Relative pairing in cyclic cohomology and divisor flows

Matthias Lesch, Henri Moscovici and Markus Pflaum

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1 Relative pairings

Given a short exact sequence of chain complexes

$$0 \longrightarrow (K_\bullet, \partial_K) \stackrel{\kappa}{\longrightarrow} (A_\bullet, \partial_A) \stackrel{\alpha}{\longrightarrow} (B_\bullet, \partial_B) \longrightarrow 0 \tag{1}$$

consider the mapping-cone

$$\widetilde{K}_k := M(\alpha)_{k+1} := A_k \oplus B_{k+1}, \quad \widetilde{\partial} := \begin{pmatrix} \partial_A & 0 \\ -\alpha & -\partial_B \end{pmatrix}.$$

Then

$$K_\bullet \to \widetilde{K}_\bullet, \quad c_k \mapsto (\kappa(c_k), 0)$$

is a quasi-isomorphism.

For a short exact sequence of cochain complexes

$$0 \longrightarrow (F^\bullet, d_F) \stackrel{\varepsilon}{\longrightarrow} (E^\bullet, d_E) \stackrel{\delta}{\longrightarrow} (Q^\bullet, d_Q) \longrightarrow 0, \tag{2}$$

the cochain complex $(\widetilde{Q}^\bullet, \widetilde{d})$ with

$$\widetilde{Q}^k := M(\varepsilon)^k := E^k \oplus F^{k+1}, \quad \widetilde{d} := \begin{pmatrix} d_E & -\varepsilon \\ 0 & -d_F \end{pmatrix}$$

is quasi-isomorphic to $(Q^\bullet, d_Q)$.

If in the short exact sequences (1) and (2) the cochain complex $E^\bullet$ is dual to $A_\bullet$ and $F^\bullet$ dual to $B_\bullet$, then $\widetilde{Q}^\bullet$ is dual to $\widetilde{K}_\bullet$. Moreover, this duality induces the natural pairing

$$\begin{aligned} \langle -, - \rangle : H^k(\widetilde{Q}^\bullet) \times H_k(\widetilde{K}_\bullet) \to \mathbb{C}, \\ ([\varphi_k, \psi_{k+1}], [a_k, b_{k+1}]) \mapsto \langle \varphi_k, a_k \rangle + \langle \psi_{k+1}, b_{k+1} \rangle. \end{aligned}$$

2 Relative cyclic (co)homology

Recall that the cyclic cohomology of a Fréchet algebra $\mathcal{A}$ is given by the cohomology of $\mathrm{Tot}^{\bullet}_{\oplus}\,\mathcal{B}C^{\bullet,\bullet}(\mathcal{A})$, the total cohomology of the bicomplex

$$\begin{array}{ccccccc}
\uparrow b & & \uparrow b & & \uparrow b & & \uparrow b \\
C^3(\mathcal{A}) & \xrightarrow{B} & C^2(\mathcal{A}) & \xrightarrow{B} & C^1(\mathcal{A}) & \xrightarrow{B} & C^0(\mathcal{A}) \\
\uparrow b & & \uparrow b & & \uparrow b & & \\
C^2(\mathcal{A}) & \xrightarrow{B} & C^1(\mathcal{A}) & \xrightarrow{B} & C^0(\mathcal{A}) & & \\
\uparrow b & & \uparrow b & & & & \\
C^1(\mathcal{A}) & \xrightarrow{B} & C^0(\mathcal{A}) & & & & \\
\uparrow b & & & & & & \\
C^0(\mathcal{A}) & & & & & &
\end{array}$$

associated to the mixed complex $(C^{\bullet}(\mathcal{A}), b, B)$.

Assume to be given a continuous surjective homomorphism $\sigma : \mathcal{A} \to \mathcal{B}$ of Fréchet algebras. Then there are mixed complexes $(C^\bullet(\mathcal{A}), b, B)$ and $(C^\bullet(\mathcal{B}), b, B)$, and one obtains an exact sequence of mixed complexes

$$0 \to C^\bullet(\mathcal{B}) \xrightarrow{\sigma^*} C^\bullet(\mathcal{A}).$$

Then one is in a relative situation, and the relative cyclic cohomology $HC^\bullet(\mathcal{A}, \mathcal{B})$ coincides with the cohomology of the sum of total complexes

$$\big(\operatorname{Tot}_\oplus^\bullet \mathcal{B}C^{\bullet,\bullet}(\mathcal{A}) \oplus \operatorname{Tot}_\oplus^{\bullet+1} \mathcal{B}C^{\bullet,\bullet}(\mathcal{B}), \widetilde{b+B}\big),$$

where the differential is given by

$$\widetilde{b+B} = \begin{pmatrix} b+B & -\sigma^* \\ 0 & -(b+B) \end{pmatrix}.$$

Dually, the relative cyclic homology $HC_\bullet(\mathcal{A}, \mathcal{B})$ is the homology of $\big(\operatorname{Tot}_\bullet^\oplus \mathcal{B}C_{\bullet,\bullet}(\mathcal{A}, \mathcal{B}), \widetilde{b} + \widetilde{B}\big)$, where $\mathcal{B}C_{p,q}(\mathcal{A}, \mathcal{B}) = \mathcal{B}C_{p,q}(\mathcal{A}) \oplus \mathcal{B}C_{p,q+1}(\mathcal{B})$, while

$$\widetilde{b} = \begin{pmatrix} b & 0 \\ -\sigma_* & -b \end{pmatrix}, \text{ and } \widetilde{B} = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}.$$

3 Relative K-theory and elliptic paths

Let $\mathcal{A}$ and $\mathcal{B}$ be two local unital Fréchet algebras and

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \xrightarrow{\sigma} \mathcal{B} \longrightarrow 0$$

an exact sequence of Fréchet algebras and unital homomorphisms. Then the canonical homomorphism

$$\begin{aligned} \kappa: \; K_1(\mathcal{A}, \mathcal{B}) &\to \pi_1\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A}); I\big), \\ I + S &\mapsto \big[[0,1] \ni s \mapsto I + sS \in \operatorname{Ell}_\infty(\mathcal{A})\big] \end{aligned}$$

is an isomorphism, where $\operatorname{Ell}_\infty(\mathcal{A}) := \sigma^{-1}(\operatorname{GL}_\infty(\mathcal{B}))$.

The paths $(a_s)_{0 \le s \le 1}$ leading to elements of $\pi_1\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A})\big)$ are called admissible elliptic paths.

4 The relative Chern character

Let $\mathcal{A}$ be a Fréchet algebra $\mathcal{A}$ and $g \in \mathrm{GL}_\infty(\mathcal{A})$. Then the odd Chern character of g is the following normalized cyclic cycle:

$$\mathrm{ch}_\bullet(g) = \sum_{k=0}^{\infty} (-1)^k k!\, \mathrm{tr}_{2k+1}\left((g^{-1} \otimes g)^{\otimes(k+1)}\right).$$

For a continuous family of invertible matrices $g_s \in \mathrm{GL}_\infty(\mathcal{A})$, $s \in [0,1]$ one has the following transgression formula by Getzler:

$$\frac{d}{ds}\,\mathrm{ch}_\bullet(g_s) = (b+B)\, \not{\mathrm{c}}\mathrm{h}_\bullet(g_s, \dot{g}_s), \tag{3}$$

where the secondary Chern character $\not{\mathrm{c}}\mathrm{h}$ is defined by

$$\begin{aligned}\not{\mathrm{c}}\mathrm{h}_\bullet(g,h) &= \mathrm{tr}_0(g^{-1}h) + \\ &+ \sum_{k=0}^{\infty}(-1)^{k+1}k! \sum_{j=0}^{k} \mathrm{tr}_{2k+2}\left((g^{-1}\otimes g)^{\otimes(j+1)} \otimes g^{-1}h \otimes (g^{-1}\otimes g)^{\otimes(k-j)}\right).\end{aligned}$$

Using the transgression formula (3) one shows that for a continuous surjective algebra homomorphism $\sigma : \mathcal{A} \to \mathcal{B}$ and a smooth admissible elliptic path $(a_s)_{0\le s\le 1}$ in $\mathfrak{gl}_N(\mathcal{A})$ the pair

$$\mathrm{ch}_\bullet\left((a_s)_{0\le s\le 1}\right) := \left(\mathrm{ch}_\bullet(a_1) - \mathrm{ch}_\bullet(a_0), -\int_0^1 \not{\mathrm{c}}\mathrm{h}_\bullet\left(\sigma_*(a_s), \sigma_*(\dot{a}_s)\right) ds\right)$$

is well-defined and defines a relative cyclic cycle, the so-called *Chern character* of the family $(a_s)_{0\le s\le 1}$.

5 Relative cycles and their characters

A *relative cycle* of degree k over $(\mathcal{A}, \mathcal{B})$ consists of the following data:

1. differential graded unital algebras (Ω, d) and $(\partial\Omega, d)$ over $\mathcal{A}$ resp. $\mathcal{B}$ together with a surjective unital homomorphism $r : \Omega \to \partial\Omega$ of degree 0,
2. unital homomorphisms $\varrho_{\mathcal{A}} : \mathcal{A} \to \Omega^0$ and $\varrho_{\mathcal{B}} : \mathcal{B} \to \partial\Omega^0$ such that $r \circ \varrho_{\mathcal{A}} = \varrho_{\mathcal{B}} \circ \sigma$,
3. a graded trace $\int$ on Ω of degree k such that

$$\int d\omega = 0\,, \quad \text{whenever} \quad r(\omega) = 0.$$

The graded trace $\int$ induces a unique closed graded trace $\int'$ on $\partial\Omega$ of degree $k-1$, such that Stokes' formula

$$\int d\omega = \int' r\omega\,, \qquad \text{for } \omega \in \Omega$$

is satisfied.

Let C be a relative cycle of degree k over $(\mathcal{A}, \mathcal{B})$. Define $(\varphi_k, \psi_{k-1}) \in C^k(\mathcal{A}) \oplus C^{k-1}(\mathcal{B})$ as follows:

$$\varphi_k(a_0, \ldots, a_k) := \frac{1}{k!} \int \varrho(a_0) d\varrho(a_1) \ldots d\varrho(a_k), \tag{4}$$

$$\psi_{k-1}(b_0, \ldots, b_{k-1}) := \frac{1}{(k-1)!} \int' \varrho(b_0) d\varrho(b_1) \ldots d\varrho(b_{k-1}). \tag{5}$$

Then (φ_k, ψ_{k-1}) is a relative cyclic cocycle in $\mathrm{Tot}^k_{\oplus} \mathcal{B}C^{\bullet,\bullet}(\mathcal{A}, \mathcal{B})$ (cf. Connes, Gorokhovsky).

6 Divisor flow associated to a relative cycle

The divisor flow of a smooth admissible elliptic path $(a_s)_{0\leq s\leq 1}$ with respect to the relative cycle C is defined to be the relative pairing between $\mathrm{ch}_\bullet\left((a_s)_{0\leq s\leq 1}\right)$ and the character $(\varphi_{2k+1},\psi_{2k})$, i.e.

$$\begin{aligned}&\mathrm{DF}_C\left((a_s)_{0\leq s\leq 1}\right) := \mathrm{DF}\left((a_s)_{0\leq s\leq 1}\right) :=\\ &:= \frac{1}{(-2\pi i)^{k+1}}\langle(\varphi_{2k+1},\psi_{2k}), \mathrm{ch}_\bullet((a_s)_{0\leq s\leq 1})\rangle\\ &= \frac{1}{(-2\pi i)^{k+1}}\big(\langle\varphi_{2k+1}, \mathrm{ch}_\bullet(a_1)\rangle - \langle\varphi_{2k+1}, \mathrm{ch}_\bullet(a_0)\rangle\big)\\ &\quad - \frac{1}{(-2\pi i)^{k+1}}\langle\psi_{2k}, \int_0^1 \not{\mathrm{ch}}_\bullet(\sigma(a_s),\sigma(\dot{a}_s))ds\rangle.\end{aligned}$$

By homotopy invariance of cyclic homology, the divisor flow even defines a map

$$\mathrm{DF}_C : K_1(\mathcal{A},\mathcal{B}) \cong \pi_1\big(\,\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A})\big) \to \mathbb{C}$$

which is additive with respect to composition of paths.

7 Parametric pseudodifferential operators

For $U \subset \mathbb{R}^n$ open denote by $\mathrm{S}^m(U;\mathbb{R}^N)$ the space of symbols of Hörmander type $(1,0)$. It consists of those $a \in C^\infty(U \times \mathbb{R}^N)$ such that for multi-indices $\alpha \in \mathbb{Z}_+^m, \gamma \in \mathbb{Z}_+^N$ and each compact subset $K \subset U$ there is an estimate

$$\left|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi)\right| \leq C_{\alpha,\gamma,K}(1+|\xi|)^{m-|\gamma|}, \quad x \in K, \xi \in \mathbb{R}^N.$$

For each fixed $a \in \mathrm{S}^m(U;\mathbb{R}^n \times \mathbb{R}^p)$ and μ_0 one has $a(\cdot,\cdot,\mu_0) \in \mathrm{S}^m(U;\mathbb{R}^n)$, hence one obtains a family of pseudodifferential operators parameterized over $\mathbb{R}^p$ by putting

$$\begin{aligned}\left[\,\mathrm{Op}(a(\mu_0))\,u\right](x) := \left[A(\mu_0)\,u\right](x) \\ := \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\, a(x,\xi,\mu_0)\,\hat{u}(\xi)\,đ\xi \\ = \int_{\mathbb{R}^n}\int_U e^{i\langle x-y,\xi\rangle}\, a(x,\xi,\mu_0)\,u(y) đy đ\xi.\end{aligned}$$

Hereby, $đ$ denotes $(2\pi)^{-n/2}$-times the Lebesgue measure on $\mathbb{R}^n$.

The resulting space of operators is $\mathrm{L}^m(U;\mathbb{R}^p)$, the space of parameter dependent pseudodifferential operators. Similarly, one defines the subspace $\mathrm{CL}^m(U;\mathbb{R}^p) \subset \mathrm{L}^m(U;\mathbb{R}^p)$ of classical parameter dependent pseudodifferential operators.

By local patching one defines the spaces $\mathrm{CL}^m(M,E;\mathbb{R}^p)$ and $\mathrm{L}^m(M,E;\mathbb{R}^p)$ of classical parameter dependent pseudodifferential operators between sections of a vector bundle E over a smooth manifold M.

8 Regularized traces for parametric pseudodifferential operators

There is a unique linear extension

$$\mathrm{TR} : \mathrm{CL}^{\infty}(M, E; \mathbb{R}^p) \to \mathrm{PS}^{\infty}(\mathbb{R}^p)/\mathbb{C}[\mu_1, \ldots, \mu_p]$$

of TR to operators of all orders such that

1. $\mathrm{TR}(AB) = \mathrm{TR}(BA)$, i.e. TR is a "trace".
2. $\mathrm{TR}(\partial_j A) = \partial_j \,\mathrm{TR}(A)$ for $j = 1, ..., p$.

This unique extension TR satisfies furthermore:

3. $\mathrm{TR}(\mu_j A) = \mu_j \,\mathrm{TR}(A)$ for $j = 1, ..., p$.
4. $\mathrm{TR}(\mathrm{CL}^m(M, E; \mathbb{R}^p)) \subset \mathrm{PS}^{m+n}(\mathbb{R}^p)/\mathbb{C}[\mu_1, \ldots, \mu_p]$.

(cf. Lesch–Pflaum)

The *extended trace* $\overline{\mathrm{TR}}$ on $\Omega := \mathrm{CL}^{\infty}(M, E; \mathbb{R}^p) \otimes \Lambda^{\bullet} T_0^* \mathbb{R}^p$ is obtained by regularizing the usual integral over $\mathbb{R}^p$. For a p-form $A(\mu) d\mu_1 \wedge \ldots \wedge d\mu_p$ it is defined by

$$\overline{\mathrm{TR}}(A(\mu) d\mu_1 \wedge \ldots \wedge d\mu_p) := \displaystyle{\not}\!\!\int_{\mathbb{R}^p} \mathrm{TR}(A)(\mu) d\mu_1 \wedge \ldots \wedge d\mu_p.$$

The *formal trace* $\widetilde{\mathrm{TR}} := d\overline{\mathrm{TR}} = \overline{\mathrm{TR}} \circ d$ is a closed graded trace of degree $p - 1$ on $\partial\Omega := \mathrm{CL}^{\infty}(M, E; \mathbb{R}^p)/\mathrm{CL}^{-\infty}(M, E; \mathbb{R}^p) \otimes \Lambda^{\bullet} T_0^* \mathbb{R}^p$.

With these data, $C := (\Omega, \partial\Omega, \overline{\mathrm{TR}}, \widetilde{\mathrm{TR}})$ is a relative cycle.

9 Melrose's divisor flow and its higher analogues

Let $A_s \in \mathrm{CL}^{\infty}_{2k+1}$, $s \in [0,1]$, be a smooth family of elliptic operators of some fixed order $m \in \mathbb{N}$ such that A_0 and A_1 are invertible. The pairing in relative cyclic homology of the character $\mathrm{ch}_\bullet\big((A_s)_{0\le s\le 1}\big)$ with the character of the relative cycle C gives the divisor flow

$$\begin{aligned}\mathrm{DF}\big((A_s)_{0\le s\le 1}\big) &= c_k\Big(\overline{\mathrm{TR}}((A_1^{-1}dA_1)^{2k+1}) - \overline{\mathrm{TR}}((A_0^{-1}dA_0)^{2k+1})\Big)\\ &\quad - (2k+1)c_k \int_0^1 \widetilde{\mathrm{TR}}(A_s^{-1}\,\partial_s A_s(A_s^{-1}\,dA_s)^{2k})\,ds,\\ &\quad \text{with } c_k := \frac{k!}{(-2\pi i)^{k+1}(2k+1)!}.\end{aligned}$$

For $k=0$ this is the divisor flow of Melrose.

The divisor flow has the following properties:

1. Its natural domain is $\pi_1\Big(\mathrm{Ell}^m_\infty(\mathrm{CL}^0_{2k+1}), \mathrm{GL}_\infty(\mathrm{CL}^0_{2k+1})\Big)$.

2. It is additive with respect to composition of paths as well as with respect to multiplication of paths.

3. The divisor flow induces an ismorphism

$$K_1(\mathrm{CL}^0_{2k+1}, \mathrm{C}\mathcal{S}^0_{2k+1}) \longrightarrow \mathbb{Z}.$$

10 Cohomological interpretation of the spectral flow

Let $(D_s)_{0\leq s\leq 1}$ be a smooth path of first order self-adjoint elliptic differential operators acting on sections of a vector bundle E over a compact riemannian manifold M.

Then the map $[0,1] \ni s \mapsto \mathcal{D}_s^{\pm}(\mu) = D_s \pm c(\mu)$ is a smooth path of elliptic elements in CL^1_{2k+1} such that $\mathcal{D}_0, \mathcal{D}_1$ are invertible. Hereby, $c : \mathbb{R}^{2k+1} \to \mathfrak{gl}_{2^k}(\mathbb{C})$ is the unique irreducible Clifford representation which sends the volume form to the identity. Then one has the equality

$$\mathrm{DF}\big((\mathcal{D}_s^{\pm})_{0\leq s\leq 1}\big) = \pm\,\mathrm{SF}\big((D_s)_{0\leq s\leq 1}\big)\,,$$

where $\mathrm{SF}\big((D_s)_{0\leq s\leq 1}\big)$ denotes the spectral flow of the operator family $(D_s)_{0\leq s\leq 1}$.