From extreme value statistics to global fluctuations

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OUTLINE

- *Introduction on fluctuations of global observables*
- *Relation between extreme value statistics and global fluctuations*
- *Stochastic cascade model, truncated* $1/f$ *noise and correlation length*
- *Conclusions*

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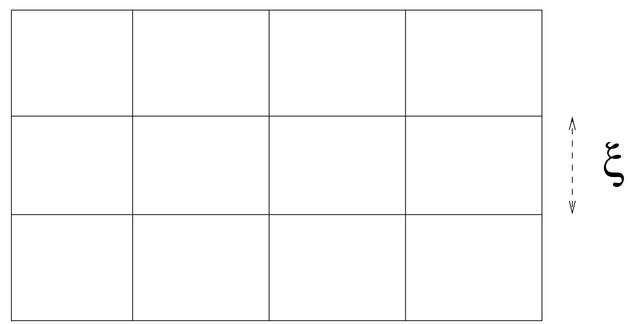
Gaussian and non-Gaussian statistics

- *Global observable*

  Quantity X integrated over the whole system

- *Systems with finite correlation length ξ*

  - Split the system into independent subsystems of linear size $\propto \xi$
  - Central limit theorem:
    Gaussian distribution for X
    in the limit $N \to \infty$ (ξ fixed)

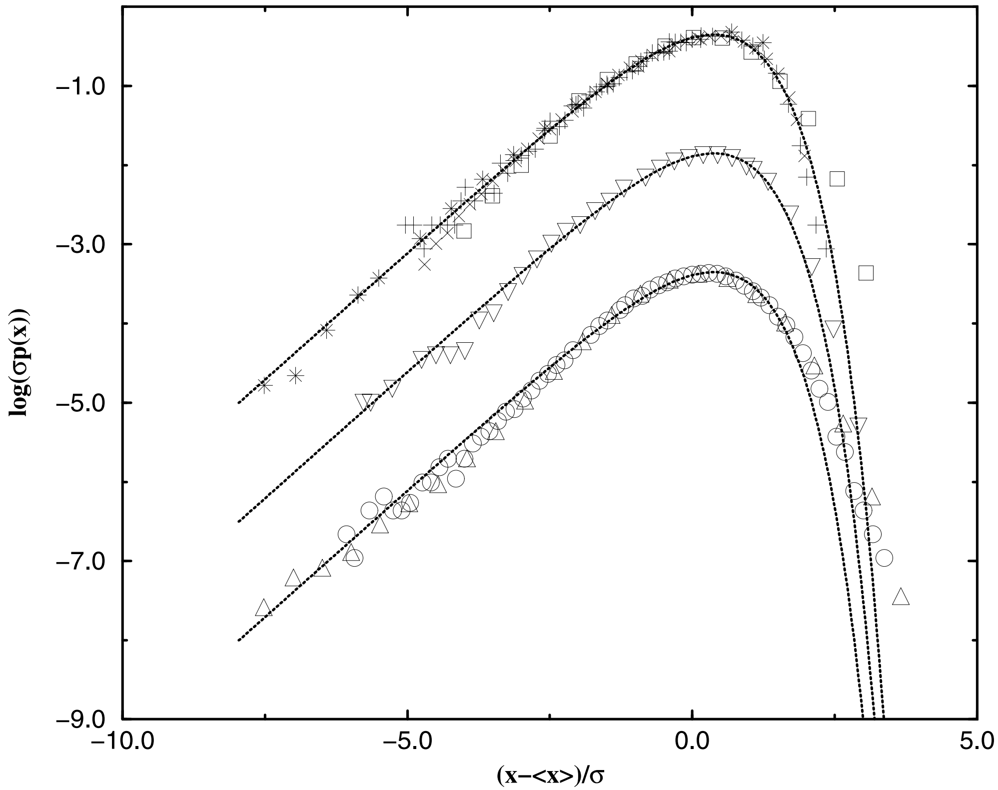


- *When does the Central Limit Theorem breaks down?*

  - Long range correlation: $\xi \propto L$ (system size)
  - Finite number $(L/\xi)^d$ of effective degrees of freedom
  - Distribution of global observables cannot converge to a Gaussian

Examples of non-Gaussian statistics

- *Critical equilibrium and non-equilibrium models*



Bramwell et. al., Phys. Rev. Lett. **84**, 3744 (2000)

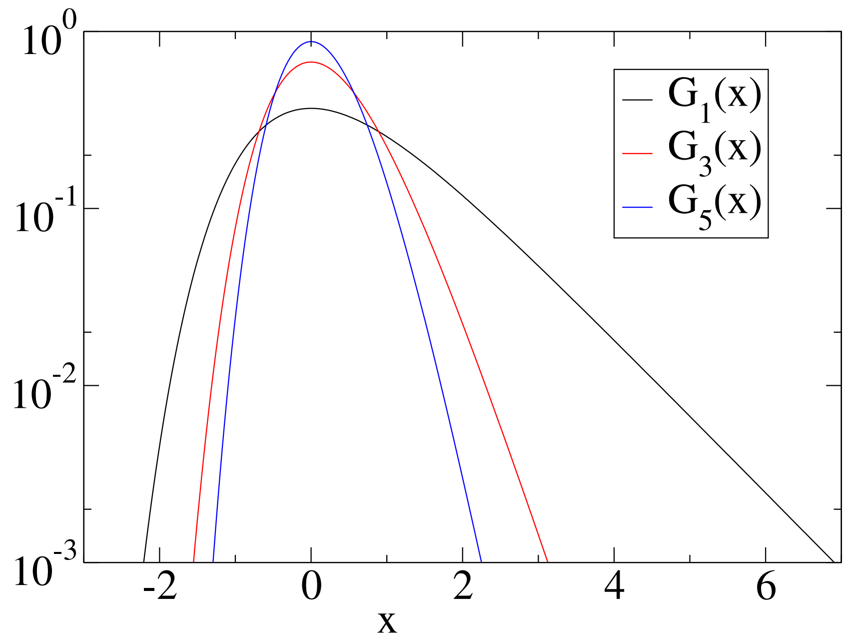
Observations

- *Generic properties of non-Gaussian distributions*
  - Often asymmetric, with an exponential tail on one side, and a rapid fall-off on the other side
  - A convenient fitting function: the generalized Gumbel distribution from extreme value statistics

$$G_a(x) = C \exp\left[-a\left(x + e^{-x}\right)\right]$$

Variable x obtained by rescaling X

a is the only fitting parameter



$1/f$ noise: an exactly solvable case

- *A simple model for correlated systems*
  - N statistically independent gaussian Fourier modes of complex amplitude c_n
  - Describes a 1D periodic system or a periodic signal
  - Distribution $P(c_n) \propto \exp(-\sigma\, n\, |c_n|^2)$; $\quad \langle |c_n|^2 \rangle \propto 1/n$

- *Global fluctuations*
  - Global observable = integrated power spectrum (total “energy”) $E = \sum_n |c_n|^2$
  - Exact analytical result: for $N \to \infty$, $P(E) \to G_1(x)$, upon rescaling E into x

Antal, Droz, Györgyi, Rácz, Phys. Rev. Lett. 87, 240601 (2001)

Open questions

- *Relation between global fluctuations in correlated systems and extreme value statistics?*
- *Is there a hidden extremal process?*
- *Interpretation of non-integer values of k in $G_k(x)$?*
- *Can one find an exactly solvable model with a generalized Gumbel distribution $G_a(x)$, $a > 0$ real?*

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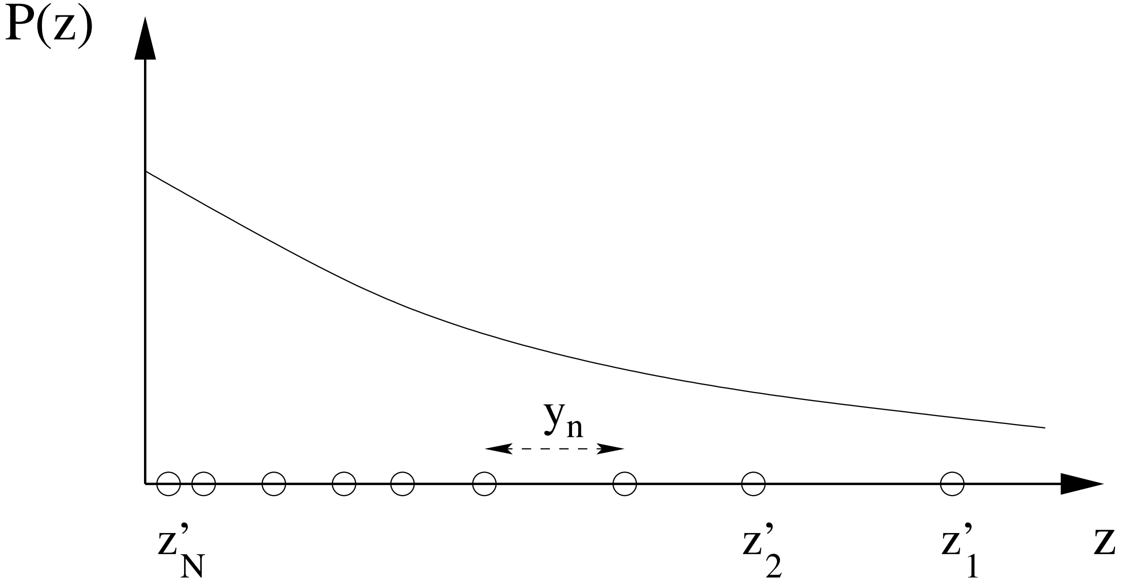
Reinterpreting EVS

- *Extreme values as random sums*
  - N random values z_j drawn from a distribution $P(z)$
  - Ordering: $z'_j = z_{\sigma(j)}$ ($\sigma =$ permutation) so that $z'_1 \geq z'_2 \geq ... \geq z'_N$
  - Intervals: $y_j = z'_j - z'_{j+1}$ and $y_N = z'_N$



  - $\max(z_j) \equiv z'_1 = \sum_{j=1}^{N} y_j$ (the y_j's are a priori correlated)

Reinterpreting EVS

- *Exponential case*
  - Simple case: $P(z) = \kappa\, e^{-\kappa z}$
  - The y_j's become statistically independent
  - Non-identical distributions: $p_n(y_n) = n\,\kappa\, e^{-n\,\kappa\, y_n}$
- *Distribution of the sum of the y_n's*
  - From EVS: distribution of $z_1' \equiv \max(z_j) \to$ Gumbel $G_1(x)$ when $N \to \infty$
  - $z_1' = \sum_{j=1}^{N} y_j \quad \Rightarrow \quad \sum y_j$ distributed according to $G_1(x)$
    Gumbel distribution $G_1(x)$ = sum of non-identical random variables
  - $1/f$ noise: $y_n = |c_n|^2$ and $p_n(y_n) = n\,\sigma\, e^{-n\,\sigma\, y_n}$
    Clear relationship between $1/f$ noise and EVS

Reinterpreting EVS

- *From the kth largest value...*
  - z'_k the kth largest value of the set (z_j)
  - From EVS: distribution of $z'_k \to G_k(x)$
- *... to a problem of sum*
  - In terms of sum: $z'_k = \sum_{j=k}^{N} y_j$
  - Relabeling: $u_j = y_{j+k-1}$, so that $z'_k = \sum_{j=1}^{N-k+1} u_j$
  - $\sum_{n=1}^{N'} u_n$ distributed according to $G_k(x)$, with $p_n(u_n) = (n+k-1)\kappa e^{-(n+k-1)\kappa u_n}$ when $N' \to \infty$

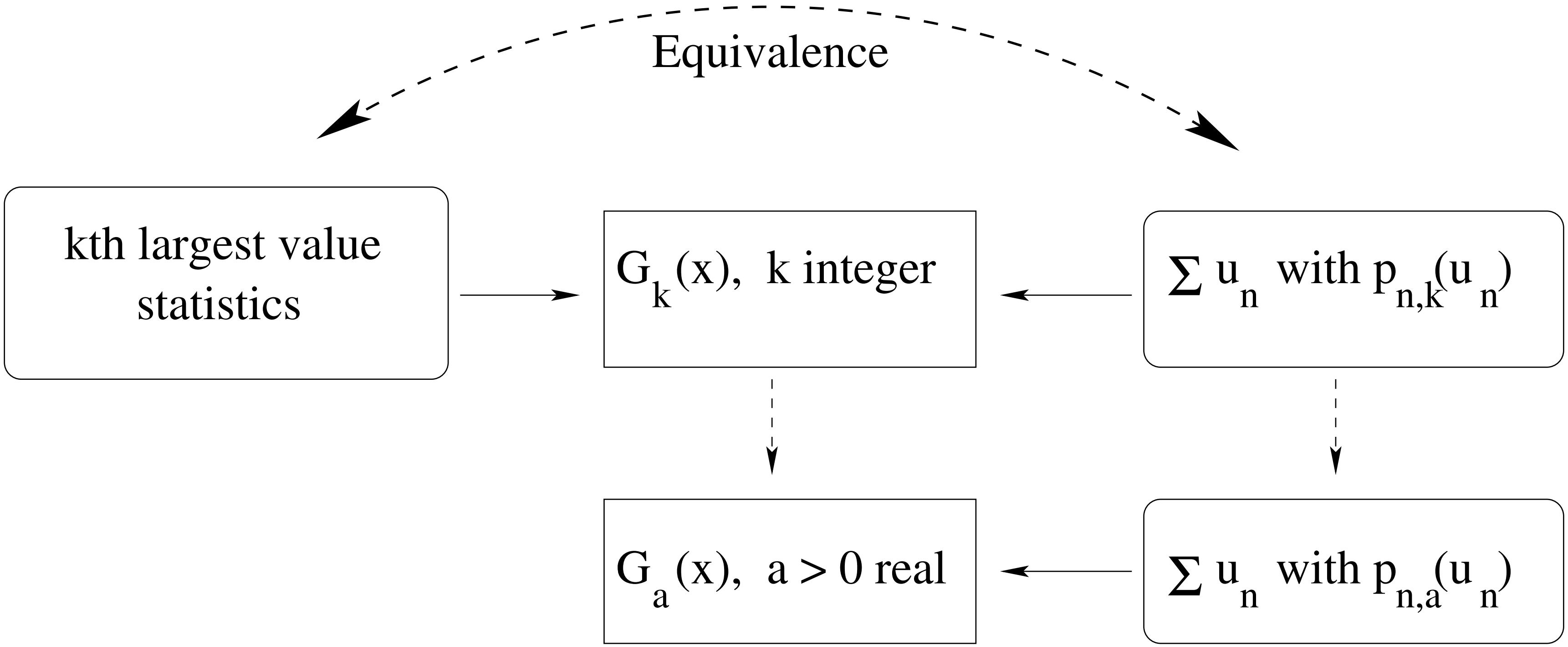
What about $G_a(x)$?

- *A more general problem of sum*
  - When considering $\sum u_n$, the parameter k appears only in $p_n(u_n)$
  - $p_n(u_n)$ easily generalized to $k = a > 0$ real:

$$p_n(u_n) = (n + a - 1)\kappa e^{-(n+a-1)\kappa u_n}$$

- *Distribution of the sum*
  - $\sum u_n$ distributed according to $G_a(x)$
  - Calculation independent of extreme value statistics

A visual summary



Beyond the exponential case

- *Gumbel class in extreme value statistics*
  - If $P(z)$ decays faster than any power, distribution of the kth largest value = Gumbel $G_k(x)$ $\quad \Rightarrow \quad$ generic limit distribution
  - Can one define an associated problem of sum for $k = a$ real for any $P(z)$ in the Gumbel class?
- *Distribution of the intervals from EVS*

$$J_k(y_k, ..., y_N) = \frac{N!}{(k-1)!} P(y_N) P(y_{N-1} + y_N) ... \times P(y_k + ... + y_N)\, F(y_k + ... + y_N)^{k-1}$$

with $F(x) = \int_x^\infty P(z)\, dz$

Sum of correlated variables u_n

- Distribution of the correlated variables u_n

$$\Phi(u_1, ..., u_N) = \frac{\Gamma(N)}{Z_N} P(u_N) P(u_{N-1} + u_N) ... \times P(u_1 + ... + u_N)\, \Omega[F(u_1 + ... + u_N)]$$

with $\Omega(F) \sim F^{a-1}$ for $F \to 0$, and $Z_N = \int_0^1 dv\, \Omega(v)(1-v)^{N-1}$

- Distribution of $\sum u_n$ is a Gumbel distribution $G_a(x)$
- Possibility to generalize to Fréchet and Weibull classes for EVS by suitably choosing $P(z)$

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A stochastic cascade model

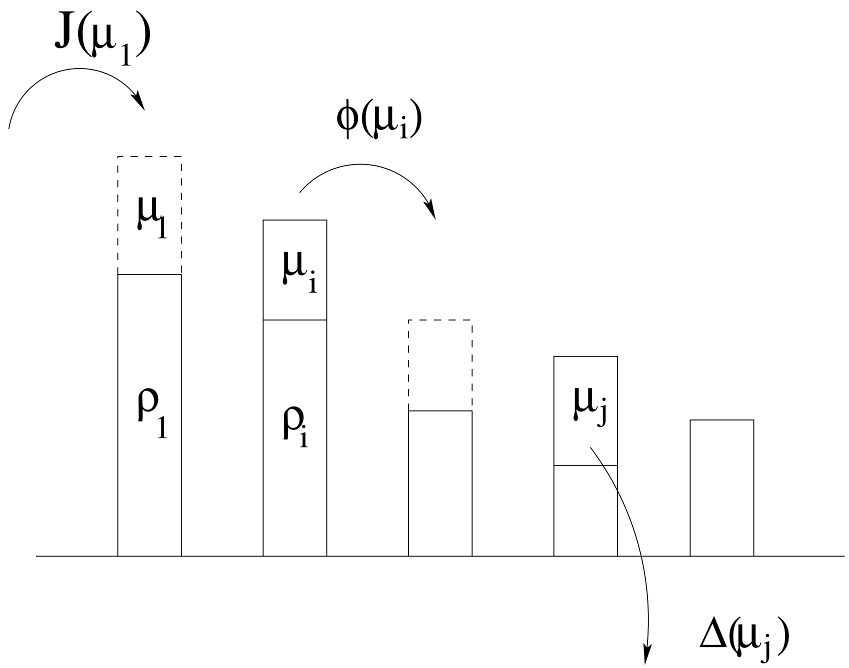
- *Motivation*
  - Propose an exactly solvable “microscopic” model with a Gumbel distribution $G_a(x)$
  - Physical interpretation of the parameter a?
- *Model and dynamical rules*

Energy ρ_i on each site

Transfer or dissipation of an amount of energy μ

$$J(\mu) = e^{-\beta\mu}\,\phi(\mu)$$

$$\Delta(\mu) = (e^{\lambda\mu} - 1)\,\phi(\mu)$$



A stochastic cascade model

- *Steady-state distribution of local variables*
  - Local energies ρ_n becomes independent in steady state
  - Distribution of ρ_n

$$p_n(\rho_n) = (\lambda n + \beta)\, e^{-(\lambda n+\beta)\rho_n}$$

- *Global observable*
  - Total energy: $E = \sum_n \rho_n$
  - Gumbel distribution $G_a(x)$ with $a = 1 + \beta/\lambda$
  - a related to the ratio between injection and dissipation
    Small dissipation $\rightarrow$ large a, and $G_a(x)$ close to a Gaussian

Correlation length

- *Truncated $1/f$ noise*
  - ρ_n interpreted as a power spectrum $|c_q|^2$, $q = 2\pi n/L$
  - Gaussian distribution $P_q(c_q) \propto \exp\left[-\left(\frac{\lambda L}{2\pi} q + \beta\right)|c_q|^2\right)$

$$\langle |c_q|^2 \rangle = \frac{1}{\lambda n + \beta} \propto \left(q + \frac{2\pi(a-1)}{L} \right)^{-1}$$

- *Power spectrum and correlation function*
  - Power spectrum = Fourier transform of the correlation function
  - Correlation length ξ defined through $\langle |c_q|^2 \rangle \sim 1/[q + \xi^{-1}]$

$$\xi = \frac{L}{2\pi(a-1)}$$

  - If $a \to 1$ ($1/f$ noise), $\xi \to \infty$; otherwise, $\xi \sim L$

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Conclusions

- *Clear relation between EVS and a class of sums of random variables with decreasing amplitudes*
- *Gumbel distribution $G_a(x)$ with a real defined from random sums*
- *Generalization of the central limit theorem to a class of correlated variables; $G_a(x)$ = limit distribution for the sum*
- *Truncated $1/f$ noise: a minimal model for correlated systems (1D, PBC, uncorrelated modes)*

E. Bertin, Phys. Rev. Lett. **95**, 170601 (2005)

E. Bertin and M. Clusel, J. Phys. A **39**, 7607 (2006)