Painlevé equations and differential Galois Theory

Cambridge September 2006

Hiroshi Umemura

§1 Historical rambling

N.H. Abel (1802-29) E. Galois (1811-32)

Galois theory of algebraic equations

S. Lie

algebraic equations $\Rightarrow$ Diff. equations

Infinite dimensional

E. Picard Galois Theory of LOD's 1896

Finite dimensional Picard-Vessiot Theory

1895 Stockholm Lessons . P. Painlevé

600 pages, 22 lessons

Pursuit of new special functions defined by 2nd order algebraic diff. equations.

Model

$$\wp'^2 = \wp^3 - g_2\wp - g_3, \quad g_2, g_3 \in \mathbb{C}$$

What are the new functions?

What are the so far known functions?

How to prove a discovered function is new?

Painlevé knew implicitely strongly normal extensions (= Galois ext. of Kolchin).

Painlevé took hold of what he had discovered.

The language were lacking!

Varieties, Algebraic groups, Lie algebras, ...

He anticipated algebraic geometers.

J. Drach, General Galois Theory of ODE's 1898

Thesis

Problematic?

Infinite dimension

Discovery of Painlevé equations 1900
Painlevé; (unexpected) success
lucky

Review of Drach's work 1900 ...

E. Vessiot Infinite dimensional

R. Fuchs P_{VI} arises from a monodromy preserving deformation. 1907

1902–03 Dispute between Painlevé and R. Liouville on the irreducibility of the 1st Painlevé equation.

Painlevé asked help for Drach's Theory which was not accepted by public.

(1) The irreducibility is proved by Drach's Theory.
(2) Drach's Theory will be admited soon.

Too optimist!

The irreducibility was proved in 1980's (or before)

by Kolchin, Kovacic, Nishioka.

Vessiot's last articles were published in 1940's.

Study of infinite dimensional diff. Galois Theory

was abandoned and forgotten for several decades.

Exception

J.-F. Pommaret, Diff. Galois Theory 1980's.

1996 Umemura, General Galois Theory of diff. field ext. Algebraic Geometrie

2000 Malgrange,

General Galois Theory of foliations

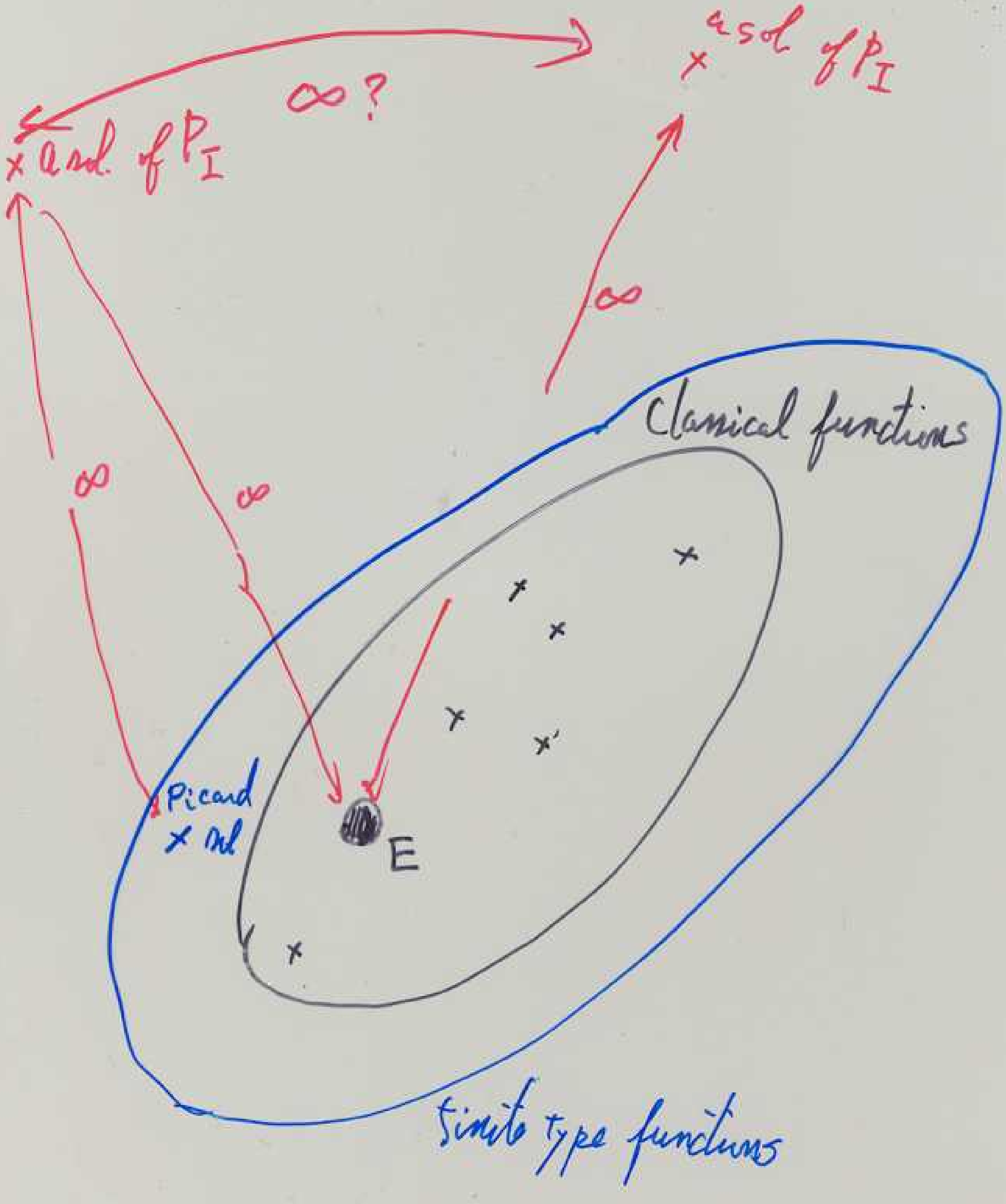
analytic Geometrie

They are equivalent in spirit.

Irreducibility of the Painlevé equations

A proof depending on general diff. Galois theory

G. Casale 2005...


a sol. of P_I
∞?
a sol of P_I
∞
∞
∞
Classical functions
Picard
× sol
E
finite type functions

4 fig

§2 Stockholm Lessons

Let's go back to the end of the 19th century.

2.1 How do we define functions?

Whittacker - Watson A course of modern analysis

Cambridge 1928

Riemann zeta function,
Gamma function,
Hypergeometric functions and their confluences
e.g. Hermite-Weber function, Bessel function..

Elliptic functions

1 Definition by series

$\tau \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$

$$\theta(z,\tau) := \sum_{n\in\mathbb{Z}} \exp(\pi\sqrt{-1}\, n^2 \tau + 2\pi\sqrt{-1}\, n z)$$

$\omega_1, \omega_2 \in \mathbb{C}$ linearly independent$/\mathbb{R}$

$$L := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$$

$$\wp(z, L) = \sum_{0 \neq \omega \in L} \left(\left(\frac{1}{z+\omega}\right)^2 - \left(\frac{1}{\omega}\right)^2 \right)$$

Every analytic function is defined by a Taylor series. A function has different Taylor expansion.

It is not easy in general to understand the properties of the function from a series expansion.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(2) Functional equation

$\theta(z,\tau)$ satisfies the functional equations

(a) $\theta(z+1,\tau)=\theta(z,\tau)$,

(b) $\theta(z+\tau,\tau)=\exp(-\pi\sqrt{-1}\,\tau-2\pi\sqrt{-1}\,z)\,\theta(z,\tau)$.

(a), (b) characterize $\theta(z,\tau)$ up to constant multiplication. Namely y is a holomorphic function on $\mathbb{C}$ such that

(a) $y(z+1)=y(z)$

(b) $y(z+\tau)=\exp(-\pi\sqrt{-1}\,\tau-2\pi\sqrt{-1}\,z)\,y(z)$

Then $\exists\, c\in\mathbb{C}$ s.t. $y(z)=c\,\theta(z,\tau)$.

(3) Differential equation

$$(\wp')^2=4\wp^3-g_2\wp-g_3,\quad (\Delta=g_2^3-27g_3^2\neq 0)$$

$(g_2, g_3\in\mathbb{C}$ determined by $L)$

If $y(z)$ is a meromorphic function,

$$y'^2 = 4y^3 - g_2 y - g_3,$$

then there exist $c \in \mathbb{C}$ such that $y = \wp(z+c, L)$

(4) Other definitions

$$\Gamma(t) := \int_0^\infty e^{-z} z^{t-1} dz \qquad \text{Euler}$$

$$\Gamma(t) = \lim_{n\to\infty} \frac{1\cdot 2\cdot \cdots (n-1)}{t(t+1)\cdots(t+n-1)} n^t$$

Adopt the view point (3) Differential eq.

Transcendental diff. eq. are too difficult.

$$y^{(3)} + \exp(\tan y'') - y' + \wp(y+y') = 0$$

We limit ourselves to algebraic diff. equations.

$F(x, y, y', \cdots, y^{(n)}) = 0$

x = indepedent variable, y = dependent var.

F is a polynomial in $y, y', \cdots, y^{(n)}$.

Objective Pursuit of special functions that generalize the $\wp$-function

$$y'^2 = 4y^3 - g_2 y - y$$

We prefer a special function meromorphic over $\mathbb{C}$.

meromorphy is difficult to check

$\Downarrow$

without movable sing. points that depends on solutions.

Essential sing. and Branch

Movable branch point

Ex. $y' = \frac{1}{2}\frac{1}{y}$ general sol $y = \sqrt{x+c}$

$x = -c$ is a branch point that depends on the parameter c. So $y' = \frac{1}{2}\frac{1}{y}$ has a movable singular point (branch).

Theorem (Fuchs, Poincaré 1884)

$F(x, y, y') = 0$ polynomial in y, y' no mouvable points.

Then $F(x, y, y') = 0$ is integrated by the so far known functions.

K be a field of meromorphic functions such that the coefficients of F are in K.

$F(x, y, y') = 0$ is an algebraic curve$/K$.

If $g = 0$, it is reduced to a Riccati equation and hence to a linear diff. eq of the second order.

If $g = 1$, it is reduced to the $\wp$-function.

If $g \geq 2$, it is reduced to an algebraic equation.

So let us study

$$F(x, y, y', \cdots, y^{(n)}) = 0$$

for $n = 2, 3, \cdots$ without mouvable singular points. One would encounter many interesting

One would encounter many interesting special functions.

Optimist!

Painlevé classified $y'' = R(x, y, y')$, where $R(x, y, y')$ is a rational function, no movable singularities.

He threw away from the classification those that he could integrate by the so far known functions.

Liste of 6 Painlevé equations.

Expectation These 6 equations are irreducible to the so far known functions.

Classification in $n = 2$ is remarkablly difficult.

(1) It is difficult to show that an algebraic diff. eq. has no movable singular points.

It is practically impossible to decide wether an algebraic differential equation has no movable singular points without explicitely writing down solutions. E. Picard

Poincaré had not expected Painlevé's success.

(2) $n=3$, Classification remains open.

$\exists$ Important diff. equation with movable singular point.

Jacobi Algebraic diff. eq. satisfied by $\theta(0,\tau)$ has movable singular point. $n=3$

R. Fuchs P_6 describes a monodromy preserving deformation of a second order linear eq. 1907

Garnier

Later developments were done in this context.

§3 Generation of functions

Painlevé   Stockholm Lessons

Functions are meromorphic over a domain $D \subset \mathbb{C}$.

Inductive definition.

We start from $\mathbb{C}(x)$ for example.

(0) If $f(x)$ is known, then $f'(x)$ is known.

(P1) If $f(x)$, $g(x)$ are known, then $f(x) \pm g(x)$, $f(x)g(x)$, $f(x)/g(x)$ $(g(x) \neq 0)$ are known.

(P2) If $\underset{\neq 0}{a_0(x)}$, $a_1(x)$, $\cdots$, $a_m(x)$ are known, and

$$a_0(x) f(x)^n + a_1(x) f(x)^{n-1} + \cdots + a_n(x) = 0,$$

then $f(x)$ is known.

(P3) $F'(x) = f(x)$ for a known $f(x)$, then $F(x)$ is known.

(P4) $a_0(x) \neq 0$, $a_1(x)$, $\cdots$, $a_m(x)$ are known and

$$a_0(x) f^{(m)}(x) + a_1(x) f^{(m-1)}(x) + \cdots + a_m(x) f(x) = 0$$

then $f(x)$ is known.

(P5) $A = \mathbb{C}^n/\Gamma$ be an Abelian variety $\quad \Gamma \simeq \mathbb{Z}^{2n}$

$a_1(x), a_2(x), \cdots, a_n(x)$ known functions

$f: A \dashrightarrow \mathbb{C}$ is a meromorphic function

or f is an Abelian function

$p: \mathbb{C}^n \to A = \mathbb{C}^n/\Gamma$ is the projection

Then $f \circ p(a_1(x), \cdots, a_n(x))$ is a known function.

Let $F: U \longrightarrow \mathbb{C}^n$

$x \longmapsto (a_1(x), a_2(x), \cdots, a_n(x))$

$$\begin{tikzcd}
U \arrow[r, dashed, "F"] \arrow[ddr, dashed, "f \circ p \circ F"'] & \mathbb{C}^n \arrow[d, "p"] \\
& \mathbb{C}^n/\Gamma \arrow[d, dashed, "f"] \\
& \mathbb{C}
\end{tikzcd}$$

$f \circ p$ is a meromorphic function on $\mathbb{C}^n$ periodic w.r.t. $\Gamma \simeq \mathbb{Z}^{2n}$

$\parallel$

$f \circ p$ is an Abelian function

Substitution into an Abelian function is a new know function.

Example $n=1$, $A=\mathbb{C}/\Gamma$ is an elliptic curve

$\Gamma \simeq \mathbb{Z}^2$ $\Gamma=(\omega_1,\omega_2)$, $\omega_1, \omega_2 \in \mathbb{C}$

$$\begin{tikzcd} U \arrow[r, dashed, "a"] & \mathbb{C} \arrow[d, "p"] \\ & A=\mathbb{C}/\Gamma \arrow[d, dashed, "f"] \\ & \mathbb{C} \end{tikzcd}$$

$\wp(t)$ is a meromorphic function on $\mathbb{C}$

$\wp(t+\omega_1)=\wp(t)$

$\wp(t+\omega_2)=\wp(t)$

$\wp(t)$ is considered as a meromorphic function on $A=\mathbb{C}/\Gamma$

f

$p\circ f=\wp$

$a(x): U\to\mathbb{C}$ is known,

then $f\circ p\circ a(x)=\wp(a(x))$ is known.

Namely one can substitute known functions into an Abelian function.

What are these operations.

(0) $\frac{d}{dx}$, (P1) $\pm$, $\times$, $\div$

$\Leftrightarrow$ Generate differential field
field closed under the derivation $\frac{d}{dx}$.