Noncommutative Geometry on Q-Spaces of $\mathbb{Q}$-Lattices

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This is joint work, in progress, with **A. Connes** on the complex NC-geometry of the quotient space of $\mathbb{Q}$-lattices in $\mathbb{C}$ modulo commensurability. It builds on our prior work on modular Hecke algebras and their Hopf symmetry, and on the **Connes-Marcolli** C^*-algebraic framework for $\mathbb{Q}$-lattice spaces.

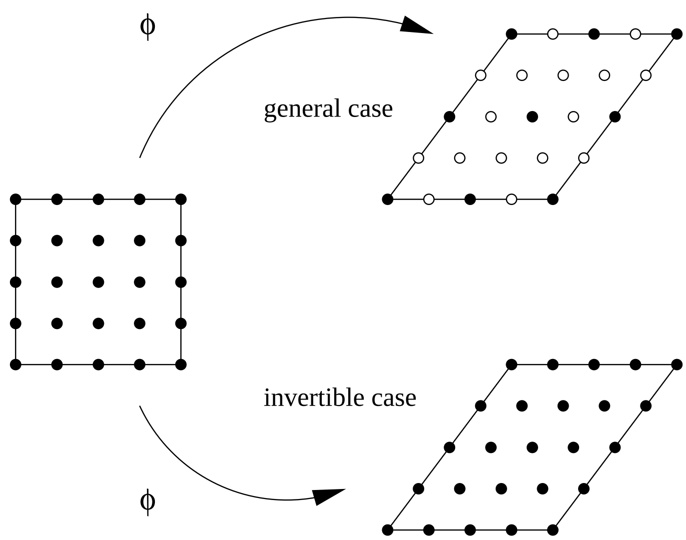
The emerging spectral-geometric picture, resembling the transverse geometry of a generic codimension 1 foliation, has notable arithmetic features.

$\mathbb{Q}$-**lattices** [Connes-Marcolli]

$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \ \omega_1, \omega_2 \in \mathbb{C}$, $\mathbb{R}$-linearly indep.

(Λ, ϕ) with $\phi : \mathbb{Q}^2/\mathbb{Z}^2 \longrightarrow \mathbb{Q}\Lambda/\Lambda$ additive.

- $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$ iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$, and $\phi_1 - \phi_2 \equiv 0 \mod \Lambda_1 + \Lambda_2$.



(Λ, ϕ) is *invertible* if ϕ is an isomorphism; two invertible $\mathbb{Q}$-lattices are commensurable iff they are equal, therefore

$$\mathcal{L}^\times = \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}) \subset \mathcal{L} = \{(\Lambda, \phi)\}/\sim$$

Groupoid description

$\mathcal{L}$ is a locally compact groupoid $\cong$ the quotient of the locally compact groupoid

$$\mathcal{U} = \{(\alpha, \rho, g) \quad \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathrm{GL}_2^+(\mathbb{R})\,; \quad \alpha\rho \in M_2(\hat{\mathbb{Z}})\}\,,$$

$$r[\alpha, \rho, g] = [\alpha\rho, \alpha g] \qquad s[\alpha, \rho, g] = [\rho, g]\,,$$

$$[\alpha_1, \rho_1, g_1] \circ [\alpha_2, \rho_2, g_2] = [\alpha_1\alpha_2, \rho_2, g_2]\,,$$

by the action of $\Gamma \times \Gamma$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

$$(\gamma_1, \gamma_2)\,(g, \rho, g) = (\gamma_1\alpha\gamma_2^{-1}, \gamma_2\rho, \gamma_2 g)\,.$$

The above isomorphism is implemented by

$$[\alpha, \rho, g] \mapsto \Big((g^{-1}\alpha^{-1}\Lambda_0, g^{-1}\rho), (g^{-1}\Lambda_0, g^{-1}\rho)\Big)$$

$\Lambda_0 = \mathbb{Z}e_1 + \mathbb{Z}e_2$, with $e_1 = 1$ and $e_2 = -i$.

Coordinates

• $C_c(\mathcal{L})$:= algebra of $\Gamma \times \Gamma$-invariant continuous functions on $\mathcal{U}$ with compact support modulo $\Gamma \times \Gamma$, with product $(f_1 * f_2)(\alpha, \rho, g) =$

$$\sum_{\beta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q}),\, \beta\rho \in M_2(\hat{\mathbb{Z}})} f_1(\alpha\beta^{-1}, \beta\rho, \beta g)\, f_2(\beta, \rho, g),$$

and involution $f^*(\alpha, \rho, g) = \overline{f(\alpha^{-1}, \alpha\rho, \alpha g)}$.

• $C^*(\mathcal{L})$:= C^*-completion in the regular representation of the groupoid (cf. *infra*).

• $\mathcal{B} := \mathbb{C}^* \backslash \mathcal{L}$, although not quite a groupoid, has $C^*(\mathcal{B}) := C^*(\mathcal{L})^{\mathbb{C}^*}$ as algebra of coordinates. Here $\lambda = a + ib \in \mathbb{C}$ is identified to $\lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$.

• Similarly, $\mathcal{L}^K := K \backslash \mathcal{L}$ has coordinate algebra $C^*(\mathcal{L}^K) := C^*(\mathcal{L})^K$, where $K := \mathrm{SO}(2)$.

• The units space $\mathcal{L}^{(0)} = \Gamma\backslash(M_2(\hat{\mathbb{Z}}) \times \mathrm{GL}_2^+(\mathbb{R}))$ is endowed with the **invariant measure**

$$d\nu(\rho, g) := d^+\rho \otimes d^+g \,,$$

where $d^+\rho, d^+g$ are additive measures on $M_2()$.

The contravariant change of variables on the finite adeles is compensated by the covariant change on 2×2-matrices over $\mathbb{R}$:

$$\begin{aligned} d^+(\beta^{-1}\rho) &= (\det \beta)^2 \, d^+\rho \\ d^+(\beta^{-1}g) &= (\det \beta)^{-2} \, d^+g \,. \end{aligned}$$

For $y = (\rho, g) \in Y := M_2(\hat{\mathbb{Z}}) \times \mathrm{GL}_2^+(\mathbb{R})$, let $\mathcal{G}_y = \{\alpha \in \mathrm{GL}_2^+(\mathbb{Q}) \,|\, \alpha y \in Y\}$, $\quad \mathfrak{H}_y = \ell^2(\Gamma\backslash\mathcal{G}_y)$,

$$(\pi_y(f)\,\xi)(\alpha) := \sum_{\beta \in \Gamma\backslash\mathcal{G}_y} f(\alpha\beta^{-1}, \beta y)\,\xi(\beta)\,.$$

• The ν-regular representation of $C^*(\mathcal{L})$ is

$$\pi(f) := \int_{\Gamma\backslash Y}^{\oplus} \pi_y(f)\, d\nu(y)\,,$$

$$||f|| := \sup_{y \in Y} ||\pi_y(f)||\,.$$

Modular forms as lattice functions

A *modular form of weight* $k \in \mathbb{Z}^+$ is a holomorphic function F on $\mathbb{H} = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$, s.t.

$$F|_k\,\gamma = F\,, \quad \forall\,\gamma \in \Gamma \equiv \Gamma(1) := \mathrm{SL}_2(\mathbb{Z})\,;$$

$$F|_k\,g\,(z) := \det(\alpha)^{k/2}\,(cz+d)^{-k}F(g\cdot z)\,,$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R}), \quad g\cdot z = \frac{az+b}{cz+d},$$

whose Fourier expansion at ∞, $\widehat{F}(q)$, $q = e^{2\pi iz}$, is holomorphic at $q = 0$. The corresponding lattice function is

$$\tilde{F}(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)) \;\; := \;\; \omega_2^{-k}\,F(\frac{\omega_1}{\omega_2})$$

$$\omega_1, \omega_2 \in \mathbb{C}^*, \qquad \mathrm{Im}\,\frac{\omega_1}{\omega_2} \;\; > \;\; 0\,,$$

or equivalently, $\quad \forall\, g \in \mathrm{GL}_2^+(\mathbb{R})$,

$$\tilde{F}(g^{-1}\Lambda_0) \;:=\; (\det g)^{\frac{k}{2}}\,(F|_k\,g)\,(i)\,.$$

Eisenstein series

- For $k > 2$ and $\mathbf{a} = (a_1, a_2) \in (\mathbb{Q}/\mathbb{Z})^2$

$$G_{\mathbf{a}}^{(k)}(z) := \sum_{\mathbf{m} \neq 0,\, \mathbf{m} \equiv \mathbf{a}\,(1)} (m_1 z + m_2)^{-k}\,.$$

When $k = 1$ or $k = 2$, replace by $G_{\mathbf{a}}^{(k)}(z, 0)$,

$$G_{\mathbf{a}}^{(k)}(z, s) := \sum (m_1 z + m_2)^{-k} |m_1 z + m_2|^{-s}\,,$$

with $G_{\mathbf{a}}^{(2)}(z)$ only quasi-holomorphic, that is

$$z \mapsto G_{\mathbf{a}}^{(2)}(z) + \frac{2\pi i}{z - \bar{z}}$$

is holomorphic in $z \in \mathbb{H}$.

For any weight $k \geq 1$ and level N,

$$E_{\mathbf{x}}^{(k)}(z) := \frac{(k-1)!}{(2\pi i N)^k} \sum_{\mathbf{a} \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2} \psi_{\mathbf{x}}(\mathbf{a}) \cdot G_{\mathbf{a}}(z)\,,$$

$$\psi_{\mathbf{x}}\left(\frac{a_1}{N}, \frac{a_2}{N}\right) := e^{2\pi i (a_2 x_1 - a_1 x_2)}\,, \qquad \frac{a_j}{N} \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}.$$

$E_0^{(2)}$ is quasi-holomorphic, more precisely

$$E_0^{(2)}(z) = -\frac{1}{2\pi i}\left(2\,\frac{d}{dz}(\log\eta) + \frac{1}{z-\bar{z}}\right).$$

- *Fourier expansion:* $E_{\mathbf{x}}^{(k)}(z) =$

$$-\ \frac{\mathbf{B}_k(x_1)}{k} + \sum_{0<r\equiv x_1(1)} \sum_{n=1}^{\infty} r^{k-1} e^{2\pi i n(x_2+rz)}$$

$$+ \sum_{0<r\equiv -x_1(1)} \sum_{n=1}^{\infty} r^{k-1} e^{2\pi i n(-x_2+rz)},$$

where $\mathbf{B}_k$ is the periodized Bernoulli function

$$\mathbf{B}_k(x) = B_k(x-[x]), \quad x \in \mathbb{R}.$$

- *Distribution Law:*

$$E_{\mathbf{x}}^{(k)} = \sum_{\mathbf{y}\cdot\alpha=\mathbf{x}} E_{\mathbf{y}}^{(k)}|\alpha, \quad \forall\,\alpha \in M_2^+(\mathbb{Z}).$$

Arithmetic multipliers

Modular Hecke algebra $\mathcal{A}(\Gamma)$: consists of finitely supported maps

$$F : \Gamma\backslash G^+(\mathbb{Q}) \to \mathcal{M}\,, \qquad \Gamma\alpha \mapsto F_\alpha\,,$$

with values in modular forms, satisfying

$$F_{\alpha\gamma} = F_\alpha|\gamma\,, \quad \forall\,\alpha \in \mathrm{GL}_2^+(\mathbb{Q})\,, \quad \forall\,\gamma \in \Gamma\,,$$

and with product defined by the rule

$$(F^1 * F^2)_\alpha := \sum_{\Gamma\beta\in\Gamma\backslash\,\mathrm{GL}_2^+(\mathbb{Q})} F^1_{\alpha\beta^{-1}}|\beta \cdot F^2_\beta\,.$$

The elements of $\mathcal{A}(\Gamma)$ supported in $\Gamma\backslash M_2^+(\mathbb{Z})$ form a subalgebra $\mathcal{A}(M_2^+(\mathbb{Z}),\Gamma)$, and the assignment $F \mapsto f^F$, $F \in \mathcal{A}(M_2^+(\mathbb{Z}),\Gamma)$,

$$f^F(\alpha,\rho,g) := \chi(F_\alpha)(g) \equiv (F_\alpha|g)\,(i)\,,$$

maps to the multiplier algebra of $C^*(\mathcal{L})$.

Canonical Hermitian 2-structure

• The fibre and the base of the $\mathbb{C}^*$-bundle

$$\mathbb{C}^* \longrightarrow \mathcal{L}^{(0)} \longrightarrow \mathcal{B}^{(0)} = \Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times \mathbb{H})$$

have tautological complex structure.

• For any $|u| < 1$, view the $\mathbb{Q}$-lattice (Λ, ϕ) as lattice in $\mathbb{C}_u$, w.r.t. to complex structure $d_u z := dz + u d\bar{z}$; one obtains germ of map $v \mapsto (\Lambda_v, \phi_v)$, where for $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$,

$$\Lambda_v := \{\Psi_v(\omega) = \omega + \frac{v}{2i(\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2)}\,\bar{\omega}; \quad \omega \in \Lambda\}$$

$$\text{and} \quad \phi_v := \Psi_v \circ \phi\,.$$

Since $\quad (\lambda\,\Lambda)_v = \lambda\,\Lambda_{\frac{v}{\lambda^2}}\,, \quad \forall\,\lambda \in \mathbb{C}^\times$, one gets

$$\mathcal{L}^{-2} \cong \mathrm{Jet}^1_{\mathbb{C}}(\mathcal{B}) \quad \Longleftrightarrow \quad \mathcal{L}^2 \simeq \mathcal{O}^\times_{\mathcal{B}} \quad \text{i.e.}$$

$\mathcal{L}$ is a *theta characteristic* ($\sim$ *spin structure*), and above $\cong$ gives canonical (running) Hermitian metric on the base $\mathcal{B}^{(0)}$.

Holomorphic connection and framing

• *Canonical affine connection* has vertical vector field

$$\mathcal{Y}(\tilde{F})(\Lambda) := \frac{1}{4}\left(\lambda\frac{d}{d\lambda}\right)\tilde{F}(\lambda^{-1}\Lambda)|_{\lambda=1};$$

and horizontal vector field

$$\mathcal{X}_{\mathbb{R}}(\tilde{F})(\Lambda) \ := \ \frac{1}{2\pi i}\frac{d}{dv}|_{v=0}\tilde{F}(\Lambda_v)\,.$$

However, if F is a modular form, in the coordinates $\quad \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \quad z = \frac{\omega_1}{\omega_2}$,

$$X_{\mathbb{R}}(F) = \frac{1}{2\pi i}\left(\frac{dF}{dz} + \frac{2Y(F)}{z-\bar{z}}\right).$$

• *Holomorphic affine connection* has horizontal vector field:

$$\mathcal{X} \ = \ \mathcal{X}_{\mathbb{R}} + 2\tilde{E}_0^{(2)}\mathcal{Y}\,,$$

$E_0^{(2)} \ = \$ quasi-holomorphic Eisenstein series.

• *Holomorphic affine framing*:

$$T^{(1,0)} = \mathbb{C}\mathcal{Y} + \mathbb{C}\mathcal{X}, \quad [\mathcal{Y},\mathcal{X}] \ = \ \mathcal{X}\,.$$

Quasi-regular representation

Let R denote the right quasi-regular representation of $\mathsf{GL}_2^+(\mathbb{R})$ on the Hilbert space

$$\mathfrak{H} := \int^{\oplus}_{\Gamma\backslash(M_2(\hat{\mathbb{Z}})\times \mathsf{GL}_2^+(\mathbb{R}))} \mathfrak{H}_{(\rho,g)}\, d^+\rho \otimes d^+g$$

of the regular representation of $C^*(\mathcal{L})$.

As basis of $\mathfrak{gl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}I$ take:

$$\begin{aligned} E &= \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad & \bar{E} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \\ H &= \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad & I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

with nonzero bracket relations

$$[H,E] = 2E\,, \quad [H,\bar{E}] = -2\bar{E}\,, \quad [E,\bar{E}] = H\,.$$

The holomorphic framing takes the form

$$\begin{aligned} \mathcal{Y} &= \frac{1}{4}\Big(R(I) + R(H)\Big), \\ \mathcal{X} &= -\frac{1}{4\pi}\,\mathrm{det}\cdot R(E) + 2\tilde{E}_0^{(2)}\mathcal{Y}\,. \end{aligned}$$

Explicitly, in standard coordinates on $\mathrm{GL}_2^+(\mathbb{R})$,

$$g = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

one has

$$\begin{aligned} R(E) &= e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}\right), \\ R(\bar{E}) &= e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right), \\ R(H) &= \frac{1}{i}\frac{\partial}{\partial\theta}, \qquad R(I) = r\,\frac{\partial}{\partial r}. \end{aligned}$$

Since $d^+g = r^3 dr \dfrac{dxdy}{y^2} d\theta = r^4\, dg$,

$$R(E)^* = -R(\bar{E}), \quad R(H)^* = R(H),$$
$$\text{while} \quad R(I)^* = -R(I) - 4\,\mathrm{Id}.$$

Hypoelliptic $\bar\partial$-operator

• The Hermitian structure on $\mathcal{L}$ is that canonically associated to the framing

$$\begin{aligned}
\mathcal{Y} &= \frac{1}{4}\Big(r\frac{\partial}{\partial r} + \frac{1}{i}\frac{\partial}{\partial\theta}\Big)\,, \\
\mathcal{X} &= -\frac{1}{4\pi}\,\mathrm{det}\cdot R(E) + 2\tilde{E}_0^{(2)}\mathcal{Y}\,; \\
\bar{\mathcal{Y}} &= \frac{1}{4}\Big(r\frac{\partial}{\partial r} - \frac{1}{i}\frac{\partial}{\partial\theta}\Big)\,, \\
\bar{\mathcal{X}} &= -\frac{1}{4\pi}\,\mathrm{det}\cdot R(\bar{E}) + 2\tilde{\bar{E}}_0^{(2)}\,\bar{\mathcal{Y}}\,.
\end{aligned}$$

• The *hypoelliptic* $\bar\partial$*-operator* on $\mathcal{L}$ is

$$Q = (\bar\partial_v^*\,\bar\partial_v - \bar\partial_v\,\bar\partial_v^*) \oplus \gamma_v(\bar\partial_h + \bar\partial_h^*)\,,$$

with vertical part $\bar\partial_v$ based on $\bar{\mathcal{Y}}$, and horizontal part $\bar\partial_h$ based on $\bar{\mathcal{X}}$. The corresponding *spectral triple* over $C^*(\mathcal{L})$ has first order operator D given by the functional equation $Q = D|D|$.

Taking the vertical component of second order compensates for the 'symmetry breaking' caused by the choice of the flat connection:

$$\mathcal{Y}(F|\alpha) = \mathcal{Y}(F)|\alpha, \quad \forall \alpha \in \mathsf{GL}_2^+(\mathbb{Q}), \qquad \text{while}$$
$$\mathcal{X}(F|\alpha) = \mathcal{X}(F)|\alpha + 2\Big(E_0^{(2)} - E_0^{(2)}|\alpha\Big)\mathcal{Y}(F)|\alpha.$$

The built-in 'bias', is dealt with by the *hypoelliptic* version of pseudodifferential calculus for Heisenberg manifolds.

• The complex structure on $\mathcal{L}$ induces a *CR-structure* on $\mathcal{L}^K$ and $\bar{\partial}$ restricts to a $\bar{\partial}_b$-operator.

• **Problem**: Compute the Connes-Chern character of the $\bar{\partial}_b$ *spectral triple*, more exactly its *local index cocycle*, together with the *transgression* to the local index cocycle of the spectral triple obtained by replacing $\mathcal{X}$ with $\mathcal{X}_{\mathbb{R}}$.

NCG Local Index Formula [Connes-Mo]

Local cocycle $\varphi_D = \{\varphi_n\}_{n=0,2,\ldots}$ in the (b,B)-bicomplex of $\mathcal{A}$ whose cyclic cohomology class gives the Connes-Chern character of $(\mathcal{A}, \mathfrak{H}, D)$:

$$\text{for} \quad n > 0, \qquad \varphi_n(a^0, \ldots, a^n) =$$

$$\sum_{k \in \mathbb{N}^n} c_{n,k} ⨍ a^0 [Q, a^1]^{(k_1)} \ldots [Q, a^n]^{(k_n)} |Q|^{-n-2|k|},$$

$$\text{where} \quad ⨍ T := \text{res}_{s=0} \text{Trace}(T|D|^{-s}),$$

$$\nabla(T) = [Q^2, a], \qquad T^{(k)} = \nabla^k(T),$$

$$|k| = k^1 + \ldots + k^n,$$

$$c_{n,k} = (-1)^{|k|} \, (k_1! \ldots k_n!)^{-1} \Gamma\left(|k| + \frac{n}{2}\right) \times$$

$$\Big((k_1+1)\ldots(k_1+\cdots+k_n+n)\Big)^{-1}.$$

Hopf algebra symmetry

The Ramanujan operator acting on the algebra $\mathcal{M}$ of modular forms of all levels satisfies

$$\mathcal{X}(F|\alpha) = \mathcal{X}(F)|\alpha + \mu_\alpha \cdot \mathcal{Y}(F)|\alpha$$

where

$$\mu_\alpha = 2\Big(E_0^{(2)} - E_0^{(2)}|\alpha\Big).$$

This gives rise to an action on the crossed product $\mathcal{A}_{\mathbb{Q}} := \mathcal{M} \ltimes \mathrm{GL}_2^+(\mathbb{Q})$ of the algebra $\mathcal{H}_1$ generated by $\{X, Y, \delta_n;\, n \geq 1\}$ modulo the relations

$$[Y, X] = X, \quad [Y, \delta_k] = k\delta_k$$
$$[X, \delta_k] = \delta_{k+1}, \qquad [\delta_j, \delta_k] = 0.$$

$\mathcal{H}_1$ is in fact a Hopf algebra, with coproduct

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y,$$
$$\Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,$$
$$\Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1,$$

counit

$$\epsilon(X) = \epsilon(Y) = \epsilon(\delta_k) = 0, \quad \epsilon(1) = 1\,,$$

and antipode

$$\begin{aligned} S(1) &= 1, \qquad S(X) = -X + \delta_1 Y \\ S(Y) &= -Y, \qquad S(\delta_1) = -\delta_1\,. \end{aligned}$$

The generators of $\mathcal{H}_1$ act on $\mathcal{A}_{\mathbb{Q}}$ as follows: $\forall\, F \in \mathcal{A}_{\mathbb{Q}}$ and $\forall\, \alpha \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\begin{aligned} \mathcal{Y}(F)_\alpha &= \mathcal{Y}(F_\alpha)\,, \qquad \mathcal{X}(F)_\alpha = \mathcal{X}(F_\alpha)\,, \\ \delta_n(F)_\alpha &= \mathcal{X}^{n-1}(\mu_\alpha)\, F_\alpha. \end{aligned}$$

• With respect to this action, $\mathcal{A}_{\mathbb{Q}}^{\mathrm{op}}$ is a Hopf module-algebra, i.e. $\forall\, h \in \mathcal{H}_1$, and with the standard notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$,

$$h(a *^{\mathrm{op}} b) = h_{(1)}(a) *^{\mathrm{op}} h_{(2)}(b), \quad \forall a, b \in \mathcal{A}(\Gamma).$$

Emerging symmetry structure

• The presence of the det factor in $\mathcal{X}$ leads to a twisted version $\mathcal{H}_1^\dagger$ of $\mathcal{H}_1$, which acquires a group-like element σ.

• The Lie algebra underlying the complex framing of $\mathcal{L}$ has additional bracket relations

$$\begin{aligned}[][\mathcal{Y}, \bar{\mathcal{Y}}] &= 0, \quad [\mathcal{Y}, \bar{\mathcal{X}}] = [\bar{\mathcal{Y}}, \mathcal{X}] = 0, \\ [\mathcal{X}, \bar{\mathcal{X}}] &\simeq \det{}^2 \cdot (\mathcal{Y} - \bar{\mathcal{Y}}).\end{aligned}$$

Note however that $\mathcal{Y} - \bar{\mathcal{Y}} = \frac{1}{2i}\frac{\partial}{\partial\theta}$ acts by 0 on the coordinates of $\mathcal{L}^K$.

• The emerging symmetry structure resembles a 'quantum double' of $\mathcal{H}_1^\dagger$, defined over the 'base' ring $\mathcal{E}(\mathbb{Q})$ of rational Eisenstein series.

• The relevant cohomology class comes from the Euler (area) class $e \in H^2(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{k}; \mathbb{R})$.

Euler class: Dedekind cocycle

- The assignment

$$\alpha \mapsto \mu_\alpha = 2\Big(E_0^{(2)} - E_0^{(2)}|\alpha\Big)$$

is a 1-cocycle on $\mathrm{GL}_2^+(\mathbb{Q})$ with values in $\mathcal{E}_2(\mathbb{Q})$:= all rational Eisenstein series of weight 2. On the other hand,

$$\Psi(\gamma)\,(\omega) := \int_{z_0}^{\gamma z_0} \omega, \quad \gamma \in \mathrm{GL}_2^+(\mathbb{Q})$$

is a 1-cocycle with values in the dual $\mathcal{E}_2(\mathbb{Q})^*$, whose class is independent of $z_0 \in \mathbb{H}$. The coupling of these two gives the 2-cocycle

$$\tau(\gamma_1, \gamma_2) := \int_{z_0}^{\gamma_2 z_0} \mu_{\gamma_1}(z)\, dz\,.$$

- $\mathrm{Re}\,\tau$ represents $2e \in H^2(\mathrm{SL}(2,\mathbb{Q}), \mathbb{R})$, where $e =$ Euler class, while $\mathrm{Im}\,\tau$ is a coboundary.

Replacing the period integrals by *modular symbols* leads to a cohomologous 2-cocycle

$$\begin{aligned}\theta(\gamma_1, \gamma_2) &= \\ \int_{z_0}^{\gamma_2 z_0} \mu_{\gamma_1}(z) dz &- z_0 \cdot \mathbf{a}_0(\mu_{\gamma_1}|\gamma_2 - \mu_{\gamma_1}) \\ &+ \int_{z_0}^{i\infty} (\widetilde{\mu_{\gamma_1}|\gamma_2} - \widetilde{\mu_{\gamma_1}})(z)\, dz,\end{aligned}$$

that depends on the choice of the cusp at ∞, and not on $z_0 \in \mathbb{H}$.

Its real part $\rho := \mathrm{Re}\,\theta \in Z^2(\mathrm{GL}_2^+(\mathbb{Q}), \mathbb{Q})$ is rational and has the following arithmetic expression. Let

$$\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in M_2^+(\mathbb{Z}),$$

represent elements in $\mathrm{PGL}_2^+(\mathbb{Q})$ *modulo* scalar matrices.

Then, for $c_2 = 0$,

$$\rho(\gamma_1, \gamma_2) = \quad -\frac{b_2}{d_2}\left(\sum_{\mathbf{x}\cdot\check{\gamma}_1=0} \mathbf{B}_2(x_1) - \mathbf{B}_2(0)\right) \ ,$$

while for $c_2 > 0$,

$$\begin{aligned}\rho(\gamma_1, \gamma_2) \qquad &= -\frac{a_2}{c_2}\left(\sum_{\mathbf{x}\cdot\check{\gamma}_1=0} \mathbf{B}_2(x_1) - \mathbf{B}_2(0)\right) \\ &-\frac{d_2}{c_2}\left(\sum_{\mathbf{x}\cdot\check{\gamma}_2\check{\gamma}_1=0} \mathbf{B}_2(x_1) - \sum_{\mathbf{x}\cdot\check{\gamma}_2=0} \mathbf{B}_2(x_1)\right) \\ &+2\sum_{\mathbf{x}\cdot\check{\gamma}_1=0}\sum_{j=0}^{c_2-1} \mathbf{B}_1\left(\frac{x_1+j}{c_2}\right)\mathbf{B}_1\left(a_2\frac{x_1+j}{c_2}+x_2\right) \\ &- 2\sum_{j=0}^{c_2-1} \mathbf{B}_1\left(\frac{j}{c_2}\right)\mathbf{B}_1\left(\frac{a_2 j}{c_2}\right);\end{aligned}$$

where $\mathbf{x} \in \mathbb{Q}^2/\mathbb{Z}^2$, $\qquad \mathbf{B}_1(x) = x - [x] - \frac{1}{2}$, $\mathbf{B}_2(x) = (x-[x])^2 - (x-[x]) + \frac{1}{6}$.

Euler class: Petersson cocycle

An alternative expression of the 1-cocycle μ is

$$\mu_\alpha(z) = \frac{1}{2\pi^2}\left(G_2^{(0)}(z) - G_2^{(0)}|\alpha\,(z) - \frac{2\pi i\, c}{cz+d}\right),$$

where $G_2^{(0)}$ is the *quasimodular* holomorphic weight 2 Eisenstein series, related to $E_2^{(0)}$ by

$$G_2^{(0)}(z) = 4\pi^2 E_2^{(0)}(z) + \frac{2\pi i}{z-\bar{z}}\,.$$

Thus, μ is equivalent (up to a factor) to

$$\begin{aligned}
\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z\right) \mapsto \frac{c}{cz+d} &= \frac{d}{dz}\log(cz+d) \\
&= \frac{d}{dz}\log(cz+d) - \frac{d}{dz}\log(c\bar{z}+d) \\
&= \frac{d}{dz}\log\frac{1}{z-\bar{z}} - \frac{d}{dz}\log\frac{1}{z-\bar{z}}|\alpha\,,
\end{aligned}$$

Therefore, replacing $\mathcal{X}$ by $\mathcal{X}_{\mathbb{R}}$ amounts to replacing μ by the equivalent cocycle

$$\nu_\alpha(z) = \frac{1}{\pi i} \cdot \frac{c}{cz+d} = \frac{1}{\pi i}\frac{d}{dz} \log j(\alpha, z).$$

The 2-cocycle

$$(\gamma_1, \gamma_2) \mapsto \frac{1}{2}\int_{z_0}^{\gamma_2 z_0} \nu_{\gamma_1}(z)\, dz$$

is clearly equivalent to

$$\begin{aligned} \epsilon(\gamma_1,\gamma_2) \;\; &:= \;\; \frac{1}{2\pi i}\Big(\log j(\gamma_2, z_0) + \log j(\gamma_1, \gamma_2 z_0) \\ &- \;\; \log j(\gamma_1\gamma_2, z_0)\Big) \in \mathbb{Z}\,, \end{aligned}$$

where the logarithm is determined by

$$\mathrm{Im} \log \in [-\pi, \pi)\,.$$

This is precisely the *integral* 2-cocycle introduced by Petersson.

Splitting formula

• Let $\gamma_1, \gamma_2 \in \mathsf{SL}(2,\mathbb{Q})$, then one has

$$\epsilon(\gamma_1,\gamma_2) - \frac{1}{2}\rho(\gamma_1,\gamma_2) = \Phi(\gamma_1\gamma_2) - \Phi(\gamma_1) - \Phi(\gamma_2),$$

with Φ the *Dedekind-Rademacher function*, defined as follows: for $\gamma \in \mathsf{SL}(2,\mathbb{Z})$ and $c > 0$

$$\Phi(\gamma) = \frac{a+d}{12\,c} - \sum_{j=0}^{c-1} \mathbf{B}_1\left(\frac{j}{c}\right)\mathbf{B}_1\left(\frac{a\,j}{c}\right) - \frac{1}{4};$$

for $\gamma \in \mathsf{SL}(2,\mathbb{Q})$, with $c = 0$ and $d > 0$,

$$\Phi\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \frac{1}{12}\frac{b}{d}.$$

• When $\gamma_1, \gamma_2 \in \mathsf{SL}(2,\mathbb{Z})$, then $\rho(\gamma_1,\gamma_2) = 0$. One recovers the classical *splitting formula* for the Petersson cocycle, displaying the triviality of the Euler class of $\mathsf{SL}(2,\mathbb{Z})$.

Concluding comments: expected outcome

• The **splitting formula** has a K-homological counterpart, consisting of a related **transgression formula** between the two *local cocycles* describing the Connes-Chern character of the $\bar{\partial}_b$ *spectral triple* over $C^*(\mathcal{L}^K)$, corresponding to the two connections $\mathcal{X}$ and $\mathcal{X}_{\mathbb{R}}$.

• The above Chern character is obtained as the image via characteristic map of a **universal class** in the **Hopf cyclic cohomology** of a *quatum double* of the Hopf algebra $\mathcal{H}_1^{\dagger}$, defined over the ring $\mathcal{E}(\mathbb{Q})$ of rational Eisenstein series for $\mathrm{GL}_2(\mathbb{A})$.

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