Symmetry Groups Underlying Bailey's Transformations for ${}_{10}\phi_9$-series

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Painlevé Equations and Monodromy Problems

Goal

- Study the group structure of the most general (two- and three-term) transformations for basic hypergeometric series.
- Give explicitly a prototype of each transformation under consideration.

Bailey's Transformation for Terminating ${}_{10}\phi_9$-Series

- Two term transformation for terminating, balanced, VWP ${}_{10}\phi_9$-series

$$ {}_{10}W_9(a;q^{-n},c,d,e,f,g,h;q,q) = \frac{(aq,aq/fg,a^2q^2/cdef,a^2q^2/cdeg;q)_n}{(aq/f,aq/g,a^2q^2/cdefg,a^2q^2/cde;q)_n} $$
$$ \times\ {}_{10}W_9\Big(\frac{a^2q}{cde};q^{-n},\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},f,g,h;q,q\Big), $$

  with n a nonnegative integer (*terminating*) and

$$ a^3q^2 = q^{-n}cdefgh. \quad (\textit{balancing requirement}) $$

- Note that the series on rhs is of same "type" as series on lhs.

Bailey's Transformation for Terminating ${}_{10}\phi_9$-series

- More symmetrically:

$$w(a;q^{-n};c,d,e,f,g,h) = w(\frac{a^2q}{cde};q^{-n};\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},f,g,h),$$

with $w(a;q^{-n};c,d,e,f,g,h)$ a rescaling of ${}_{10}W_9$.

- Trivial symmetries: permutation of last six parameters.
- Thus, e.g.

$$w(a;q^{-n};c,d,e,f,g,h) = w(\frac{a^2q}{cde};q^{-n};\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},f,g,h)$$
$$= w(\frac{aq^{1-n}}{cde};q^{-n};\frac{fq^{-n}}{a},\frac{gq^{-n}}{a},\frac{hq^{-n}}{a},\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd}) = \cdots$$

Transformation for Non-Terminating Series

- Not for a series, but for a linear combination of two series:

$$\Phi(a;b;c,d,e,f,g,h) = \cdots \times {}_{10}W_9(a;b,c,d,e,f,g,h;q,q)$$
$$+\cdots \times {}_{10}W_9(\frac{b^2}{a};b,\frac{bc}{a},\frac{bd}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a};q,q).$$

- Equivalent with Bailey's four-term transformation formula for balanced ${}_{10}\phi_9$-series.
- $\Phi(a;b;c,d,e,f,g,h) = \Phi(\frac{a^2q}{cde};b;\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},f,g,h)$ with $a^3q^2 = bcdefgh$.
- Structurally equivalent with transformation for terminating series. Also permutations of last six parameters are allowed.

Invariance Group of Two Term Transformation

- Consider six transformations acting on $(a, b, c, d, e, f, g, h, q)$:

  $$r_1 \equiv c \leftrightarrow d,\ r_2 \equiv d \leftrightarrow e,\ r_3 \equiv e \leftrightarrow f,\ r_4 \equiv f \leftrightarrow g,\ r_5 \equiv g \leftrightarrow h,$$

  and

  $$r_6(a, b, c, d, e, f, g, h, q) = \left(\frac{a^2q}{cde}, b, \frac{aq}{de}, \frac{aq}{ce}, \frac{aq}{cd}, f, g, h, q\right).$$

- Using GAP, one shows that

  $$|H| = |\langle r_1, r_2, r_3, r_4, r_5, r_6\rangle| = 51840 = |W(E_6)|.$$

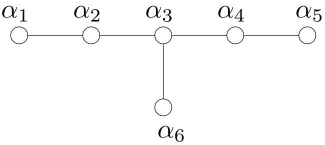
- Is it really the Weyl group of type E_6? We will show this now.

Description of Root System and Weyl Group

- Real vector space $\mathbb{R}^8$; orthonormal basis vectors ϵ_i; $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$.
- Roots of E_6 are elements of $V \subset \mathbb{R}^8$ consisting of elements $\sum_{i=1}^{8} c_i \epsilon_i$ with $c_1 + c_2 = 0$ and $c_3 + \cdots + c_8 = 0$.
- Simple roots: $\alpha_i = \epsilon_{i+2} - \epsilon_{i+3}$ $(1 \leq i \leq 5)$ and

$$\alpha_6 = \frac{1}{2}(-\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8),$$

with Dynkin diagram:



Description of Root System and Weyl Group

- Weyl group $W(E_6)$ generated by six reflections $\tilde{r}_i \equiv \tilde{r}_{\alpha_i}$, $1 \leq i \leq 6$, with

$$\tilde{r}_i(x) = x - 2\frac{\langle x, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i = x - \langle x, \alpha_i \rangle \alpha_i.$$

- More in particular $\tilde{r}_i$, $1 \leq i \leq 5$ swaps coordinates in positions $i+2$ and $i+3$.
- Let $x' = \tilde{r}_6(x)$ then

$$\begin{aligned} x_1' &= x_1 + y, & x_2' &= x_2 - y, & x_3' &= x_3 + y, & x_4' &= x_4 + y, \\ x_5' &= x_5 + y, & x_6' &= x_6 - y, & x_7' &= x_7 - y, & x_8' &= x_8 - y, \end{aligned}$$

with $y = \frac{1}{4}(-x_1 + x_2 - x_3 - x_4 - x_5 + x_6 + x_7 + x_8) = \frac{1}{2}(x_2 + x_6 + x_7 + x_8) = -\frac{1}{2}(x_1 + x_3 + x_4 + x_5)$.

Connection between Group and Transformation

- Let x_i be 8 variables with $\prod_{i=1}^{8} x_i = 1$ and let

$$a = q^{1/2}x_1^2, \quad b = q^{1/2}x_1x_2, \quad c = q^{1/2}x_1x_3, \quad d = q^{1/2}x_1x_4,$$
$$e = q^{1/2}x_1x_5, \quad f = q^{1/2}x_1x_6, \quad g = q^{1/2}x_1x_7, \quad h = q^{1/2}x_1x_8.$$

Then

$$\frac{a^2q}{cde} = \frac{q^{1/2}x_1}{x_3x_4x_5}, \quad b = q^{1/2}x_1x_2, \quad \frac{aq}{de} = \frac{q^{1/2}}{x_4x_5}, \quad \frac{aq}{ce} = \frac{q^{1/2}}{x_3x_5},$$
$$\frac{aq}{cd} = \frac{q^{1/2}}{x_3x_4}, \quad f = q^{1/2}x_1x_6, \quad g = q^{1/2}x_1x_7, \quad h = q^{1/2}x_1x_8.$$

- This corresponds to the action of $\tilde{r}_6$.
- Swapping e.g. c and d corresponds to swapping x_3 and x_4, i.e. to $\tilde{r}_1$, etc.

Theorem

- Let: $\tilde{\Phi}(x) \equiv \Phi(q^{1/2}x_1^2; q^{1/2}x_1x_2; q^{1/2}x_1x_3, \cdots, q^{1/2}x_1x_8)$.
- Let x_1 up to x_8 be eight variables satisfying $\prod_{i=1}^{8} x_i = 1$. The function $\tilde{\Phi}(x)$ is then invariant under the Weyl group of type E_6 acting multiplicatively on the variables $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$. Stated otherwise, it holds that

$$\tilde{\Phi}(x) = \tilde{\Phi}\big(\tilde{h}(x)\big),$$

for each element $\tilde{h}$ of the group $H = \langle \tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5, \tilde{r}_6 \rangle \cong E_6$.

The 72 distinct forms of the Φ-series

- The 72 distinct forms of the Φ-series are:

$$\begin{aligned}
\Phi(a;b;c,d,e,f,g,h) &= \Phi(a;b;c,d,e,f,g,h) && 1\\
&= \Phi(\frac{b^2}{a};b;\frac{bc}{a},\frac{bd}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a}) && 1\\
&= \Phi(\frac{a^2q}{cde};b;\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},f,g,h) && 20\\
&= \Phi(\frac{abq}{cde};b;\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a}) && 20\\
&= \Phi(\frac{bc}{d};b;c,\frac{bc}{a},\frac{aq}{de},\frac{aq}{df},\frac{aq}{dg},\frac{aq}{dh}) && 30
\end{aligned}$$

- Note that the second parameter is fixed!

A Three Term Transformation

- In [D.P. Gupta and D.R. Masson, *Trans. Amer. Math. Soc.* **350**(2):769–808, 1998] one finds the following three term transformation:

$$C_1\,\Phi(a;c;b,d,e,f,g,h)+C_2\,\Phi(\frac{q}{a};\frac{q}{h};\frac{q}{b},\frac{q}{c},\frac{q}{d},\frac{q}{e},\frac{q}{f},\frac{q}{g})$$
$$+C_3\,\Phi(\frac{c^2}{a};\frac{bc}{a};c,\frac{cd}{a},\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a})=0.$$

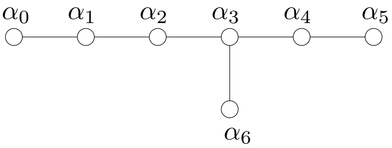
- Consider r_0 with:

$$r_0(a,b,c,d,e,f,g,h,q)=(\frac{c^2}{a},\frac{bc}{a},c,\frac{cd}{a},\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a},q).$$

- Let $G=\langle r_0,H\rangle$. Using GAP one sees $|G|=2903040=|W(E_7)|$.

Weyl Group of Type E_7

- Take the simple roots of E_6 and add $\alpha_0 = \epsilon_1 - \epsilon_3$. Dynkin diagram:



- Using the same realization as before one sees that indeed $\tilde{r}_0$ corresponds to r_0. So $G \cong W(E_7)$.

Description of the 56 E_6 orbits in E_7

- Three term transformation can be seen as transformation between three *sets of series*, each of size 51840:

$$C_1\,\Phi^{c} + C_2\,\Phi^{q/h} + C_3\,\Phi^{bc/a} = 0.$$

- There are $56 = |W(E_7)|/|W(E_6)|$ such sets.
- We have $c = q^{1/2}x_1x_3$, $q/h = q^{1/2}/x_1x_8$ and $bc/a = q^{1/2}x_2x_3$. Thus, we could say that:

$$C_1\,\tilde{\Phi}_{(1,3)} + C_2\,\tilde{\Phi}_{(1,8)^*} + C_3\,\tilde{\Phi}_{(2,3)} = 0.$$

- We thus have 28 series $\tilde{\Phi}_{(i,j)}$ and 28 series $\tilde{\Phi}_{(i,j)^*}$ with $1 \leq i < j \leq 8$.

Equivalent Three Term Transformations

- The 56 series correspond to the 56 solutions of a second order difference equation. [Gupta and Masson, 1998]
- Any three series are thus connected by a transformation, and hence $\binom{56}{3} = 27720$ transformations.
- What is the connection between Φ^b, Φ^c and $\Phi^{bd/a}$ (or stated otherwise between $\tilde{\Phi}_{(1,2)}$, $\tilde{\Phi}_{(1,3)}$ and $\tilde{\Phi}_{(2,4)}$.)
- Applying

$$a \to \frac{acq}{bdh},\ b \to \frac{aq}{dh},\ c \to c,\ d \to \frac{aq}{bh},\ e \to \frac{ce}{a},\ f \to \frac{cf}{a},\ g \to \frac{cg}{a},\ h \to \frac{aq}{bd},$$

  on the three term transformation between Φ^c, $\Phi^{bc/a}$ and $\Phi^{q/h}$ gives the desired transformation.
- It is, however, impossible to find in this way the transformation between Φ^b, Φ^c and $\Phi^{bc/a}$.

Number of Different Three Term Transformations

- There are 56 cosets of E_6 in E_7. How many different orbits are there when E_7 acts on sets consisting of three cosets each?
- GAP code:

```
E7 := Group(m0,m1,m2,m3,m4,m5,m6);
E6 := Group(m1,m2,m3,m4,m5,m6);
rc := RightCosets(E7,E6);;
P2 := Action(E7, rc, OnRight);;
comb := Combinations([1..56],3);;
P3 := Action(P2,comb,OnSets);;
orb := Orbits(P3,[1..27720],OnPoints);;
```

and

```
gap> Size(orb);
5
gap> [Size(orb[1]), Size(orb[2]), Size(orb[3]), Size(orb[4]),
Size(orb[5])];
[ 4032, 7560, 12096, 1512, 2520 ]
```

Characterization of the five orbits

Orbit 1	$(i,j),(i,k),(i,l)$	$(i,j)^*,(i,k)^*,(i,l)^*$	$8\times\binom{7}{3}=280$
(size 4032)	$(i,j),(i,k),(j,k)$	$(i,j)^*,(i,k)^*,(j,k)^*$	$\binom{8}{3}=56$
	$(i,j),(i,k),(l,m)^*$	$(i,j)^*,(i,k)^*,(l,m)$	$\binom{8}{2}\times 6\times\binom{5}{2}=1680$
Orbit 2	$(i,j),(i,k),(j,l)$	$(i,j)^*,(i,k)^*,(j,l)^*$	$\binom{8}{2}\times 6\times 5=840$
(size 7560)	$(i,j),(i,k),(j,l)^*$	$(i,j)^*,(i,k)^*,(j,l)$	$8\times 7\times 6\times 5=1680$
	$(i,j),(k,l),(m,n)^*$	$(i,j)^*,(k,l)^*,(m,n)$	$\binom{8}{2}\times\binom{6}{2}\times\binom{4}{2}/2=1260$
Orbit 3	$(i,j),(j,k),(l,m)$	$(i,j)^*,(j,k)^*,(l,m)^*$	$8\times\binom{7}{2}\times\binom{5}{2}=1680$
(size 12096)	$(i,j),(i,k),(j,k)^*$	$(i,j)^*,(i,k)^*,(j,k)$	$8\times\binom{7}{2}=168$
	$(i,j),(j,k),(j,l)^*$	$(i,j)^*,(j,k)^*,(j,l)$	$8\times 7\times\binom{6}{2}=840$
	$(i,j),(k,l),(i,m)^*$	$(i,j)^*,(k,l)^*,(i,m)$	$8\times 7\times 6\times\binom{5}{2}=3360$
Orbit 4	$(i,j),(k,l),(m,n)$	$(i,j)^*,(k,l)^*,(m,n)^*$	$\binom{8}{2}\times\binom{6}{2}\times\binom{4}{2}/3!=420$
(size 2520)	$(i,j),(k,l),(i,k)^*$	$(i,j)^*,(k,l)^*,(i,k)$	$\binom{8}{2}\times 6\times 5=840$
Orbit 5	$(i,j),(i,k),(i,j)^*$	$(i,j)^*,(i,k)^*,(i,j)$	$8\times 7\times 6=336$
(size 1512)	$(i,j),(k,l),(i,j)^*$	$(i,j)^*,(k,l)^*,(i,j)$	$\binom{8}{2}\times\binom{6}{2}=420$

Prototypes of the transformations

- How to find the actual coefficients in the three term transformations?
- Limiting process to three term transformations between ${}_8W_7$-series, which have already been studied.
- Simplify and rewrite the coefficients involved so that all obvious symmetries are there.
- For the first orbit (of size 4032) we have:

$$bd\,\left(\frac{cd}{a},\frac{c}{d},\frac{dq}{c},\frac{aq}{be},\frac{aq}{bf},\frac{aq}{bg},\frac{aq}{bh};q\right)_\infty \Phi(a;b;c,d,e,f,g,h)$$
$$+\,bc\,\left(\frac{bd}{a},\frac{d}{b},\frac{bq}{d},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{cg},\frac{aq}{ch};q\right)_\infty \Phi(a;c;d,b,e,f,g,h)$$
$$+\,cd\,\left(\frac{bc}{a},\frac{b}{c},\frac{cq}{b},\frac{aq}{de},\frac{aq}{df},\frac{aq}{dg},\frac{aq}{dh};q\right)_\infty \Phi(a;d;b,c,e,f,g,h)=0.$$

- Symmetric in $\{e,f,g,h\}$ and cyclic permutations of $\{b,c,d\}$.

Moving Around in the Orbits

- How to find transformations like

$$a \to \frac{acq}{bdh},\ b \to \frac{aq}{dh},\ c \to c,\ d \to \frac{aq}{bh},\ e \to \frac{ce}{a},\ f \to \frac{cf}{a},\ g \to \frac{cg}{a},\ h \to \frac{aq}{bd},$$

  for transforming one identity into another.
- 70 of the 126 roots of E_7 are given by:

$$\frac{1}{2}(\sum_{i=1}^{8}(-1)^{s_i}\epsilon_i), \quad (s_i \in \{0,1\};\quad \sum_{i=1}^{8} s_i = 4).$$

- Let $x' = \tilde{r}(x)$, the reflection through such a root. Then either $x_i' + x_j' = x_i + x_j$ when s_i and s_j are different, or $x_i' + x_j' = -(x_k + x_l)$ with k and l such that $s_i = s_j = s_k = s_l$.
- In this way a table for transitions between patterns in orbits can be set up, using one such a reflection, sometimes followed by $x \to -x$, followed by a permutation of the parameters.

Example of a Transistion

- Previous transformation between Φ^b, Φ^c and Φ^d has the pattern $(i,j),(i,k),(i,l)$ with $i=1$, $j=2$, $k=3$ and $l=4$.
- The goal transformation connects Φ^b, Φ^c, $\Phi^{bc/a}$ or $(1,2),(1,3),(2,3)$ which has the pattern $(i',j'),(i',k'),(j',k')$.
- From table: transition: $s_i = s_j = s_k = s_l$ followed by $x \to -x$.
- Thus $s_1 = s_2 = s_3 = s_4 = 1$

$$(1,2),(1,3),(1,4) \overset{\tilde{r}}{\Rightarrow} (3,4)^*,(2,4)^*,(2,3)^*$$
$$\overset{x\longrightarrow -x}{\Rightarrow} (3,4),(2,4),(2,3) \overset{x_1\leftrightarrow x_4}{\Rightarrow} (1,2),(1,3),(2,3)$$

- Thus $x_i' = x_i + y$ $(i=1,\ldots,4)$ and $x_i' = x_i - y$ $(i=5,\ldots,8)$ with $y = (-x_1 - x_2 - x_3 - x_4 + x_5 + x_6 + x_7 + x_8)/4 = -(x_1+x_2+x_3+x_4)/2$.
- $x_i'' = -x_i'$
- $x_i''' = x_i''$, $(i \neq 1,4)$, $x_1''' = x_4''$ and $x_4''' = x_1''$.

- Translating back in terms of the variables a up to h means that:

$$a \to \frac{bc}{d},\ b \to c,\ c \to b,\ d \to \frac{bc}{a},\ e \to \frac{aq}{de},\ f \to \frac{aq}{df},\ g \to \frac{aq}{dg},\ h \to \frac{aq}{dh}.$$

- After a minor rewriting of the coefficients, one gets:

$$\begin{aligned} &b\,(\frac{a}{b}, \frac{bq}{a}, \frac{cd}{a}, \frac{ce}{a}, \frac{cf}{a}, \frac{cg}{a}, \frac{ch}{a}; q)_\infty\, \Phi(a; b; c, d, e, f, g, h) \\ &-c\,(\frac{a}{c}, \frac{cq}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}, \frac{bg}{a}, \frac{bh}{a}; q)_\infty\, \Phi(a; c; b, d, e, f, g, h) \\ &+c\,(\frac{b}{c}, \frac{cq}{b}, d, e, f, g, h; q)_\infty\, \Phi(\frac{b^2}{a}; \frac{bc}{a}; b, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}, \frac{bg}{a}, \frac{bh}{a}) = 0. \end{aligned}$$

The Other Orbits

- A prototype for each of the orbits was determined.
- Expressions are more involved as not all coefficients can be completely factored.
- Second orbit; one coefficient is a difference of two products.
- Third orbit; two coefficients are a difference of two products.
- Fourth orbit; three coefficients are a difference of two products.
- Fifth orbit; one coefficient factors completely, one is a difference of two products and one is a difference of three products.
- We believe that for each orbit the prototype given is as simple as possible.