PHASE TRANSITION IN THE ALDOUS - SHIELDS MODEL OF GROWING TREES

DAVID DEAN — L.P.T. University of Toulouse III

SATYA MAJUMDAR — L.P.T.M.S. University of Paris XI

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GROWING CLUSTERS

FRACTALS IN NATURE - SNOWFLAKES, SOOT

ARTIFICIAL STRUCTURES - NETWORKS
INTERNET
SOCIAL NETWORKS

DIFFERENT GROWTH RULES LEAD TO A HUGE VARIETY OF DIFFERENT STRUCTURES

Eden Model
Invasion percolation
Diffusion Limited Aggregation
Growing network models
Insect constructions

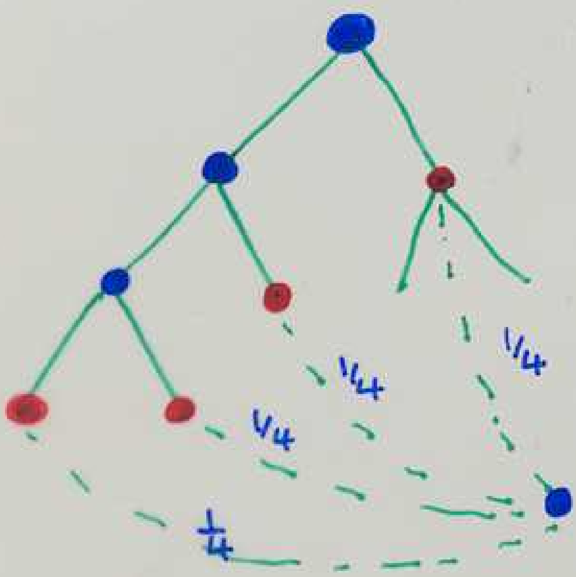
OFTEN NEED UNDERLYING LATTICE STRUCTURE

MANY MODELS STUDIED ON CAYLEY TREES

- No loops → exact solution
- Mimmicks large d lattices (mean field)
- Models occur exactly in search tree constructions

EDEN MODEL ON BINARY CAYLEY TREE $\equiv$ RANDOM BINARY SEARCH TREE

EDEN MODEL



● occupied site

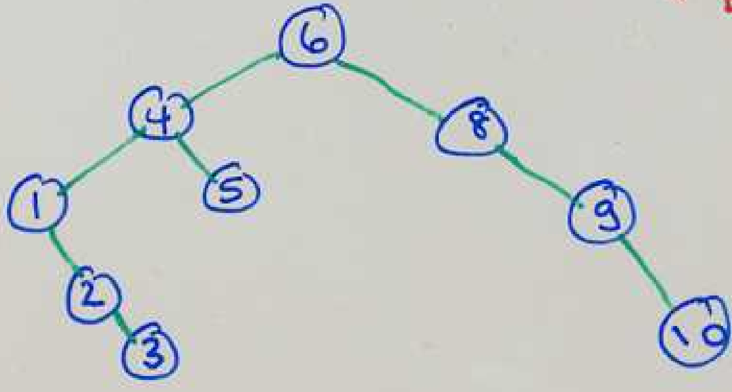
● potential growth site - perimiter.

● - new particle
- absorbed at perimeter site with equal probability.

RBST

$\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

Bell series



DATA ARRIVES AS RANDOM PERMUTATION OF N ELEMENTS

$P_\sigma = \frac{1}{N!}$

RBST AND EDEN GROWTH GENERATE STATISTICALLY IDENTICAL TREE ENSEMBLES

RBST Tree growth $\mathscr{G} = \{T_1, T_2, T_3 \cdots T_N\}$

↖ History

$$P_{\mathscr{G}} = \frac{1}{N!}$$ - determined by σ

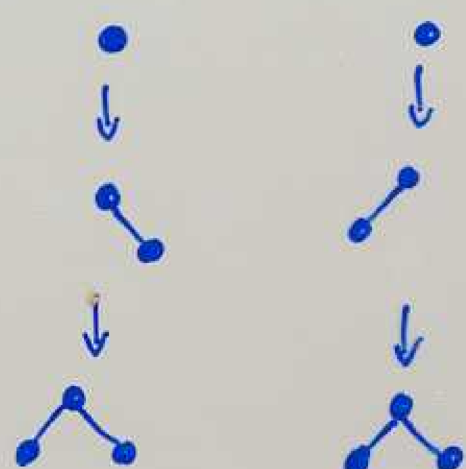
EDEN After $n-1$ particles arrive there are n perimeter sites

(each particles which arrives destroys one perimeter site but creates 2)

$$P_{\mathscr{G}} = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{N} = \frac{1}{N!}$$

Note different growth histories can give the same final tree.

$\{2,1,3\}$ + $\{2,3,1\}$



DIGITAL SEARCH TREE AND DIRECTED DIFFUSION LIMITED AGGREGATION

DDLA - BRADLEY AND STRENSKI (1985)

RANDOM WALK DOWN TREE UNTIL UNOCCUPIED SITE FOUND

$p = 1/2$ $\quad$ $p = 1/2$

perimiter site a at depth l_a

Prob next particle absorbed at $a \propto 2^{-l_a}$

Equivalence DDLA - DST - Ziv Lampel data Compression algorithm (Majumdar 2003)

Generalisation

$$p_a \propto c^{-l_a} \qquad c > 0$$

$c = 1$ Eden model / RBST.

$c = 2$ DST / DDLA

CANONICAL → GRAND CANONICAL

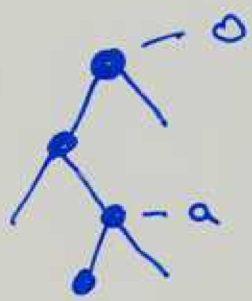
ALDOUS SHIELDS (1988) CONTINUOUS TIME VERSION $(m=2)$

r_a = rate at which particles occupie perimiter site a $\quad \boxed{r_a = c^{-l_a}}$

FOR THE GENERALISED ALDOUS SHIELDS GROWTH MODEL WHAT IS $n(t)$ – THE NUMBER OF OCCUPIED SITES AT TIME t ?

BACKWARD FOKKER-PLANCK APPROACH.

$n_a(t)$ = # particles in a subtree rooted at a (including a)



$n_0 = 4$

$n_a = 2$

Let t be the time after which a becomes a potential growth site ($t = 0$ for $a = 0$).

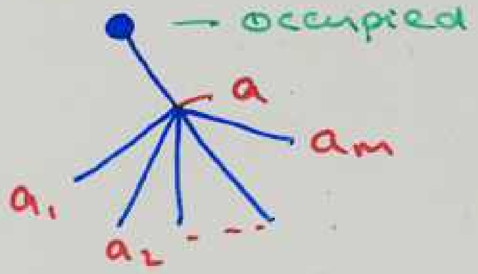
Have $n_a(0) = 0$

$$\boxed{P(n_a, t) = f(l_a, t)}$$

Dist^n f^n of n_a

M-ary tree

Subtree rooted at site a at "time" $t=0$



In the next time interval Δt

(i) No particle arrives at a

$$\text{Prob} = 1 - e^{-l_a}\,\Delta t$$

(ii) A particle arrives at a

$$\text{Prob} = e^{-l_a}\,\Delta t$$

Reccurrence – after first time step Δt

nothing happens $t-\Delta t$ time left

$$n_a(t) = n_a(t-\Delta t)(1-I) + \Big[1 + \sum_{a} n_{a_i}(t-\Delta t)\Big] I$$

particle arrives

$t-\Delta t$ left for new growth sites.

$$I = \begin{cases} 1 & \text{Prob } e^{-l_a}\,\Delta t \\ 0 & \text{Prob } 1-e^{-l_a}\,\Delta t \end{cases}$$

Generating Function

$$G_a(\mu,t) = \langle \exp(-\mu n_a(t)) \rangle$$
$$= G(\mu, la, t).$$
$$= G(\mu, 0, c^{-la} t) \equiv G(\mu, t)$$

time slowed down at level a by c^{-la} wrt root.

$$\langle e^{-\mu n_a(t)} \rangle = \left\langle \left\langle e^{-\mu\left[I\left(1 + \sum_{i=1}^{m} n_{a_i}(t-\Delta t) - n_a(t-\Delta t)\right)\right]} \cdot e^{-\mu n_a(t-\Delta t)} \right\rangle_I \right\rangle$$

$$= \left\langle c^{-la} e^{-\mu\left(1 + \sum_{i=1}^{m} n_{a_i}(t-\Delta t)\right)} \right\rangle \Delta t$$
$$+ (1 - c^{-la} \Delta t) \langle e^{-\mu n_a(t-\Delta t)} \rangle$$

$$G(\mu, la, t) = c^{-la} \Delta t \, e^{-\mu} G^m(\mu, la+1, t-\Delta t)$$
$$+ (1 - c^{-la}\Delta t) \, G(\mu, la, t-\Delta t)$$

time slowing down.

$$= c^{-la} \Delta t \, e^{-\mu} G^m\left(\mu, la, \frac{t-\Delta t}{c}\right)$$
$$+ (1 - c^{-la}\Delta t) \, G(\mu, la, t-\Delta t)$$

$la = 0$

$$\Rightarrow \boxed{\frac{\partial G}{\partial t}(\mu, t) = e^{-\mu} G^m(\mu, t/c) - G(\mu, t)}$$

MOMENTS

$M(t) = \langle n(t) \rangle$: $\frac{d}{dt} M = 1 - M(t) + m M\left(\frac{t}{c}\right)$

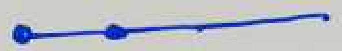
$V(t) = \langle n^2(t) \rangle_c$: $\frac{dV}{dt} = \left(\frac{dM}{dt}\right)^2 - V(t) + m V(t/c)$

In general non-local equations (except $c=1$).

Exact solution for Eden Growth ($c=1$)

$$G(\mu, t) = e^{-t}\left[1 - e^{-\mu}\left(1 - e^{-(m-1)t}\right)\right]^{-\frac{1}{m-1}}$$

$m > 1$ exponential increase of $\langle n(t) \rangle$

Check $m = 1$ line 

$$G(\mu, t) = e^{-t} e^{-e^{-\mu} t}$$

$\Rightarrow$ $P(n,t) = \frac{e^{-t} t^n}{n!}$ – Poisson process.

Exact solution for DST ($c = m$) ($m > 1$)

$$P(n,t) = \frac{e^{-t} t^n}{n!}$$

Solution $G(\mu, t) = e^{-at}$ works.

SELF CONSISTENT SCALING APPROACH FOR LEADING ASYMPTOTIC BEHAVIOUR OF $M(t)$ $V(t)$.

Mean

Ansatz: $M(t) \simeq A t^{\alpha}$

Match coefficients of t^{α}.

$$\frac{m}{c^{\alpha}} - 1 = 0 \qquad \boxed{\alpha = \frac{\ln(m)}{\ln(c)}}$$

($\frac{d}{dt}$ - neglected)

Note cannot determine amplitude A this way.

Variance

Ansatz: $V(t) \simeq B t^{\beta}$

Matching coefficients - two choices

$$-Bt^{\beta} + Bm\frac{t^{\beta}}{c^{\beta}} + A^2\alpha^2 t^{2\alpha-2} = 0$$

(i) If $\beta > 2\alpha - 2$ $\quad c^{\beta} = m$

$\Rightarrow \beta = \frac{\ln(m)}{\ln(c)} = \alpha$

B not determined $\boxed{\alpha < 2}$

(ii) $\beta = 2\alpha - 2$ $\quad \Rightarrow B = \frac{A^2\alpha^2}{(1 - c^{2-\alpha})}$

Here $V(t) > 0$ so must have

$2 - \alpha < 0 \quad \Rightarrow \quad \boxed{\alpha > 2}$

SPECIAL CASE $\alpha \in \mathbb{N}$

ANSATZ $M(t) = \sum_{k=1}^{\infty} b_n t^n$ (no $k=0$ as $M(0)=0$)

MATCH POWERS OF t

$$b_1 = 1$$

$$b_{k+1} = b_k \frac{1}{k+1}\left(\frac{m}{c^k} - 1\right) \qquad k > 1$$

SO IF $\exists$ $k^* \in \mathbb{N}$: $k^* = \frac{\ln(m)}{\ln(c)}$ $(=\alpha)$ THEN

$$b_{km} = 0 \quad \forall \quad k > k^*$$

AT LATE TIMES

$$M(t) \sim \frac{t^\alpha}{\alpha!}\left(\frac{m}{c^{\alpha-1}} - 1\right)\left(\frac{m}{c^{\alpha-2}} - 1\right)\cdots\left(\frac{m}{c} - 1\right)$$

$$= \frac{t^\alpha}{\alpha!}\left(c^{\alpha-1} - 1\right)\left(c^{\alpha-2} - 1\right)\cdots(c-1)$$

$m = c^\alpha$

e.g. $\boxed{\alpha = 2}$ $M(t) = \frac{c-1}{2} t^2 \Rightarrow \boxed{A = \frac{c-1}{2}}$

ANSATZ $V = B t^2 u(t)$

Get both amplitudes

$$\Rightarrow \frac{4A^2}{B} = u(t) - u(t/c)$$

$u(t) = \ln t$ $\boxed{B = \frac{(c-1)^2}{\ln c}}$

Point $\alpha_c = 2$ separates regions of

NORMAL GROWTH $\alpha < 2$ $V(t) \sim M(t)$

FAST GROWTH $\alpha > 2$ $V(t) \sim M(t)^{2-2/\alpha}$

ALDOUS SHIELDS $\alpha = 2$

RIGOROUS PROBABILISTIC RESULT ON LIMITING DISTRIBUTION

$c < \sqrt{2}$ ANOMALOUSLY LARGE FLUCTUATIONS

$c > \sqrt{2}$ LIMITING GAUSSIAN DISTRIBUTION

SUMMARY

$V(t) \sim t^{\alpha} \sim M(t)$ $\alpha < 2$

$\sim t^2 \ln(t) \sim M(t) \ln M(t)$ $\alpha = \alpha_c = 2$

$\sim t^{2-\alpha} \sim M(t)^{2-2/\alpha}$ $\alpha > 2$

WHAT DOES THIS MEAN FOR
THE CANONICAL CASE

GEOMETRIC INTERPRETATION OF
DYNAMICAL RESULT

NORMAL CASE $\alpha < 2$

TREES "BALANCED"

ANORMAL CASE

TREES "UNBALANCED"

STILL OPEN QUESTON

WHAT'S THE RIGHT GEOMETRIC
ORDER PARAMETER?

GENERAL SOLUTION FOR $M(t)$

(- useful method Flajolet and Richmond 1992)

$$\tilde{M}(s) = \int_0^\infty dt\, e^{-st} M(t)$$

$$\tilde{M}(s) = \frac{1}{s(s+1)} + \frac{mc}{(s+1)} \tilde{M}(cs)$$

$M(s) \to 0$ as $s \to \infty$

$c > 1$ so $M(c^k s) \to 0$ for $k \to \infty$

LOOK FOR AN ITERATIVE SOLUTION

$$\tilde{M}(s) = \frac{1}{s} \sum_{j=0}^{\infty} \frac{m^j}{(1+s)(1+cs) \cdots (1+c^j s)}$$

LARGE $t \to$ SMALL s

BUT LIMIT $s \to 0$ IS DELICATE !

$$\sum_{j=0}^{\infty} \frac{m^j}{1 \cdot 1 \cdot 1 \cdot 1}$$ divergent for $m > 1$

DEFINE $$Q(u) = \prod_{l=0}^{\infty} \left(1 + \frac{u}{c^l}\right)$$

$$Q(s/c) = (1 + s/c)(1 + s/c^2) \cdots$$

$$Q(c^j s) = (1 + c^j s)(1 + c^{j-1} s) \cdots (1+s)\, Q(s/c)$$

$$\tilde{M}(s) = \frac{Q(s/c)}{s} H(s)$$

$$H(s) = \sum_{j=0}^{\infty} \frac{m^j}{Q(c^j s)}$$

MELLIN TRANSFORM OF H

$$H^*(x) = \int_0^\infty ds\ s^{x-1} H(s)$$

$$= \sum_{j=0}^{\infty} m^j \int_0^\infty \frac{s^{x-1}}{Q(c^j s)} ds$$

$$= \sum_{j=0}^{\infty} (m c^{-x})^j \int_0^\infty \frac{\sigma^{x-1}\, d\sigma}{Q(\sigma)}$$

$$= \frac{h^*(x)}{1 - m c^{-x}}$$

$$h^*(x) = \int_0^\infty d\sigma \frac{\sigma^{x-1}}{Q(\sigma)}$$

In evaluation of sum assumed $Re(m c^{-x}) < 1$

$$Re(x) > \frac{\ln(m)}{\ln(c)} = \alpha$$

$h^*(x)$ no poles for $Re(x) > 0$

$\frac{1}{1-mc^{-x}}$ poles at $x_k = \alpha + \frac{2\pi i k}{\ln(c)}$

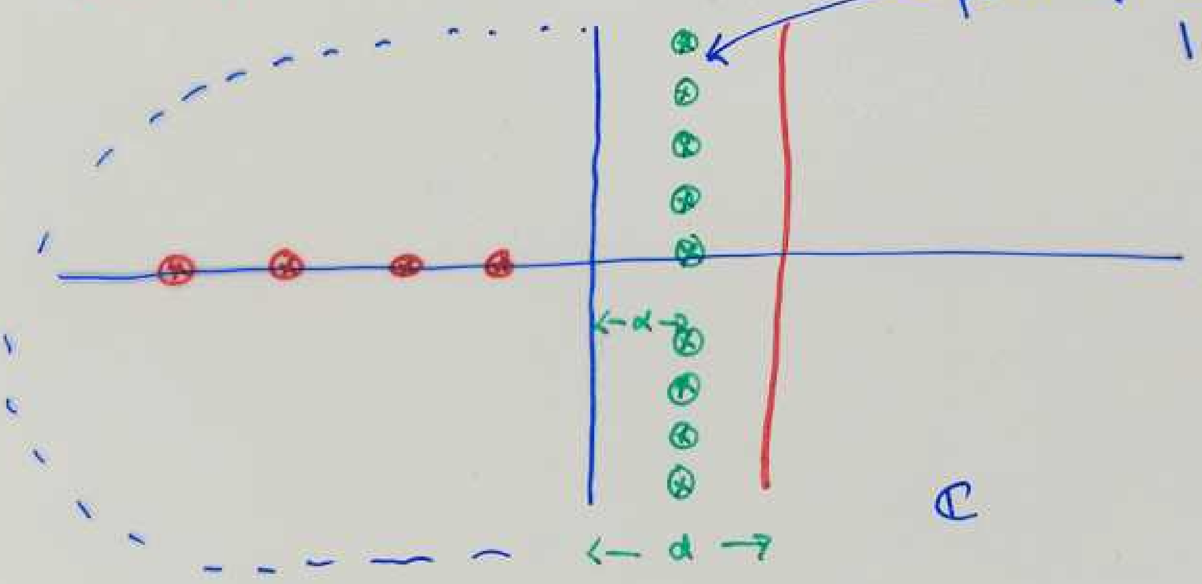
$k = 0, \pm 1, \pm 2 \ldots$

INVERSION FORMULA FOR MELLIN TRANSFORM

$$H(s) = \frac{1}{2\pi i} \int_{-i\infty + d}^{i\infty + d} dx \; H^*(x)\, s^{-x}$$

All poles to left of contour

poles of $\frac{1}{1 - mc^{-x}}$



Close contour on the left

CAUCHY : $H(s) = \sum_{\text{poles}} \text{Res}\left[\frac{h^*(x)\, s^{-x}}{1 - \exp(\ln(m) - x\ln(c))}\right]$

Large $t \leftrightarrow$ small s

Poles ⊗ dominate small s behavior.

$$\tilde{H}(s) \sim \frac{1}{s^{\alpha+1}\ln(c)}\left[h^*(\alpha) + \sum_{k \neq 0} h^*\!\left(\alpha + \frac{2\pi i k}{\ln(c)}\right) s^{\frac{2\pi i k}{\ln(c)}}\right]$$

used $Q(0) = 1$

RAMANUJAN IDENTITY

$$h^*(\alpha) = \int_0^\infty \frac{\sigma^{\alpha-1}\, d\sigma}{(1+\sigma)(1+\sigma/c)(1+\sigma/c^2)\cdots}$$

$$= \frac{\pi}{\sin(\pi\alpha)} \prod_{k=1}^{\infty} \frac{1-c^{\alpha-k}}{1-c^{-k}}$$

$$\tilde{M}(s) = \tilde{M}_p(s) + \tilde{M}_\ell(s)$$

$$M_p(s) = \frac{1}{s^{\alpha+1}\ln(c)} h^*(\alpha)$$

$M_p(t)$ has pure power law growth

$$M_p(t) \sim A t^\alpha$$

$$A = \frac{\pi}{\ln(m)\Gamma(\alpha)\sin\pi\alpha} \prod_{k=1}^{\infty} \frac{1-c^{\alpha-k}}{1-c^{-k}}$$

use $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$ and $\alpha = \dfrac{\ln(m)}{\ln(c)}$

AGREES WITH POWER SERIES SOLUTION

FOR $\alpha \in \mathbb{N}$

$M_e(t)$ - log periodic component

$M_e(t) \sim A t^{\alpha} g(\ln t)$ g periodic function.

$M(t) \sim A t^{\alpha} (1 + g(\ln(t)))$

CANNOT SAY MUCH ABOUT g - not there for $\alpha \in \mathbb{N}$.

CALCULATION FOR VARIANCE

$$\tilde{V}(s) = \frac{Q(s/c)}{s^{1+\alpha}\ln(c)}\left[h_1^*(1+\alpha) + \sum_{k\neq 0} h_1^*\left(1+\alpha - \frac{2\pi i k}{\ln(c)}\right)\right]$$

$$h_1^*(x) = \int_0^\infty \frac{S(s)}{Q(s)} s^{x-1}\, ds$$

$$S(s) = \int_0^\infty dt\, e^{-st} \left(\frac{dM}{dt}\right)^2$$

source term in V equation

LATE TIMES IF $h_1^*\left(1+\alpha - \frac{2\pi i k}{\ln c}\right)$ exists

$$V(t) \sim B' t^{\alpha}\left[1 + G(\ln(t))\right]$$ - normal regime

periodic

$$B = \frac{h_1^*(1+\alpha)}{\Gamma(1+\alpha)\ln(c)}$$

WHEN IS THIS VALID ?

$$\left(\frac{dH}{dt}\right)^2 \simeq A^2 \alpha^2 t^{2\alpha-2}\left[1+g(\ln(t))\right]$$

$$S(s) = \int_0^\infty e^{-st}\left(\frac{dH}{dt}\right)^2$$

$$\simeq A^2\alpha^2 \int_0^\infty t^{2\alpha-2}\left[1+g(\ln t)\right] e^{-st}\,dt$$ for

small s.

$$S(s) \simeq C_0 s^{-2\alpha+1} \qquad \alpha > 1/2$$

$$\simeq -\ln(s) \qquad \alpha = 1/2$$

$$\simeq A_1 \qquad \alpha < 1/2$$

UP TO LOG PERIODIC CORRECTIONS.

$A_1 = \int_0^\infty \left(\frac{dH}{dt}\right)^2 dt$ – depends on the whole history of $H(t)$.

$$C_0 = A^2\alpha^2\Gamma(2\alpha-1)$$

$$h_1^\alpha(1+\alpha) = \int_0^\infty \frac{S(s)}{Q(s)} s^\alpha\, ds$$

$\downarrow$

$s^\alpha \cdot s^{-2\alpha+1}$ $\qquad \alpha > 1/2$

$s^\alpha \; -\ln(s)$ $\qquad \alpha = \frac{1}{2}$ ✓ OK

need $\alpha < 2$

s^α $\qquad \alpha < 1/2$ ✓ OK

RESULT VALID FOR $\alpha < 2$ NORMAL PHASE

WHEN $\alpha > 2$

$$\tilde{V}(s) = \sum_{j=0}^{\infty} \frac{(\mu c)^j \; S(c^j s)}{(1+s)(1+cs) - (1+c^j s)}$$

$$S(s) \simeq C_0 \, s^{-(2\alpha-1)}$$

$$\tilde{V}(s) \simeq C_0 \sum_{j=0}^{\infty} \frac{(\mu c)^j \; (c^j)^{1-2\alpha} \; s^{1-2\alpha}}{(1+s)(1+cs) --- (1+c^j s)}$$

$$\simeq \frac{C_0}{s^{2\alpha-1}} \sum_{j=0}^{\infty} (c^{2-\alpha})^j$$

Converges for $\alpha > 2$!
Easier !

$$= \frac{C_0}{s^{2\alpha-1}(1 - c^{2-\alpha})}$$

$$\Rightarrow \quad V(t) \simeq B \, t^{2\alpha-2}$$

$$B = \frac{\alpha^2 A^2}{1 - c^{2-\alpha}}$$

CONCLUSIONS

1) Generalised A.S. model has a phase transition normal → anomalous fluctuations

- shows up in the variance
- similar transition found in m-ary search tree (see Majumdar 2002)

Diffusion normal → anomalous

$$\langle x_t \rangle = 0$$
$$\langle x^2_t \rangle \approx t^{2\nu} \qquad \nu = 1/2 \text{ normal.}$$

2) Growth on finite dimensional lattices?

3) Can one find the geometric order parameter? passage grand canonical → canonical

Good geometric order parameter?