

The differences of spectral projections and scattering matrix

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1 Motivation

Let H_0 and H be self-adjoint operators in \mathcal{H} , $\delta \subset \mathbb{R}$ be an interval; assume that $\sigma(H_0)$ and $\sigma(H)$ are purely a.c. in δ . Consider a direct integral decomposition which diagonalizes the part of H_0 corresponding to the interval δ :

$$\mathcal{H}_0(\delta) \equiv E_{H_0}(\delta)\mathcal{H} = \int_{\delta}^{\oplus} \mathfrak{h}_0(\lambda)d\lambda, \quad H_0|_{\mathcal{H}_0(\delta)}: f(\lambda) \rightarrow \lambda f(\lambda). \quad (1)$$

In the same way, one can consider the diagonalization of the part of H :

$$\mathcal{H}(\delta) \equiv E_H(\delta)\mathcal{H} = \int_{\delta}^{\oplus} \mathfrak{h}(\lambda)d\lambda, \quad H|_{\mathcal{H}(\delta)}: f(\lambda) \rightarrow \lambda f(\lambda). \quad (2)$$

Suppose we would like to compare these two diagonalizations. Informally speaking, we would like to compare the relative position of $\mathfrak{h}_0(\lambda)$ and $\mathfrak{h}(\lambda)$ in \mathcal{H} .

What does scattering theory tell us about this? Assume $\exists W_{\pm} = s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} E_{H_0}(\delta)$; we have $W_{\pm}H_0 = HW_{\pm}$.

Consider the (local) scattering operator: $S = W_{+}^*W_{-}$; $SH_0 = H_0S$. Thus, H_0 and S are simultaneously diagonalisable; i.e. in the decomposition (1), we have

$$S|_{\mathcal{H}_\delta}: f(\lambda) \rightarrow S(\lambda)f(\lambda),$$

where $S(\lambda) : \mathfrak{h}_0(\lambda) \rightarrow \mathfrak{h}_0(\lambda)$ is the scattering matrix.

$S(\lambda)$ is unitary; typically $S(\lambda) = I + \text{compact}$, so $\sigma(S(\lambda)) = \{e^{i\theta_n}\}$, $\theta_n \rightarrow 0$ as $n \rightarrow \infty$; we list eigenvalues counting their multiplicity.

This construction is motivated by physics; from the point of view of spectral analysis, the definition of $S(\lambda)$ seems to be a rather special one. Suppose we don't want to know

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anything about the wave operators and instead just look at the spectral decompositions of H and H_0 and try to compare them. Do the eigenvalues $e^{i\theta_n}$ appear naturally?

I cannot provide a complete answer to this problem but I will offer some observations which confirm that these questions are not vacuous.

I shall discuss the spectrum of the operator $f(H) - f(H_0)$ for some particular choices of f and describe this spectrum in terms of the eigenvalues $e^{i\theta_n}$. Informally speaking, I will try to convince you that $\theta_n/2$ can be viewed as regularised angles between the "eigenspaces" $\mathfrak{h}(\lambda)$ and $\mathfrak{h}_0(\lambda)$.

2 Angles between subspaces

Let me recall some well known facts about relative locations of subspaces in a Hilbert space. These facts go back at least to the paper [1], but they might be even older.

Let $u, v \in \mathcal{H}$ with norm one and $|(u, v)| = \cos \theta$. Let $P = (\cdot, u)u$, $Q = (\cdot, v)v$; then $\sigma(P - Q) = \{0, \sin \theta, -\sin \theta\}$.

Now let P, Q be orthogonal projections of finite rank in a Hilbert space. Then the relative location of $\text{Ran } P$ and $\text{Ran } Q$ is encoded in $\sigma(P - Q)$. More precisely, we have the following facts.

$$\begin{aligned} \sigma(P - Q) &\subset [-1, 1]; \\ (P - Q)f = f &\Leftrightarrow f \in \text{Ran } P \cap \text{Ker } Q = \mathcal{H}_{+1}; \\ (P - Q)f = -f &\Leftrightarrow f \in \text{Ran } Q \cap \text{Ker } P = \mathcal{H}_{-1}; \\ \mathcal{H} &= \mathcal{H}_{+1} \oplus \mathcal{H}_{-1} \oplus \mathcal{H}_0. \\ (P - Q)|_{\mathcal{H}_0} &\text{ is unitarily equivalent to } (Q - P)|_{\mathcal{H}_0}. \\ \sigma((P - Q)|_{\mathcal{H}_0}) &= \{0\} \cup (\cup_n \{-\sin \theta_n, \sin \theta_n\}). \end{aligned}$$

One can choose basis u_n in $\text{Ran } P \cap \mathcal{H}_0$ and basis v_n in $\text{Ran } Q \cap \mathcal{H}_0$ such that $u_n \perp v_m$ for $n \neq m$ and $|(u_n, v_n)| = \cos \theta_n$.

If P, Q are not of finite rank, but $P - Q$ is compact, one can still denote the eigenvalues of $P - Q$ by $-\sin \theta_n, \sin \theta_n$ and interpret θ_n as the angles between $\text{Ran } P$ and $\text{Ran } Q$. Note that in this case the spectrum of $P - Q$ (together with multiplicities of eigenvalues) uniquely determines the pair of subspaces $\text{Ran } P$ and $\text{Ran } Q$ up to unitary equivalence.

3 Spectrum of $E_H - E_{H_0}$

Suppose that the difference $H - H_0$ is compact (or, more generally, the difference of resolvents of H and H_0 is compact). Then, by a well known argument, $f(H) - f(H_0)$ is compact for all $f \in C_0(\mathbb{R})$. For f with discontinuities, this is in general false. In the abstract setting, an example of this kind was given by M.G.Krein in [2]; he found a pair of operators H, H_0 such that the difference $H - H_0$ has rank one yet the difference $f(H) - f(H_0)$ with a discontinuous f is not compact. (In fact, Krein showed that this difference is not Hilbert-Schmidt, but it is not difficult to see that in his example this difference is not compact; see recent preprint [3]).

Let us consider the simplest f with discontinuity: $f = \chi_{(-\infty, \lambda)}$; so $\chi_{(-\infty, \lambda)}(H) = E_H(-\infty, \lambda)$.

The results below are valid for fairly general pairs of self-adjoint operators H_0, H . However, instead of introducing abstract assumptions on H_0 and H , for the purposes of this presentation, I shall focus on a concrete model. Let $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$ and let $H = -\Delta + V$ where $|V(x)| \leq C(1 + |x|)^{-\rho}$ with $\rho > 1$.

For a.e. $\lambda > 0$, the SM $S(\lambda)$ is well defined and $\sigma(S(\lambda)) = \{e^{i\theta_n}\}$, $\theta_n \rightarrow 0$.

Theorem 1. *For any $\lambda \geq \lambda_0 > 0$, (here $\lambda_0 = \lambda_0(V)$) $\sigma_{\text{ess}}(E_H(-\infty, \lambda) - E_{H_0}(-\infty, \lambda)) = [-a, a]$ where $a = \|S(\lambda) - I\|/2$.*

Corollary. *$(E_H(-\infty, \lambda) - E_{H_0}(-\infty, \lambda))$ is compact iff $S(\lambda) = I$.*

Theorem 2. *Assume $\rho > d$ (trace class assumption). Then for any $\lambda \geq \lambda_0 > 0$, the a.c. part of $(E_H(-\infty, \lambda) - E_{H_0}(-\infty, \lambda))$ is unitarily equivalent to the operator of multiplication by x in the direct sum $\oplus_n L^2([- \sin(\theta_n/2), \sin(\theta_n/2)], dx)$.*

Remarks:

1. The two theorems agree: $\|S(\lambda) - I\|/2 = \max_n |e^{i\theta_n} - 1|/2 = \max_n \sin(\theta_n/2)$.
2. The quantity $\sin(\theta_n/2)$ is invariant with respect to the change $\theta_n \mapsto 2\pi - \theta_n$. This is to be expected, as swapping H_0 and H replaces $S(\lambda)$ by $S(\lambda)^*$.
3. In particular, if $S(\lambda)$ has an eigenvalue -1 , this corresponds to the eigenspaces of H and H_0 at λ being "orthogonal" to each other.
4. If f has several discontinuities, one can apply Theorem 1 in an obvious way, combining information on the values of the scattering matrix at each discontinuity.

4 Spectrum of $f(H) - f(H_0)$

How can one measure the angles between the eigenspace of H_0 at λ and the eigenspace of H at λ ? One can try considering the spectrum of the difference $E_H(\lambda - \epsilon, \lambda + \epsilon) - E_{H_0}(\lambda - \epsilon, \lambda + \epsilon)$ for $\epsilon \rightarrow +0$. But previous theorems show that this difference will have some essential spectrum. Instead, consider the spectrum of the difference $f((H - \lambda)/\epsilon) - f((H_0 - \lambda)/\epsilon)$ where f is some continuous positive even function which decays at infinity.

Theorem 3. *Let $f(x) = \frac{2}{1+x^2}$; then for any $\lambda \geq \lambda_0 > 0$, we have*

$$\sigma(f((H - \lambda)/\epsilon) - f((H_0 - \lambda)/\epsilon)) \rightarrow \cup_n \{-\sin(\theta_n/2), \sin(\theta_n/2)\}$$

as $\epsilon \rightarrow 0$ in the following sense. There exists a compact self-adjoint operator A with $\sigma(A) = \{0\} \cup (\cup_n \{-\sin(\theta_n/2), \sin(\theta_n/2)\})$ and a family of unitaries $U(\epsilon)$ such that

$$\|f((H - \lambda)/\epsilon) - f((H_0 - \lambda)/\epsilon) - U(\epsilon)AU(\epsilon)^*\| \rightarrow 0.$$

What about general f 's? Consider the Hankel type integral operator in $L^2(0, \infty)$ with the integral kernel $\widehat{f}(x + y)$ (here \widehat{f} is the Fourier transform of f). This is a compact self-adjoint operator; denote its eigenvalues by α_m .

Theorem 4. For any $\lambda \geq \lambda_0 > 0$, we have

$$\sigma(f((H - \lambda)/\epsilon) - f((H_0 - \lambda)/\epsilon)) \rightarrow \cup_{n,m} \{-2\alpha_m \sin(\theta_n/2), 2\alpha_m \sin(\theta_n/2)\}$$

as $\epsilon \rightarrow 0$ in the same sense as in the previous theorem.

The function $f(x) = \frac{2}{1+x^2}$ is special because in this case the corresponding Hankel type operator has rank one with the only non-zero eigenvalue being $1/2$.

References

- [1] P. R. Halmos, *Two subspaces* Trans. Amer. Math. Soc. **144** (1969), 381–389.
- [2] M. G. Krein, *On the trace formula in perturbation theory (Russian)* Mat. Sb. **33 (75)**, no. 3 (1953), 597–626.
- [3] V. Kostykin, K. Makarov, *On Krein's Example*, preprint 2006, arxiv.math.SP/0606249