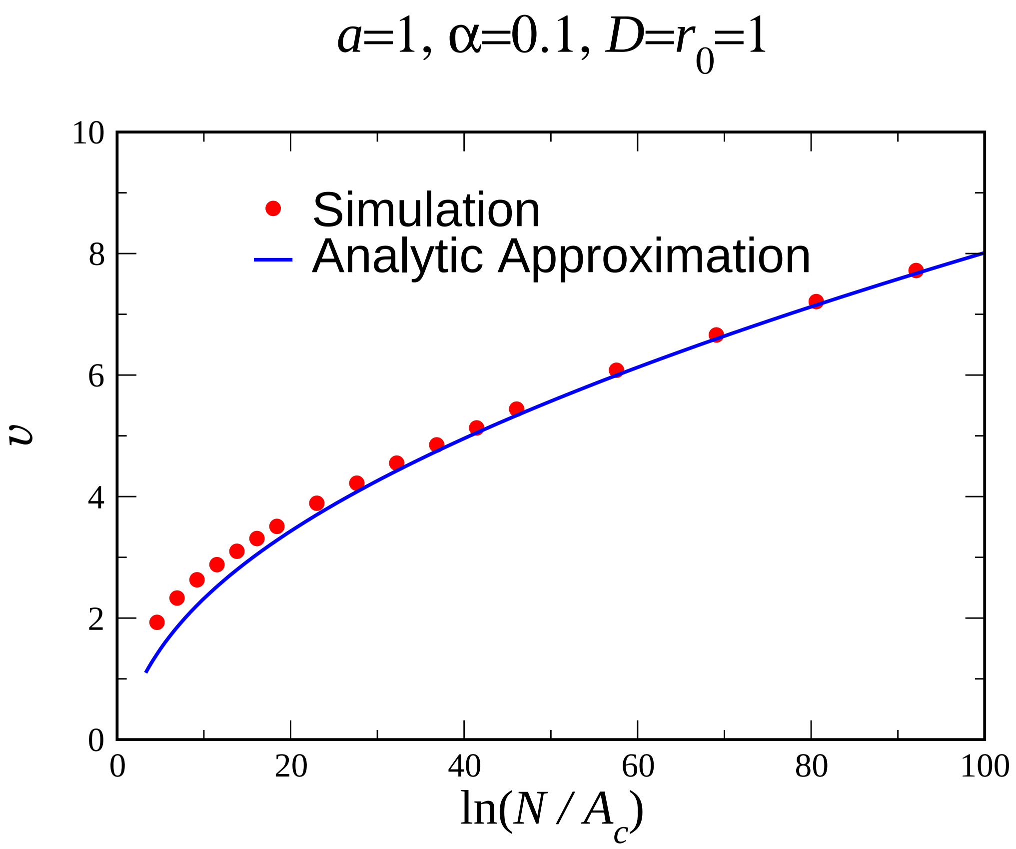
Unsurprisingly, velocity asymptotically only depends on the front location in Cutoff RDE, also
Surprisingly, not true without cutoff



Cutoff regularizes theory, Responsible for attractor
Leading edge becomes more and more unstable $\Rightarrow$ Cutoff Essential

Theory vs. Simulation of Cutoff Quasistatic RDE

Can analytically solve for small cutoff quasistatic case



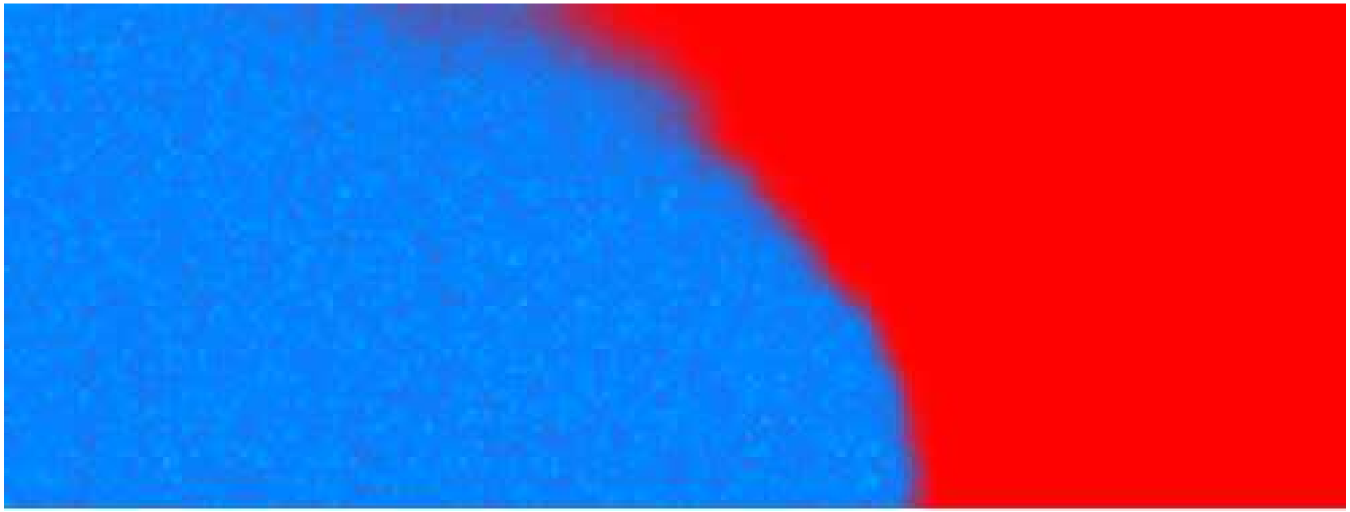
Two Dimensions

Things become even more interesting in 2D

Planar front is unstable

Deterministic Cutoff Model develops into a growing finger

Stochastic Model develops a well-defined finger for large enough N



$N = 1000, \; D = 0.25, \; \alpha = 0.06, \; L_y = 61, \; a = 1$

Take-Home Messages

Discreteness of particles is a singular perturbation --- controls pattern selection

Few ‘‘pioneer’’ particles at leading edge control dynamics

Need better mathematical tools