Asymptotics of the Painlevé Equations

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11 September 2006

Outline

Introduction
- Preliminaries
- Linear Special Functions
- Riccati Equation

The First Painlevé Equation $\mathrm{P_I}: y''(x) = 6y^2(x) - x$
- Solutions
- Asymptotic Behaviours
- Global Information
- Analytic Information
- Summary

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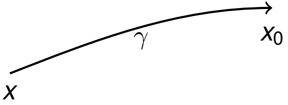
Asymptotic Notation



$f, g \colon \mathbb{C} \mapsto \mathbb{C}$, $|g| > \epsilon > 0$

- $f(x) \ll g(x)$ as $x \underset{\gamma}{\to} x_0$ iff $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.
- $f(x) \sim g(x)$ as $x \underset{\gamma}{\to} x_0$ iff $\lim_{x \to x_0} \frac{f(x)-g(x)}{g(x)} = 0 \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$
- $f(x) = O\big(g(x)\big)$ as $x \underset{\gamma}{\to} x_0$ iff $\exists\, M$ s.t. $\left|\frac{f(x)}{g(x)}\right| < M$ as $x \underset{\gamma}{\to} x_0$.

Exercise: Prove that $\sinh x \sin x = O(e^x)$ as $x \underset{\mathbb{R}}{\to} +\infty$

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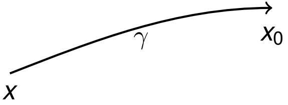

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Exercise: Prove that $\sinh x \sin x = O(e^x)$ as $x \underset{\mathbb{R}}{\to} +\infty$

Asymptotic Series

- f is *asymptotic to a series* $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ as $x \to x_0$, or

$$f(x) \sim \sum_{n=0}^{\infty} a_n(x - x_0)^n, \text{ as } x \to x_0$$

iff for each integer $N \geq 0$ we have

$$\lim_{x \to x_0} \frac{f(x) - \sum_{n=0}^{N} a_n(x - x_0)^n}{(x - x_0)^N} = 0.$$

Example: $\exists$ K s.t.

$$\frac{\exp(x) - \sum_{n=0}^{N} x^n/n!}{x^N} \leq K\, x, \text{ as } x \to 0$$

by Taylor's theorem.

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More Asymptotic Series

$$Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt \quad \sim \quad \frac{e^x}{x} \sum_{n=0}^{\infty} \frac{n!}{x^n} \qquad as\ x \to +\infty, \text{ (try integration by parts)}$$

- Let $F(x) := xe^{-x} Ei(x)$.
- MAPLE $\underset{20 digits}{\Rightarrow}$: $F(100) = 1.0102062527748357112$

$$\sum_{n=0}^{99} \frac{n!}{100^n} = 1.0102062527748357112$$

$$\sum_{n=0}^{260} \frac{n!}{100^n} = 1.0108296120102260952$$

$$\sum_{n=0}^{281} \frac{n!}{100^n} = 732496.06921461904157$$

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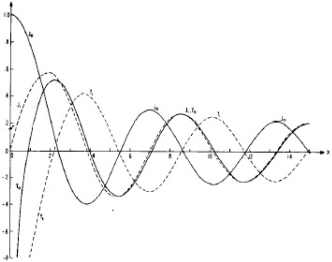
Bessel Functions



$$z^2\, w_{zz} + z\, w_z + (z^2 - \nu^2)\, w = 0$$

- Solutions $J_\nu(z)$ are bounded as $z \to 0$.
- $J_\nu(z) = \sqrt{2/(\pi z)}\left\{\cos(z - \nu\pi/2 - \pi/4) + e^{\Im(z)}\mathcal{O}(1/z)\right\}$ as $|z| \to \infty$, $|\arg z| < \pi$.
- $J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}}\left(e\,z/(2\nu)\right)^\nu$ as $\nu \to +\infty$.
- The zeroes of $J_{\nu+1}(z)$ interlace those of $J_\nu(z)$.

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Nonlinear Model: $y_x = y^2 + x^{2q}$

Under $y(x) = x^q\, u(z)$, $z = x^{q+1}/(q+1)$, we get

$$u' + 2p\frac{u}{z} = u^2 + 1, \quad q = 2p/(2p-1),\ p \neq 1/2 \tag{1}$$

- This is a Riccati equation (it has the Painlevé property).
- It can be linearized:

$$u(z) = -\frac{\psi'(z)}{\psi(z)} \quad \Rightarrow \quad \psi'' + 2p\frac{\psi'}{z} + \psi = 0$$

which can in turn be transformed via $\psi(z) = z^{1/2-p} w(z)$ to

$$w'' + \frac{w'}{z} + \left(1 - \frac{(p-1/2)^2}{z^2}\right) w = 0,$$

Bessel's equation.

Balances

Definition

- An asymptotic limiting form of an equation in which only the largest terms remain is called a *dominant balance*.
- If the set of largest terms is maximal for the original equation the balance is called *maximal*. Otherwise, it is called *submaximal*.

Example

The ODE

$$u' + 2p\frac{u}{z} = u^2 + 1,$$

has the maximal dominant balance

$$u' \sim u^2 + 1, \quad as\ z \to \infty$$

and a submaximal balance

$$u^2 \sim -1 \quad as\ z \to \infty$$

Local Asymptotics

- As $z \to \infty$, the largest terms of $u' + 2p\frac{u}{z} = u^2 + 1$ are

$$u' \sim u^2 + 1 \quad \Rightarrow \quad u(z) \sim \tan(z - z_0).$$

- How accurate is this asymptotic behaviour?
- What other local behaviours are possible?
- Boutroux (1913) answered these questions.
- The function tan is implicitly defined by

$$\int_{\eta}^{u} \frac{dv}{v^2 + 1} = z - z_0$$

where $\tan(z_0) = \eta$. Its period π is given by

$$i \oint_C \frac{dv}{v^2 + 1}$$

where C is a closed contour enclosing one of the roots of $v^2 + 1$ in the v-plane.

Asymptotic Estimates

Theorem

Assume Γ is a path of finite length l in the u-plane on which $|u| < h, k < |u^2 + 1|$. Suppose $0 < \epsilon < 1/2$ and $|z| > 1/\epsilon$. Then, we have

$$|\tan^{-1}(u - \eta) - z + z_0| < \frac{2|p|hl\epsilon}{k}.$$

Proof.

We have

$$\int_\eta^u \frac{dv}{v^2 + 1} = z - z_0 - 2p \int_\eta^u \frac{u}{z(u^2 + 1)}\, dz$$

and

$$\left| 2p \int_\eta^u \frac{u}{z(u^2 + 1)}\, dz \right| \leq 2|p|\, l\, \frac{h}{k} \epsilon.$$

□

Details

- We can extend this to the case when l and h are large and k is quite small: e.g. $l < 1/\epsilon^{3/8}$, $h \leq |\log \epsilon|$, $k > |\epsilon|^{3/8}$.
- When k becomes arbitrarily small, we get the balance $u \sim \pm i \Rightarrow$

$$u(z) \sim \pm\, i \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \;\; a_0 = 1$$

  with $a_n \sim b\, c^n\, n!$ for some b, c and $n \to \infty$.
- If z_0, z_1 are two successive zeroes of u then

$$|z_1 - z_0 - \pi| < \frac{2|p|hl\epsilon}{k}.$$

- Successive poles Z_1, Z_0 of u satisfy (consider $1/u$ in place of u)

$$|Z_1 - Z_0 - \pi| = O(\epsilon).$$

- Consider the *picket fence* of poles

$$\ldots, Z_{-2}, Z_{-1}, Z_0, Z_1, Z_2, \ldots$$

  This line (or pair of rays) is asymptotically parallel to the real z-axis.

Stokes' Phenomenon

Wasow's thm (1963 (Thm 12.1)) shows that there exist true solutions asymptotic to formal series solutions, in each sector $\mathcal{S}$, with vertex at origin, of angle less than π. Uniqueness? We can prove uniqueness if $u \sim u_f$ holds in a wider sector.

- `Example:` $u_z = u^2 + 1 - u/3z$.
- Let u_1, u_2 be two solutions with asymptotic expansion $u_f = i\sum_{k=0}^{\infty} a_k z^{-k}$, $a_0 = 1$, in the sector $|\arg(z)| < \pi$.

Let $v = u_1 - u_2 \Rightarrow v' = {u_1}' - {u_2}'$

$$\begin{aligned} v' &= {u_1}^2 - {u_2}^2 - \frac{u_1 - u_2}{3z} \\ &= \left(u_1 + u_2 - \frac{1}{3z}\right) v \\ &= (2i + h(z))\, v \end{aligned}$$

where $h(z) = O(1/z^2)$.
Integration shows

$$v = c\frac{e^{2iz}}{z^r}\left(1 + O\left(1/z\right)\right),\ r = (i+1)/3$$

But along the negative real axis $\Rightarrow$ contradiction unless $c = 0$.

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Introduction

Preliminaries

Linear Special Functions

Riccati Equation

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Solutions

Asymptotic Behaviours

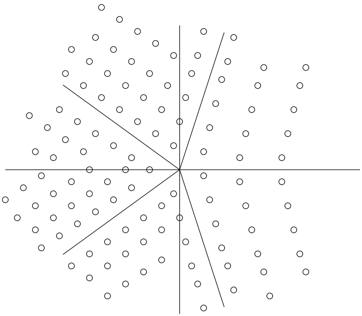
Global Information

Analytic Information

Summary

Generic Solutions

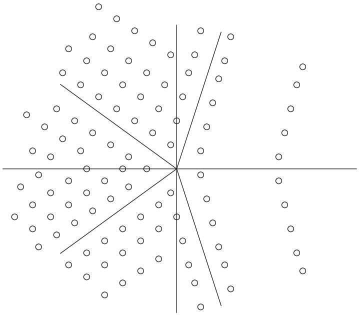
In the complex x-plane:



- If $y(x)$ is a solution, so is $\omega^2\, y\big(\omega x\big)$, where $\omega^5 = 1$.
- There are an infinite number of movable double poles. (The locations of poles change with initial conditions.)

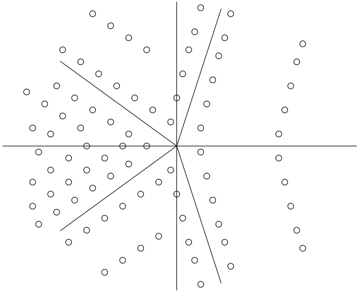
One-parameter or *tronquée* solution

The “pole-free” or *tronquée* solution:



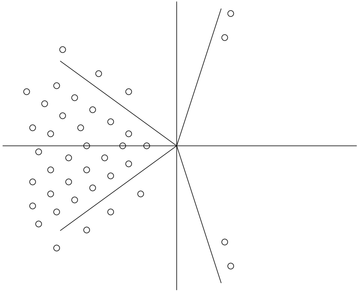
More pole-free solutions

Approaching a *tri-tronquée* solution:



Tri-tronquée solution

Limit to a *tri-tronquée* solution:



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Balances

- The transformation $y = \sqrt{x}\, u(z)$, $z = 4x^{5/4}/5 \Rightarrow$

$$y_{xx} = 6y^2 - x \quad \mapsto \quad u_{zz} = 6u^2 - 1 - \frac{u_z}{z} + \frac{4}{25}\frac{u}{z^2}$$

- Generic (2-parameter) solutions satisfy

$$u_{zz} \sim 6u^2 - 1, \Rightarrow \; u \sim \wp(z; 2, g_3) \quad \textit{as } z \to \infty$$

  in each quadrant in the z-plane. (z-quadrants $\Leftrightarrow$ x-sectors of angle $2\pi/5$.)

- Tronquée (1-parameter) solutions satisfy

$$u \sim \left(\frac{1}{6}\right)^{1/2}, \text{ as } z \to \infty$$

  in two adjacent quadrants.

- Tritronquée (0-parameter) solutions have such expansions in *four* adjacent quadrants.

Generic Conditions

- Define $P(u) := 4u^3 - 2u + 2E$. The roots of $P(u)$ coincide for some $E = D_k$.
- Given small $0 < |\epsilon|$, bounded $0 < B$, $|z_0| > 1/\epsilon$, s.t. z lies in the domain

$$\mathcal{Z} := \left\{ z \,\middle|\, |z - z_0| < B, |z| > 1/\epsilon \right\}$$

  Assume $|\eta|$, $|\eta'|$ are bounded above by $|\log \epsilon|$, $|\eta'| > |\epsilon|^{1/7}$, and E defined by

$$2E = \eta'^2 - 4\eta^3 + 2\eta - \frac{4\eta^2}{25z_0^2}$$

  satisfies $|E - D_k| > 2|\epsilon|^{2/7}$, $|E| < |\log \epsilon|^4$.
- Suppose $u(z)$ satisfies the initial conditions $u(z_0) = \eta$, $u'(z_0) = \eta'$. and γ is a path joining z_0 to z in $\mathcal{Z}$ of length $l < 2\pi B$ such that its image Γ under u lies in the domain

$$\mathcal{U} := \left\{ u \,\middle|\, |u| < |\log \epsilon|, |P(u)| > |\epsilon|^{1/7} \right\}.$$

Generic Solutions

Theorem

Under the generic conditions, $\exists$ *positive* ϵ_0 *s.t.* $\forall\ 0 < |\epsilon| \leq \epsilon_0$, *the generic solution satisfies*

$$\left|u'^2 - P(u)\right| < 8B|\epsilon||\log\epsilon|^4,$$

and

$$\left|\int_{\underset{\Gamma}{\eta}}^{u} \frac{dv}{\sqrt{P(v)}} - (z - z_0)\right| < 16\pi B^2|\epsilon|^{6/7}|\log\epsilon|^4,$$

in $\mathcal{Z}$*. Moreover, if* z_0 *and* z_{10} *are two successive points in* $\mathcal{Z}$ *where* $u = \eta$, *then for* $j = 1$ *or* 2,

$$|(z_{10} - z_0) - \omega_j| < 16\pi B^2|\epsilon|^{6/7}|\log\epsilon|^4.$$

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Asymptotic Behaviours
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Slow Modulation

Suppose z_0 is a zero of u. Let Ω_j to be the next zero and

$$E(z) := \frac{1}{2}\left(u'^2 - 4u^3 + 2u\right)$$

Let $\mathcal{L}$ denote any term satisfying

$$\left|\mathcal{L}\right| < k|\epsilon|^{1+m/7}|\log\epsilon|^n$$

for some nonnegative m, n, then

$$E(z_0 + \Omega) - E(z_0) = -\frac{\tilde{\omega}}{z_0} + \mathcal{L},$$

where $\tilde{\omega}$ is the elliptic integral

$$\tilde{\omega} = \oint \sqrt{4v^3 - 2v + 2E}\, dv$$

Many results follow from this observation.

$\star$ Boutroux: As we step along a line of zeroes (or poles), $\tilde{\omega} \to 0$.

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Preliminaries
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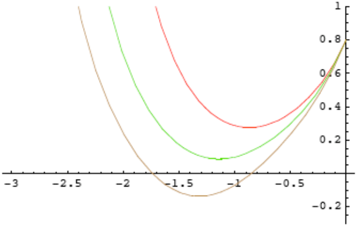
Riccati Equation

The First Painlevé Equation $P_I : y''(x) = 6y^2(x) - x$
Solutions
Asymptotic Behaviours
Global Information
Analytic Information
Summary

Asymptotic ⇒ Analytic Information

Any real solution of P_I has a pole on the negative real semi-axis.



Sketch: Let $y(x) = u(t)$, $t = -x > 0$. Then

$$u_{tt} = 6u^2 + t \;\Rightarrow\; u_{tt} > 6u^2$$

$\Rightarrow u'(t)^2/2 > 2u(t)^3 + N(u_0, u_1)$, $N(u_0, u_1) := u_1^2/2 - 2u_0^3$. Choose $t_1 > t_0$ s.t. $3u(t)^3/2 + N(u_0, u_1) > 0$, then

$$u'(t)^2 > u(t)^3 \;\Rightarrow\; -\frac{1}{\sqrt{u(t)}} + \frac{1}{\sqrt{u(t_1)}} > \frac{t - t_1}{2}$$

Further Analytic Information

$X_{min}(x_0) := \inf_{y_0 \in \mathbb{R}, y_1 \geq 0} x_{min}$, is finite $\forall\, x_0 \leq 0$. Moreover,

$$x_0 - \frac{C}{|x_0|^{1/4}} < X_{min}(x_0) < x_0 < 0$$

PROOF: Let $t_0 = -x_0 > 0$, $t_{min} = -x_{min}$, $u(t_0) = y_0 \equiv u_0$, and $u'(t_0) = -y_1 \equiv u_1 < 0$. We use monotonicity of $u(t)$ on $[t_0, t_{min}]$. Integration gives

$$-u'(t) = \sqrt{4u^3 + 2N(u_0, u_1) + 2\int_{t_0}^{t} su'(s)ds}$$

$N(u_0, u_1) = u_1^2/2 - 2u_0^3$, and

$$t - t_0 = \int_{u(t)}^{u_0} \frac{du}{\sqrt{4(u^3 - u_{min}^3) + 2\int_{u_{min}}^{u} t(\hat{u})d\hat{u}}}$$

where the inverse $t(\hat{u}) \geq t_0 \Rightarrow t_{min} - t_0 < I/{t_0}^{1/4}$, I being an elliptic integral. X_{min} is one of many transcendental functions arising in the study of P_I.

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Summary

- Interesting solutions of the Painlevé equations have subtle asymptotics.

  These have been tackled through many approaches.

- These solutions can be continued to finite regions. However, finite properties of the Painlevé equations have not received as much attention. They deserve as much scrutiny as special functions we all know.

  Such as Airy, Bessel, Whittaker, ... functions.

- We can describe real solutions of the first Painlevé equation completely for $x \leq 0$. But there still remain open problems for $x > 0$.

  There are many transcendental functions that give uniform intervals of existence for the solutions.

  www.maths.usyd.edu.au/u/nalini/papers/