Introduction to the Painlevé equations

Kazuo OKAMOTO

Sep. 11 2006
12

@Newton Institute

6 Painlevé equations

$$1898 \leqq \frac{\text{Painlevé}}{\text{Gambier}} \leqq 1906$$

Classification of the 2^{nd} order nonlinear ODE without movable critical pts.

ex. $$\frac{d^2y}{dx^2} = \frac{k+1}{y}\left(\frac{dy}{dx}\right)^2$$

$$y = (ax+b)^{-\frac{1}{k}}$$

$$a, b \in \mathbb{C}.$$

Motivation:

Find a new-transcendental function (in Math. Phy) defined by a differential equation.

$$e^x$$

$$\sin x$$

$$\log x$$

$$J_\nu(x)$$

$$F(a, b, c; x)$$

$$\cdots$$

$$\wp(x; \omega_1, \omega_3)$$

Key Words:

Irreducibility

Hamiltonian System

Holonomic Deformation
(Isomonodromic Deformation)

Birational Transformation
(Bäcklund Transformation)
Weyl Group
Roots Systems ..

τ- function & Bilinear form

Space of initial conditions.

Hamiltonian System :

$$\delta = f(t)\frac{d}{dt} \qquad f(t) \in \mathbb{C}(t)$$

$$H(t; q, p) \in \mathbb{C}[t, q. p]$$

$$\delta q = \frac{\partial H}{\partial p}, \qquad \delta p = -\frac{\partial H}{\partial q} \qquad (*)$$

If H is of 2^{nd} deg. w.r.t. p

then, by eliminating p from the system,

we obtain :

$$\delta^2 q = R(t; q, \delta q)$$

R : rational

Assumption : $(*)$ is without movable critical pts.

Ex. $\delta = \frac{d}{dt}$, $H = \frac{1}{2}p^2 - 2q^3 + \frac{1}{2}g_2 q$

$g_2 \in \mathbb{C}$

Hamiltonian system defines Weierstraß $\wp$-fnc.

$K / \mathbb{C}(t)$ differential field extention

$L = K(q, p)$: diff. field w.r.t. δ

through $\delta q = \frac{\partial H}{\partial p}$, $\delta p = -\frac{\partial H}{\partial q}$.

Theorem (Painlevé)

Under the assumption.

trans. $\deg_K L = 2$

$\Rightarrow$ Hamiltonian system is reduced to one of the Painlevé equations,

$P_{I}, P_{II}, \cdots, P_{VI}$.

Ex. $H = \frac{1}{2}p^2 - (2q^3 + tq)$

$\delta = \frac{d}{dt}$, $\delta q = \frac{\partial H}{\partial p} = p$

$\delta p = -\frac{\partial H}{\partial q} = 6q^2 + t$

$\Rightarrow \delta^2 q = 6q^2 + t \quad (P_{\mathrm{I}})$

	P_{I}	P_{II}	P_{III}	P_{IV}	P_{V}	P_{VI}
δ:	$\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$t(t-1)\frac{d}{dt}$

Ex. $H = 2q^2p^2 - (2tq^2 + (2\theta_0 + 1)q - 2t)p + (\theta_0 + \theta_\infty)tq$

$\delta = t\frac{d}{dt}$

$\Rightarrow P_{\mathrm{III}}'$

Holonomic deformation

(Monodromy Preserving deformation)

Problem: Given

$$\frac{d^2y}{dx^2} + P_1(x;t)\frac{dy}{dx} + P_2(x;t)y = 0,$$

Find two rational functions in x

$$a_1(x;t),\ a_2(x,t)$$

such that

$$\begin{cases} \dfrac{\partial^2 y}{\partial x^2} + P_1\dfrac{\partial y}{\partial x} + P_2 y = 0 \\ \dfrac{\partial y}{\partial t} = a_1 y + a_2 \dfrac{\partial y}{\partial x} \end{cases}$$

is completely integrable.

extended system.

Monodromy Preserving deformation

(R.Fuchs, R.Garnier, K.O)

(L. Scheleginger. T. Miwa & M. Jimbo)

$$\frac{d^2y}{dx^2} + p_1(x,t)\frac{dy}{dx} + p_2(x,t)y = 0$$

$$p_1 = \frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta}{x-t} - \frac{1}{x-q}$$

$$p_2 = \frac{\kappa}{x(x-1)} - \frac{H}{x(x-1)(x-t)} + \frac{q(q-1)p}{x(x-1)(x-q)}$$

$x=q$: apparent singularity

$$\Downarrow$$

$$H = q(q-1)(q-t)p^2 - \{\kappa_0(q-1)(q-t) + \kappa_1 q(q-t) + (\theta-1)q(q-1)\}p + \kappa(q-t)$$

$$(*)\begin{cases} \dfrac{\partial^2 y}{\partial x^2} + p_1 \dfrac{\partial y}{\partial x} + p_2 y = 0 \\[2ex] \dfrac{\partial y}{\partial t} = a_1(x:t) y + a_2 \dfrac{\partial y}{\partial x} \end{cases}$$

$\exists\, a_1(x:t),\ a_2(x;t)$ rational in x

s.t $(*)$ is completely integrable

$$\Updownarrow$$

$$\delta q = \frac{\partial H}{\partial p}, \quad \delta p = -\frac{\partial H}{\partial q}. \qquad P_{VI}$$

Riemann Scheme

$$\begin{array}{ccccc} x=0 & x=1 & x=t & x=q & x=\infty \\ 0 & 0 & 0 & 0 & \nu \\ \kappa_0 & \kappa_1 & \theta & 2 & \nu+\kappa_\infty \end{array}$$

$$2\nu + \kappa_0 + \kappa_1 + \theta + \kappa_\infty = 1$$

$$\kappa = \nu(\nu + \kappa_\infty)$$

Confluence of singularities

$$t :\Rightarrow 1+\epsilon t$$

$$\kappa_1 :\Rightarrow \epsilon^{-1}+\theta+1$$

(α-method)

$$\theta :\Rightarrow -\epsilon^{-1}$$

$$\epsilon \to 0$$

$$p_1(x;t) = \frac{1-\kappa_0}{x} + \frac{t}{(x-1)^2} + \frac{1-\theta}{x-1} - \frac{1}{x-q}$$

$$p_2(x;t) = \frac{\kappa}{x(x-1)} - \frac{H}{x(x-1)^2} + \frac{q(q-1)p}{x(x-1)(x-q)}$$

$$H = q(q-1)^2p^2 - \{\kappa_0(q-1)^2 + \theta q(q-1) - tq\}p$$
$$+\kappa(q-1)$$

$$\delta q = \frac{\partial H}{\partial p}, \quad \delta p = -\frac{\partial H}{\partial q} \quad : P_{\mathrm{V}}$$

$$\delta = t\frac{d}{dt}$$

$$1+1+1+1$$
$$|$$
$$2+1+1$$

$$\begin{tikzcd}
P_{VI} \arrow[r] & P_{V} \arrow[r] \arrow[dr] & P_{III} \arrow[dr] & & \\
& & P_{IV} \arrow[r] & P_{II} \arrow[r] & P_{I}
\end{tikzcd}$$

$$\begin{tikzcd}
1+1+1+1 \arrow[r] & 2+1+1 \arrow[r] \arrow[dr] & 2+2 \arrow[dr] & & \\
& & 3+1 \arrow[r] & 4 \arrow[r] & \frac{7}{2}
\end{tikzcd}$$

$$H = 2qp^2 - (q^2 + 2tq + 2\kappa_0)p + \theta_\infty q$$

$$\delta = \frac{d}{dt}$$

$$\delta q = \frac{\partial H}{\partial p}, \qquad \delta p = -\frac{\partial H}{\partial q} \qquad P_{IV}$$

$$H = H(\kappa_0, \theta_\infty)$$

$$q_1 = q\,\frac{qp - \theta_\infty}{qp - \kappa_0} \qquad H_1 = H$$

$$p_1 = p\,\frac{qp - \kappa_0}{qp - \theta_\infty}$$

$$H_1 = H(\theta_\infty, \kappa_0)$$

$$\bar{p} = p - \frac{1}{2}q - t, \qquad \bar{A} = q\bar{p} \qquad \bar{H} = H + q.$$

$$q_2 = 2\bar{p}\,\frac{\bar{A} - \kappa_0}{\bar{A} - \kappa_0 + \theta_\infty + 1}$$

$$p_2 = -\frac{1}{2}p \cdot \frac{\bar{A} - \kappa_0 + \theta_\infty + 1}{\bar{A}} \qquad H_2 = \bar{H}$$

$$H_2 = H(\kappa_0, \theta_\infty + 1)$$

Group of Transformations:

For P_{II}

$$\sigma_1 : \alpha \mapsto 1-\alpha, \quad \sigma_2 : \alpha \mapsto -\alpha$$

$$G = \langle \sigma_1, \sigma_2 \rangle \cong \widetilde{W}_a(A_1)$$

affine Weyl Group.

P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
$-$	A_1	$A_1 \times A_1$	A_2	A_3	D_4

Special Solutions

$$H = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q$$

P_{II}

$$\frac{d^2q}{dt^2} = 2q^3 + tq + \alpha$$

if $\alpha \in \mathbb{R}$

$C : [-\frac{1}{2}, 0]$ fundamental region of G.

C : Weyl chamber

when $\alpha \in \partial C$, we have :

$H(-\frac{1}{2})$ $\quad p \doteq 0 \quad \frac{dq}{dt} = -q^2 - \frac{1}{2}t$

$$q = \frac{d}{dt}\log u$$

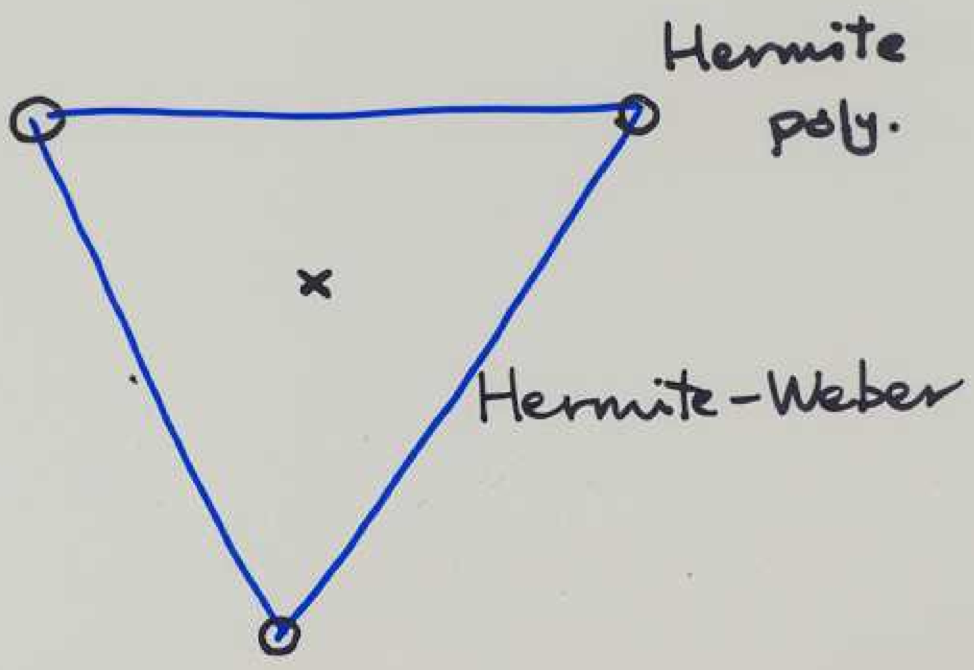
$$\frac{d^2u}{dt^2} + \frac{1}{2}tu = 0$$

$H(0)$ $\quad q = 0, \quad p = \frac{t}{2}$

Riccati-typ sol. & rational sol.

P_{I}	$\times$
P_{II}	Airy
P_{III}	Bessel
P_{IV}	Hermite - Weber
P_{V}	Confluent - Hypergeometric
P_{VI}	Gauß Hypergeometric.

P_{IV}



Bäcklund Transformation

$$H = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q$$

$$H = H(\alpha), \quad \delta = \frac{d}{dt}$$

$$\delta q = \frac{\partial H}{\partial p}, \quad \delta p = -\frac{\partial H}{\partial q}$$

P_{II}

$$\frac{d^2 q}{dt^2} = 2q^3 + tq + \alpha.$$

$$q_1 = q + \frac{\alpha + \frac{1}{2}}{p} \qquad p_1 = p$$

$$\delta q_1 = \frac{\partial H_1}{\partial p_1} \qquad \delta p_1 = -\frac{\partial H_1}{\partial q_1} \qquad H_1 = H$$

$$H_1 = H(-\alpha - 1)$$

$$p = \bar{p} + 2q^2 + t, \quad H = \bar{H} - q$$

$$q_2 = -q + \frac{\alpha - \frac{1}{2}}{\bar{p}} \qquad p_2 = -\bar{p} \qquad H_2 = \bar{H}$$

$$H_2 = H(\alpha - 1)$$

τ- functions

$$\delta^{*} \log \tau = H.$$

$$P_{I} : \quad q = -\frac{d^2}{dt^2} \log \tau$$

$$P_{II} : \quad q = \frac{d}{dt} \log \frac{\tau(\alpha-1)}{\tau(\alpha)}$$

ℓ : translation in G.

$$\ell^{*} : \tau_m \longrightarrow \tau_{m+1}$$

τ- sequence.

$$\delta^2 \log \tau_m = \frac{\tau_{m-1} \tau_{m+1}}{\tau_m^2}$$

up to elementary change

of Hamiltonian function.

Painlevé Analysis.

$$\frac{d^2q}{dt^2} = 2q^3 + tq + \alpha$$

$$q = \frac{1}{z} - \frac{t_0}{6}z - \frac{\alpha+1}{4}z^2 + hz^3 + \cdots$$

$$z = t - t_0 \quad , \quad h \in \mathbb{C}$$

$$\left(q = -\frac{1}{z} + \frac{t_0}{6}z - \frac{\alpha-1}{4}z^2 + hz^3 + \cdots \right)$$

$$q = Q^{-1},$$

$$Q = z\left[1 + \frac{t_0}{6}z^2 + \frac{\alpha+1}{4}z^3 + \left(\frac{t_0^2}{36} - h\right)z^4 + \cdots\right]$$

$$z = Q\left[1 - \frac{t_0}{6}Q^2 - \frac{\alpha+1}{4}Q^3 + \left(\frac{t_0^2}{18} + h\right)Q^4 + \cdots\right]$$

$$\frac{dq}{dt} + q^2 + \frac{t}{2} = -\left(\alpha + \frac{1}{2}\right)z + \left(\frac{t_0^2}{36} + 5h\right)z^2 + \cdots$$

$$= -\left(\alpha + \frac{1}{2}\right)Q - Q^2 P$$

$$\frac{dQ}{dt} = 1 + \frac{t}{2}Q^2 + \left(\alpha + \frac{1}{2}\right)Q^3 + Q^4 P.$$

$$\Big\downarrow \quad \frac{d^2Q}{dt^2} = \frac{2}{Q}\left(\frac{dQ}{dt}\right)^2 - \frac{2}{Q} - tQ - \alpha Q$$

$$\frac{dQ}{dt} = \frac{\partial H}{\partial P}, \qquad \frac{dP}{dt} = -\frac{\partial H}{\partial Q}.$$

$$H = \frac{1}{2}Q^4P^2 + \left[\alpha + \frac{1}{2}Q^3 + \frac{t}{2}Q^2 + 1\right]P$$
$$+ \frac{2\alpha+1}{4}\left[\left(\alpha + \frac{1}{2}\right)Q^2 + tQ\right].$$

$$q = Q^{-1}, \qquad p = -\left(\alpha + \frac{1}{2}\right)Q - Q^2P$$

$$H = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q.$$

Space of initial conditions

$$H = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q$$

$q(t_0) = q_0,\ p(t_0) = p_0 \qquad (q_0, p_0) \in \mathbb{C}^2$

$$q(t) = \frac{1}{z} - \frac{t_0}{6}z - \frac{\alpha+1}{4}z^2 + hz^3 + \cdots$$

$$\mathbb{C}^2 \cup \{h\}_{h \in \mathbb{C}}.$$

$\overline{X}$: compactification of $\mathbb{C}^2$

Hamiltonian System defineds

foliation in $\overline{X}$

(with sing.)

$(\bar{X}, D)$

$X = \bar{X} \setminus D$

$\forall x \in X \quad \exists_1$ sol. of the System.

$\forall$ sol. of the system. $\exists_1\ x \in X$

D : set of vertical leaves.

$P_{I} \quad P_{II} \quad P_{III} \quad P_{IV} \quad P_{V} \quad P_{VI}$

$\tilde{\hat{E}}_8 \quad \tilde{\hat{E}}_7 \quad \tilde{\hat{D}}_6 \quad \tilde{\hat{E}}_6 \quad \tilde{\hat{D}}_5 \quad \tilde{\hat{D}}_4$

$\Downarrow$

Sakai-san's theory.

$$\begin{array}{ccccccccc}
\tilde{\hat{D}}_4 & \longrightarrow & \tilde{\hat{D}}_5 & \longrightarrow & \tilde{\hat{D}}_6 & \longrightarrow & \tilde{\hat{D}}_7 & \longrightarrow & \tilde{\hat{D}}_8 \\
 & & & \searrow & & \searrow & & & \\
 & & & & \tilde{\hat{E}}_6 & \longrightarrow & \tilde{\hat{E}}_7 & \longrightarrow & \tilde{\hat{E}}_8
\end{array}$$

$$H = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q$$

$$\frac{dH}{dt} = -\frac{1}{2}p$$

$$\frac{d^2H}{dt^2} = -qp - \frac{1}{2}\left(\alpha + \frac{1}{2}\right)$$

$$\left(\frac{d^2H}{dt^2}\right)^2 + 4\left(\frac{dH}{dt}\right)^3 - 2\frac{dH}{dt}\left(H - t\frac{dH}{dt}\right) - \frac{1}{4}\left(\alpha + \frac{1}{2}\right)^2 = 0$$

$$\bar{H} = \frac{1}{2}\bar{p}^2 + \left(q^2 + \frac{t}{2}\right)\bar{p} - \left(\alpha - \frac{1}{2}\right)q$$

$$\left(\frac{d^2\bar{H}}{dt^2}\right)^2 + 4\left(\frac{d\bar{H}}{dt}\right)^3 - 2\frac{d\bar{H}}{dt}\left(\bar{H} - t\frac{d\bar{H}}{dt}\right) - \frac{1}{4}\left(\alpha - \frac{1}{2}\right)^2 = 0$$

Bilinear forms

$$\frac{d^3H}{dt^3} + 6\left(\frac{dH}{dt}\right)^2 + 2t\frac{dH}{dt} - H = 0$$

$$D^4\tau\cdot\tau + 2t D^2\tau\cdot\tau - 2\tau'\cdot\tau = 0$$

$$H = \frac{d}{dt}\log\tau.$$

$$\bar{H} = \frac{d}{dt}\log\bar{\tau}$$

$$D^2\bar{\tau}\cdot\tau + \frac{t}{2}\bar{\tau}\cdot\tau = 0$$

$$D^3\bar{\tau}\cdot\tau + \frac{t}{2}D\bar{\tau}\cdot\tau = \alpha\bar{\tau}\cdot\tau$$