Extreme value problems in Random Matrix Theory, Spin Glasses and Directed Polymers

J.P Bouchaud

with G. Biroli, P. Cizeau, M. Mézard, M. Potters

Fat tails and the Central Limit Theorem

- Sum of random variables: $S_N = \sum_i x_i$, with

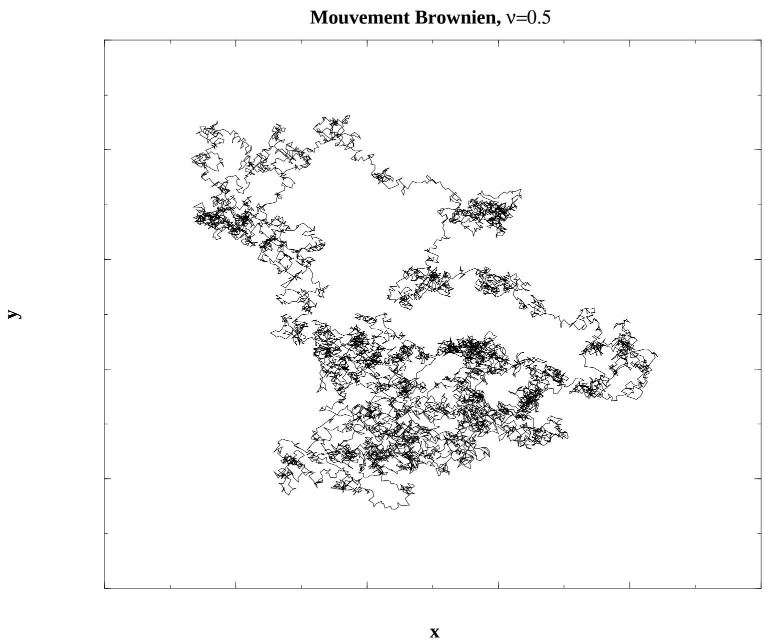
$$\rho(x) \sim_{|x|\to\infty} |x|^{-1-\mu}$$

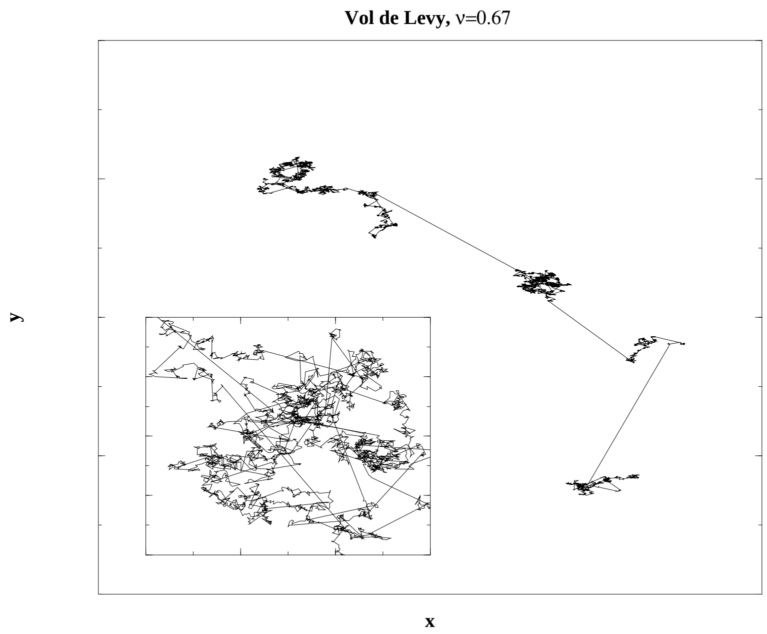
  - $\mu > 2$, finite variance $\rightarrow S_N \sim N^{1/2}$ and Gaussian
  - $\mu < 2$, infinite variance $\rightarrow S_N \sim N^{1/\mu}$ and Lévy distributed

- Random walks and Lévy flights

- Epidemic propagation [Brockmann et al.]

Brownian and Lévy flights





Fat tails and the Central Limit Theorem

- From sums to max: $S_N = [\sum_i x_i^q]^{1/q}$

- $q \to \infty$, finite N: $S_N = x_{\text{max}}$

- Interesting crossover in the N, q plane

- Example: $x = e^y$, y Gaussian:

$$\mu = \frac{\sqrt{2 \ln N}}{q\sigma}$$

- Note the analogy with the REM: $q \to \beta$, low temperature dominated by deep energy states

Extreme value statistics

- Extreme value distributions
  - Bounded variables: Weibull
  - Exponential variables: Gumbel $- G(u) = \exp[-u - \exp[-u]]$
  - Power-law variables: Fréchet $- F(u) = \mu \exp[-u^{-\mu}]/u^{1+\mu}$

- The Random Energy Model: Gumbel statistics, equivalent to 1-Step Replica Symmetry Breaking

- Application: Decaying Burgers' Turbulence and FRG of pinned manifolds, Shocks and Cusps [cf. Balents, JPB, MM; Le Doussal, Wiese]

Fat tails and Random Matrix Theory

- Eigenvalue statistics of large real symmetric matrices with iid elements x_{ij}

- Eigenvalue density:

  - $\mu > 2 \rightarrow$ Wigner semi-circle

  - $\mu < 2 \rightarrow$ unbounded density with tails $\rho(\lambda) \sim \lambda^{-1-\mu}$

- Note: $\mu < 2$ non trivial statistics of eigenvectors (localized/delocalized) ([PC,JPB])

Fat tails and Random Matrix Theory

- Largest Eigenvalue statistics ([GB,MP,JPB])
  - $\mu > 4$: $\lambda_{\max} - 2 \sim N^{-2/3}$ with a Tracy-Widom distribution (max of strongly correlated variables)
  - $2 < \mu < 4$: $\lambda_{\max} \sim N^{\frac{2}{\mu}-\frac{1}{2}}$ with a *Fréchet* distribution (although the density goes to zero when $\lambda > 2$!!)
  - $\mu = 4$: $\lambda_{\max} \geq 2$ but remains $O(1)$, with a new distribution:

$$P(\lambda_{\max}) = w\delta(\lambda_{\max} - 2) + (1-w)F(s) \quad \lambda_{\max} = s + \frac{1}{s}$$

- Note: The case $\mu > 4$ still has a power-law tail for finite N

Fat tails and Random Matrix Theory

- Empirical correlation matrices of finite time series

$$C_{ij} = \frac{1}{T} \sum_t x_i^t x_j^t$$

- Suppose true correlations are absent: $C_{ij}(T \to \infty) = \delta_{ij}$

- Empirical spectrum for $N \to \infty$, $Q = N/T$ finite: Marcenko-Pastur distribution – $\lambda_{\text{min,max}} = (1 \pm \sqrt{Q})^2$

- $\mu > 4$: $\lambda_{\text{max}} - (1 + \sqrt{Q})^2 \sim N^{-2/3}$

- $\mu < 4$: $\lambda_{\text{max}} \sim N^{\frac{4}{\mu}-1}$ – important for applications in finance.

Fat tails and Spin-glasses

- Mean-field model of spin-glasses: $H = \sum_{ij} J_{ij}\sigma_i\sigma_j$

- J_{ij} Gaussian: SK model; Order parameter: $q_{EA} = \overline{\langle S_i \rangle^2}$

- Naive guess: ‘disguised ferromagnet’, replica symmetric solution leads to $S(T = 0) < 0$

- Spin-glass transition temperature coincides with RSB: non trivial in the whole low temperature phase – multiple metastable states, valleys within valleys with Gumbel statistics

- $S(T) \sim_{T\to 0} T^2$ – very few ‘mad spins’ in the ground states

Fat tails and Spin-glasses

- Power-law interactions in space (like RKKY) $\to \rho(J) \sim J^{-1-\mu}$, $\mu = d/\alpha$.

- $\mu > 2$ $\to$ identical to the Gaussian case

- $\mu < 2$ $\to$ new physics – strong bonds dominate, less frustration ([PC,JPB])

  – Two transition temperatures: $T_{AT} < T < T_c$, $q_{EA} > 0$ but only two (time reversed) phases; $T < T_{AT}$: instability of the 'RS' phase

  – Generalisation of the RSB solution to Frechet statistics??

  – Reentrance of the RS solution at $T = 0$? $S(T) \sim_{T \to 0} T^{\mu}$

Fat tails and Directed Polymers

- Directed path that optimizes a sum of local 'bounties'

$$E(\mathcal{C}) = \sum_{t=1}^{N} e(x_t, t)$$

- In $1+1$ dimension for exponential disorder: exactly solved problem! [Johansson]
  - $E_{\text{max}} = e^* N + \varepsilon N^{1/3}$; ε: Tracy-Widom distribution (!?)
  - $W = x_N \sim N^{2/3}$: superdiffusion to catch favorable sites
  - Mapping to the KPZ equation + many other problems
  - Higher dimensions ?? [Tree limit: Derrida-Spohn, 1 step RSB]

Fat tails and Directed Polymers

- Case where $\rho(e) \sim e^{-1-\mu}$? Naive guess: $\mu = 2$ should play a role

- In fact: the Derrida-Spohn solution loses its meaning as soon as $\mu < \infty$...

- Simple argument in 1+1: The path distorts to grasp extreme bounties

$$e_{\text{max}} \sim (WN)^{1/\mu} \quad \text{should balance} \quad W^2/N \to W \sim N^{\frac{1+\mu}{2\mu-1}}$$

  suggests $\mu = 5$!

- Numerical simulations suggest the argument is exact: $\delta E_{\text{max}} = \varepsilon N^{\frac{3}{2\mu-1}}$ with ε distributed as a geometric sum of Fréchet distribution ([GB,MP,JPB])

Fat tails and Directed Polymers

- Note 1: No direct connection with the statistics of largest eigenvalue in this case?

- Note 2: Temperature chaos for $\mu > 7/2$, absent for $\mu < 7/2$

- Open problem: how to capture these effects within a perturbative FRG formalism?

- Open problem: how to extend RSB to Fréchet (/Weibull) distributions?