§3 Generation of functions

Painlevé Stockholm Lessons

Functions are meromorphic over a domain $D \subset \mathbb{C}$.

Inductive definition.

We start from $\mathbb{C}(x)$ for example.

(0) If $f(x)$ is known, then $f'(x)$ is known.

(P1) If $f(x)$, $g(x)$ are known, then $f(x) \pm g(x)$, $f(x)g(x)$, $f(x)/g(x)$ $(g(x) \neq 0)$ are known.

(P2) If $\underset{\neq 0}{a_0(x)}, a_1(x), \cdots, a_n(x)$ are known, and

$$a_0(x) f(x)^n + a_1(x) f(x)^{n-1} + \cdots + a_n(x) = 0,$$

then $f(x)$ is known.

(P3) $F'(x) = f(x)$ for a known $f(x)$, then $F(x)$ is known.

(P4) $a_0(x) \neq 0$, $a_1(x)$, $\cdots$, $a_m(x)$ are known and

$$a_0(x) f^{(m)}(x) + a_1(x) f^{(m-1)}(x) + \cdots + a_m(x) f(x) = 0$$

then $f(x)$ is known.

(P5) $A = \mathbb{C}^n/\Gamma$ be an Abelian variety $\quad \Gamma \cong \mathbb{Z}^{2n}$

$a_1(x), a_2(x), \cdots, a_n(x)$ known functions

$f : A \dashrightarrow \mathbb{C}$ is a meromorphic function

or f is an Abelian function

$p : \mathbb{C}^n \to A = \mathbb{C}^n/\Gamma$ is the projection

Then $f \circ p(a_1(x), \cdots, a_n(x))$ is a known function.

Let $F : U \longrightarrow \mathbb{C}^n$

$x \longmapsto (a_1(x), a_2(x), \cdots, a_n(x))$

$$\begin{tikzcd} U \arrow[r, dashed, "F"] \arrow[rdd, "f\circ p\circ F"'] & \mathbb{C}^n \arrow[d, "p"] \\ & \mathbb{C}^n/\Gamma \arrow[d, dotted, "f"] \\ & \mathbb{C} \end{tikzcd}$$

$f \circ p$ is a meromorphic function on $\mathbb{C}^n$ periodic w.r.t. $\Gamma \cong \mathbb{Z}^{2n}$

$\|$

$f \circ p$ is an Abelian function

substitution into an Abelian function is a new know function.

Example $n=1$, $A = \mathbb{C}/\Gamma$ is an elliptic curve

$\Gamma \simeq \mathbb{Z}^2$ $\quad \Gamma = (\omega_1, \omega_2)$, $\omega_1, \omega_2 \in \mathbb{C}$

$$\begin{tikzcd} U \arrow[r, dashed, "a"] & \mathbb{C} \arrow[d, "p"] \\ & A = \mathbb{C}/\Gamma \arrow[d, dashed, "f"] \\ & \mathbb{C} \end{tikzcd}$$

$\wp(t)$ is a meromorphic function on $\mathbb{C}$

$\wp(t+\omega_1) = \wp(t)$

$\wp(t+\omega_2) = \wp(t)$

$\wp(t)$ is considered as a meromorphic function on $A = \mathbb{C}/\Gamma$
f

$a(x) : U \to \mathbb{C}$ is known, $\quad p \circ f = \wp$

then $f \circ p \circ a(x) = \wp(a(x))$ is known.

Namely one can substitute known functions into an Abelian function.

What are these operations.

(0) $\frac{d}{dx}$, (P1) $\pm, \times, \div$

$\Longleftrightarrow$ Generate differential field
field closed under the derivation $\frac{d}{dx}$.

If we start from $\mathbb{C}(x)$.

x is known so that $\wp(x)$ is known and hence

$\wp(\wp(x))$ is known for example.

Remark $f(x)$ is a known function holomorphic on a domain U and $g(x)$ is a known function holomorphic on $f(U)$. Then $g \circ f$ is a known function on U.

These operations are not independent

Lemma (0), (P1), (P4) $\Rightarrow$ (P2), (P3)

(0) $\frac{d}{dx}$, (P1) $\pm, \times, \div$, (P4) linear diff. eq.
(P2) Quadrature, (P3) Algebraic equation

Proof (P2) $F'(x) = f(x)$, $f(x)$ being known.

By (0) $f'(x)$ is known.

$$0 = ff' - f'f = fF'' - f'F'$$ so by (P4)

F is known.

Now we show (P2). Let

$$a_0(x) \neq 0, \quad a_1(x), \quad \cdots, \quad a_n(x)$$

be known functions and

$$(*) \quad a_0 f^n + a_1 f^{n-1} + \cdots + a_n = 0.$$

The field

$$K := \mathbb{C}(a_0, a_1, \cdots, a_n, a_0', a_1', \cdots, a_n', a_0'', \cdots)$$

consists of known functions by (0) and (P1).

We may assume (*) is irreducible/K.

$$L := K(f)$$

Then L is closed under the derivation. Because char $K = 0$.

In deed let

$$F(y) := a_0 y^n + a_1 y^{n-1} + \cdots + a_n$$

so that $F(f) = 0$.

Differentiating $F(f)=0$

$$\left.\frac{\partial F}{\partial y}\right|_{y=f} f' + {}^{D}F(f) = 0$$

where

$${}^{D}F(y) = a_0' y^n + a_1' y^{n-1} + \cdots + a_n'.$$

$$\left.\frac{\partial F}{\partial y}\right|_{y=f} = n a_0 f^{n-1} + (n-1) a_1 f^{n-2} + \cdots + a_{n-1} \neq 0$$

$F(f)=0$ is *minimal polynomial* of f/K.

So

$$f' = -\frac{D_F(f)}{\left.\frac{\partial F}{\partial y}\right|_{y=f}} \in K(f)$$

Now $(K(f):K) = \deg F = n$.

$f^{(n)}, f^{(n-1)}, \cdots, f \in K(f)$

So that $\exists$ non-trivial K-linear relation.

$\exists\, b_0, b_1, b_2, \cdots, b_n \in K$ s.t.

$$b_0 f^{(n)} + b_1 f^{(n-1)} + \cdots + b f = 0$$

So by (P3) f is known

We can unify (P4) and (P5).

They are both related with algebraic groups: the general linear group $GL_m(\mathbb{C})$ and an Abelian variety.

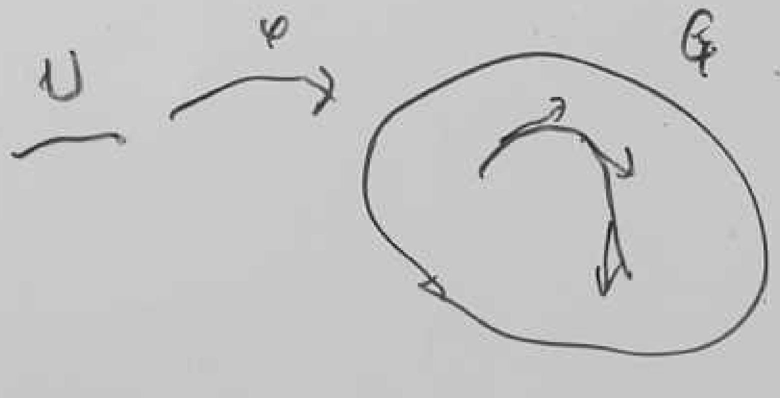
Let G be an algebraic group$/\mathbb{C}$. For example $G = GL_m(\mathbb{C})$.

$U \subset \mathbb{C}$ is a domain

$\varphi : U \to G$

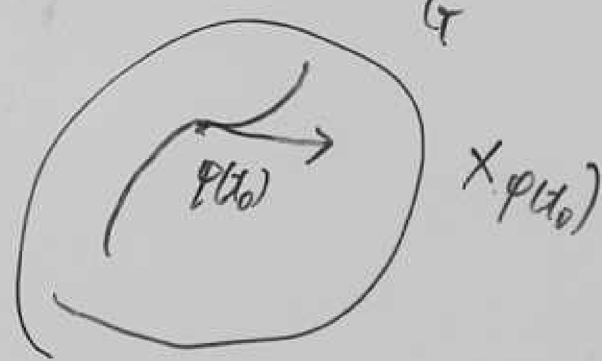
is a holomorphic map

φ is a holomorphic curve in G. So that we can consider the vector field X_φ along the curve φ.



Definition of $X_{\varphi(x_0)}$ at $\varphi(x_0)$, $\quad x_0 \in U$

Let f be a regular function in a nbd of $\varphi(x_0) \in G$.



We set

$$X_{\varphi(x_0)} f := \lim_{\varepsilon \to 0} \frac{f(\varphi(x_0+\varepsilon)) - f(\varphi(x_0))}{\varepsilon}$$

$X_{\varphi(x_0)}$ is a tangent vector at $\varphi(x_0) \in G$.
Now we translate the tangent vector $X_{\varphi(x_0)}$ by the right translation $R_{\varphi(x_0)^{-1}}$ so that

$$R_{\varphi(x_0)^{-1}} * X_{\varphi(x_0)}$$

That is a tangent vector at $1 \in G$

$$R_{\varphi(x_0)^{-1}\,*}\, X_{\varphi(x_0)} \in T_{1,G} = \mathrm{Lie}\, G$$

$$\|$$

$l\delta\varphi(x_0)$ The logarithmic derivative of

$$l\delta\varphi : \overset{U}{\cancel{G}} \to \mathrm{Lie}\, G \qquad \varphi : U \to G.$$
$$x \mapsto l\delta\varphi(x_0)$$

Examples 1. $G = \mathbb{G}_a = \mathbb{C}, +$ additive group

$\varphi : U \to \mathbb{C}$ Let t be the coordinate of $\mathbb{C}$

$$X_{\varphi(x_0)}(t) = \lim_{\varepsilon \to 0} \frac{t(\varphi(x_0+\varepsilon)) - t(\varphi(x_0))}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\varphi(x_0+\varepsilon) - \varphi(x_0)}{\varepsilon}$$

$$= \varphi'(x_0)$$

So $\quad X_{\varphi(x_0)} = \varphi'(x_0) \left(\frac{d}{dt}\right)_{t=\varphi(x_0)}$

$$R_{-\varphi(x_0)*}\ \varphi'(x_0)\left(\frac{d}{dt}\right)_{t=\varphi(x_0)} = \varphi'(x_0)\left(\frac{d}{dt}\right)_{t=0}$$

$$l\delta\varphi : U \to \mathrm{Lie}\,G \simeq \mathbb{C}$$

$$x_0 \longmapsto \varphi'(x_0)\left(\frac{d}{dt}\right)_{t=0} \leftrightarrow \varphi'(x_0)$$

$$l\delta\varphi = \varphi'(x).$$

2 $G = \mathbb{G}_m = \mathbb{C}^*$ multiplicative

$\varphi : U \to G = \mathbb{C}^*$

$$X_{\varphi(x_0)} = \varphi'(x_0)\left(\frac{d}{dt}\right)_{t=\varphi(x_0)}$$

$\frac{d}{dt}$ is not translation (by multiplication) inv.

$$= \frac{\varphi'(x_0)}{\varphi(x_0)}\left(t\frac{d}{dt}\right)_{t=\varphi(x_0)}$$

↑

translation inv.

$$R_{\varphi(x_0)^{-1}*}\ X_{\varphi(x_0)} = \frac{\varphi'(x_0)}{\varphi(x_0)}\left(t\frac{d}{dt}\right)_{t=1}$$

$$l\delta\varphi : U \to \mathrm{Lie}\, G \simeq \mathbb{C}$$

$$x_0 \mapsto \frac{\varphi'(x_0)}{\varphi(x_0)}\left(t\frac{d}{dt}\right)\Big|_{t=1} \leftrightarrow \frac{\varphi'(x_0)}{\varphi(x_0)}$$

3. $G = GL_m(\mathbb{C})$

$\varphi : U \to G = GL_m(\mathbb{C})$ $\quad$ $\varphi(x_0) \in GL_m(\mathbb{C})$ is a reg. matrix.

$$l\delta\varphi : U \to \mathrm{Lie}\, G = gl_m = M_m(\mathbb{C})$$

$$x \mapsto l\delta\varphi = \varphi'(x)\,\varphi^{-1}(x)$$

4. $A = \mathbb{C}^n/\Gamma$ is an Abelian variety.

$$\begin{tikzcd} U \arrow[r, "\tilde{\varphi}"] \arrow[dr, "\varphi"'] & \mathbb{C}^n \arrow[d, "p"] \\ & A = \mathbb{C}^n/\Gamma \end{tikzcd}$$

Assume $\varphi : U \to A$ is given by $\tilde{\varphi} : U \to \mathbb{C}^n$

$$x \mapsto (a_1(x), a_2(x), \cdots, a_n(x))$$

Then $l\delta\varphi : U \to \mathrm{Lie}\, A \simeq \mathbb{C}^n$

$$x \mapsto (a_0'(x), a_1'(x), \cdots, a_n'(x)).$$

Definition (Operation Q) Let G be an algebraic group and

$$\varphi : U \to G$$

a holomorphic map. If $l\delta\varphi : U \to \mathrm{Lie}\, G \cong \mathbb{C}^n$ is given by known functions $a_1, a_2, \cdots, a_n$

$$l\delta\varphi : U \longrightarrow \mathrm{Lie}\, G \cong \mathbb{C}^n$$
$$x \longmapsto (a_1(x), a_2(x), \cdots, a_n(x)),$$

then for $\forall$ rational function f on G

$$\begin{tikzcd} f\circ\varphi : U \arrow[r, "\varphi"] \arrow[rr, bend right] & G \arrow[r, dashed, "f"] & \mathbb{C} \end{tikzcd}$$

is known.

Intuitively, if the tangent vector field $X_{\varphi(x)}$ is described by known functions, we allow integration along the curve $\varphi : U \to G$.

Examples 1 $G=\mathbb{C}$ $\quad \varphi : U \to G = \mathbb{C}$

$l\delta\varphi = \varphi'(x)$.

$\varphi'(x)$ is known $\Rightarrow \forall$ rational function of $\varphi(x)$ is known

2. $G = \mathbb{C}^\times$ $\quad \varphi : U \to G = \mathbb{C}^\times$

$l\delta\varphi = \varphi'/\varphi$.

φ'/φ is known, say $\varphi'/\varphi = a$

$$\varphi' = a\varphi$$

One can solve the linear diff. eq. $\varphi' = a\varphi$ for the known a.

3. $G = GL_n(\mathbb{C})$, $\quad \varphi : U \to G$

$l\delta\varphi = \varphi'\varphi^{-1}$, when $\varphi'\varphi^{-1} = a$ is known

or every entry of $\underset{\overset{\shortparallel}{a}}{\varphi'\varphi^{-1}}$ is known,

we can solve $\varphi' = a\varphi$

(P4)

4 Let A be an Abelian variety

$$\begin{array}{l} \varphi : U \to \mathbb{C}^n \\ \quad\ \downarrow \\ \quad \mathbb{C}^n/\Gamma \end{array}$$ is given by $a_1(x), a_2(x), \cdots, a_m(x)$.

s.t. $\mathrm{l}\delta\varphi : U \to \mathbb{C}^n$
$x \mapsto (a_1'(x), a_2'(x), \cdots, a_n'(x))$

is given by known functions $a_1'(x), a_2'(x), \cdots, a_n'(x)$,

then for $\forall$ Abelian function f
$f(a_1(x), a_2(x), \cdots, a_m(x))$
is known.

Equivalent to (P5) modulo (O).

Theorem Equivalence of operations

(1) (O), (P1), (P2), (P3), (P4), (P5)

(2) (O), (P1), (Q).

Proof We have seen (2) $\Rightarrow$ (1)

To see (1) $\Rightarrow$ (2). We have seen
(1) $\Rightarrow$ (Q) restricted to linear alg. gr and Ab. var.

To prove (1) ⇒ (2) for every algebraic group G.

It is sufficient to notice

$\exists$ closed normal subgroup $L \subset G$ s.t.

L is a linear algebraic group and the quotient G/L is an Abelian variety.

L is isomorphic to a closed subgroup of $GL_N(\mathbb{C})$ for some N.

$$1 \to L \to G \to A \to 0.$$

(Chevalley)

We reduce to linear case and Abelian case.

Proposition Let G be an algebraic group $\varphi : U \to G$, $\psi : U \to G$ two holomorphic maps.

$$l\delta(\varphi\psi) = l\delta\varphi + (\mathrm{Ad}\,\varphi)\, l\delta\psi.$$

Proof $G = GL_n$

$$l\delta(\varphi\psi) = (\varphi\psi)'(\varphi\psi)^{-1} = (\varphi'\psi + \varphi\psi')(\varphi\psi)^{-1}$$

$$=(\varphi'\psi+\varphi\psi')(\varphi\psi)^{-1}=(\varphi'\psi+\varphi\psi')\psi^{-1}\varphi^{-1}$$

$$=\varphi'\varphi^{-1}+\varphi(\psi'\psi^{-1})\varphi^{-1}$$

$$=\ell\delta\varphi+\ \mathrm{Ad}\,\varphi(\ell\delta\psi)$$

Corollary $\varphi : U \to G$, $\psi : U \to G$ s.t.
$\ell\delta\varphi = \ell\delta\psi$. Then $\exists\, c \in G(\mathbb{C})$ s.t.
$\varphi = \psi c$.

$\mathrm{Ad}\,\psi^{-1}$

Lemma $\ell\delta\psi^{-1} = -\mathrm{Ad}\,\psi^{-1}(\ell\delta\psi)$

Proof $0=\ell\delta 1=\ell\delta(\psi^{-1}\psi)=\ell\delta\psi^{-1}+\mathrm{Ad}\,\psi^{-1}(\ell\delta\psi)$

$\therefore\ \ell\delta\psi^{-1}=-\mathrm{Ad}\,\psi^{-1}(\ell\delta\psi)$

Proof of Cor.

$$\ell\delta(\psi^{-1}\varphi)=\ell\delta\psi^{-1}+\mathrm{Ad}\,\psi^{-1}(\ell\delta\varphi)$$

$$=-\mathrm{Ad}\,\psi^{-1}(\ell\delta\psi)+\mathrm{Ad}\,\psi^{-1}\ell\delta\varphi$$

$$=-\mathrm{Ad}\,\psi^{-1}(\ell\delta\varphi)+\mathrm{Ad}\,\psi^{-1}\ell\delta\varphi$$

$$=0$$

$$l\delta \Psi^{-1}\varphi = 0$$

$$C = \Psi^{-1}\varphi \in G.$$

$$\varphi = \Psi C.$$

Definition (Classical functions) A meromorphic function is called classical if it is obtained from the rational function field $\mathbb{C}(x)$ by a finite iteration of the operations (0), (P1), (P2), (P3), (P4), (P5).

Theorem (Painlevé ...) No solution of the first Painlevé equation $y'' = 6y^2 + x$ is classical.

To prove the Theorem, we had better work in a more general setting.

All the fields that we consider are of char. 0.

A differential field (K, δ), where K is a field and

$$\delta : K \to K$$

is a derivation.

Namely

For every $a, b \in K$

$$\delta(a+b) = \delta a + \delta b,$$
$$\delta(ab) = (\delta a) b + a \delta b.$$

$c \in K$ is a constant if $\delta c = 0$.

$$C_K := \{ a \in K \mid \delta a = 0 \} \subset K$$

a subfield

Examples 1 $\left(\mathbb{C}(x), \frac{d}{dx}\right)$

2 $\mathbb{C}(U) = \{$ meromorphic functions on $U \}$

derivation $\frac{d}{dx}$

a subfield of

Abstract setting

$K = (K, \delta)$ is a differential field

$C \subseteq C_K$ the field of constants of K

G an algebraic group defined over C

Let $\varphi : \operatorname{Spec} K \to G$ be a K-valued point

or a morphism of C-schemes.

Let $\varphi(\operatorname{Spec} K) = x \in G$.

$$\mathcal{O}_x \xrightarrow{\varphi^*} K \qquad C\text{-algebra homomorphism}$$

Then $\mathcal{O}_x \xrightarrow{\varphi^*} K \xrightarrow{\delta} K$ is a tangent

vector at x. If we set

$$X_\varphi := \delta \circ \varphi^*,$$

then $X_\varphi \in T_{x,G}$.

So we can define

$$l\delta\varphi = R_{\varphi(x)^{-1}*} X_\varphi \in T_{1,G} \otimes_C K \simeq \operatorname{Lie} G \otimes_C K$$

Proposition $\varphi : \mathrm{Spec}\, K \to G$, $\psi : \mathrm{Spec}\, K \to G$.

Then $l\delta(\varphi\psi) = l\delta\varphi + \mathrm{Ad}\,\varphi\; l\delta\psi$

Corollary In the above proposition, we assume $l\delta\varphi = l\delta\psi$. $\varphi^{-1}\psi : \mathrm{Spec}\, K \to G$ factors through $\mathrm{Spec}\, C \to G$. Namely

$$\begin{array}{ccc} \mathrm{Spec}\, K & \xrightarrow{\varphi^{-1}\psi} & G \\ \downarrow & \exists \nearrow & \\ \mathrm{Spec}\, C & & \end{array}$$

Algebraically speaking

let $x = \mathrm{Im}\,\varphi(\mathrm{Spec}\, K) \in G$, $y = \mathrm{Im}\,\psi(\mathrm{Spec}\, K)$.

so we have

$\mathcal{O}_x \xrightarrow{\varphi^*} K$, $\mathcal{O}_y \xrightarrow{\psi^*} K$.

$\varphi^*(\mathcal{O}_x), \psi^*(\mathcal{O}_y) \subset K$.

Let

M = the diff subfield generated by $\varphi^*(\mathcal{O}_x)$ in K

N = the diff. subfield generated by $\psi^*(\mathcal{O}_y)$ in K

Then the composite field MN is generated over M by constants.

Lemma (Kolchin, Kovacic, Nishioka) Let $L \supset K \supset \mathbb{C}(x)$ be a tower of diff. field extension.

Let $y \in L$ s.t.

$$\delta^2 y = 6y^2 + x.$$

If $\mathrm{tr.d.}[K(y, y') : K] \leq 1$, then y is algebraic over K.

Key to the proof of Lemma. Y, Y' variables$/K$.
We introduce a diff. op.

$$\mathcal{D} : K[Y, Y'] \to K[Y, Y']$$

$$\mathcal{D} := \delta + Y' \frac{\partial}{\partial Y} + (6Y^2 + x) \frac{\partial}{\partial Y'}$$

and show that there is no $F(Y, Y'), G(Y, Y') \in K[Y, Y']$

s.t.

$(*)$ $\mathcal{D}F = G(Y, Y')F$ in $K[Y, Y']$

except for $F(Y, Y') = c \in C_K \subset K[Y, Y']$ and $G = 0$