Yang-Baxter maps and integrable difference equations

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based on joint work with

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Painleve Equations and Monodromy Problems (PEW01)
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Motivation

Integrable lattice equations and IVP on the lattice
(Nijhoff, Capel, Quispel, P)

Multidimensional consistency as integrability-Classification
(Nijhoff, Adler-Bobenko-Suris)

Yang Baxter maps-dynamical point of view (Veselov)

Yang Baxter maps-Classification $F_I - F_V$(Adler-Bobenko-Suris)

Plan

Quantum Yang Baxter equation and YB maps

Equations on quad-graphs and the 3D consistency property

Multi-dimensional consistency and YB maps

Multi-component YB maps

Coupled YB maps and discrete Painlevé equations

Quantum Yang-Baxter Equation (QYBE)(constant form)

$$R : V \otimes V \to V \otimes V$$

where V is a vector space over a field k and R a linear map .

QYBE

$$R_{(1,2)}\, R_{(1,3)}\, R_{(2,3)} = R_{(2,3)}\, R_{(1,3)}\, R_{(1,2)}$$

in End$(V \otimes V \otimes V)$

The maps $R_{(i,j)} \in$ End$(V \otimes V \otimes V)$, $1 \leq i < j \leq 3$ are defined as follows

$$R_{(1,2)} = R \otimes 1_V, \qquad R_{(2,3)} = 1_V \otimes R$$

$$R_{(1,3)} = (1_V \otimes T_{V,V})(R \otimes 1_V)(1_V \otimes T_{V,V})$$

where $T_{V,V} : V \otimes V \to V \otimes V$ is the twist map

$$T_{V,V}(m \otimes n) = n \otimes m$$

Set theoretic solutions of QYBE (YB maps)

$\mathbb{X} \rightarrow$ a set (variety)

$R \rightarrow$ a map (birational) of $\mathbb{X} \times \mathbb{X}$ into itself.

For any $x, y \in \mathbb{X}$

$$R(x,y) = \big(f(x,y), g(x,y)\big).$$

$R_{(i,j)} : \mathbb{X}^n \to \mathbb{X}^n \qquad n \geq 2$ and $1 \leq i, j \leq n$, $i \neq j$

$R_{(i,j)}(x^1, x^2, \ldots x^n) =$

$$\begin{cases} (x^1, \ldots, x^{i-1}, f(x^i, x^j), x^{i+1}, \ldots, x^{j-1}, g(x^i, x^j), x^{j+1}, \ldots x^n) & i < j, \\ (x^1, \ldots, x^{j-1}, g(x^i, x^j), x^{j+1}, \ldots, x^{i-1}, f(x^i, x^j), x^{i+1}, \ldots x^n) & i > j \end{cases}$$

E.g.

$$R_{(1,2)} = R$$

$$R_{(2,1)}(x,y) = \big(g(y,x), f(y,x)\big) = T\,R\,T$$

- *A map R is called a* YB map *if it satisfies*

$$R_{(1,2)}\,R_{(1,3)}\,R_{(2,3)} = R_{(2,3)}\,R_{(1,3)}\,R_{(1,2)}$$

regarded as an equality of maps in $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$.

Two parameter form

$$\begin{aligned} R_{(1,2)}(\alpha_1, \alpha_2)\,R_{(1,3)}(\alpha_1, \alpha_3)\,R_{(2,3)}(\alpha_2, \alpha_3) &= \\ R_{(2,3)}(\alpha_2, \alpha_3)\,R_{(1,3)}(\alpha_1, \alpha_3)\,R_{(1,2)}(\alpha_1, \alpha_2) \end{aligned}$$

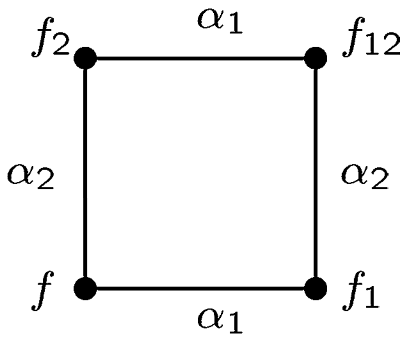
$\alpha_i \in S$ where S is a set.

Integrable equations on quad-graphs

Equations associated to planar graphs with elementary quadrilateral faces (e.g. the complex defined by Z^2)

$f : \mathbb{Z}^2 \to \mathbb{C}$

$$\mathcal{E}(f, f_1, f_2, f_{12}; \alpha_1, \alpha_2) = 0\,,$$



$$f_1 := f(n_1 + 1, n_2)\,, \qquad f_{12} := f(n_1 + 1, n_2 + 1)\,, e.t.c$$

Integrable equations from the ABS classification:

$$(f_{12} - f)(f_1 - f_2) = \alpha_1 - \alpha_2\,,$$

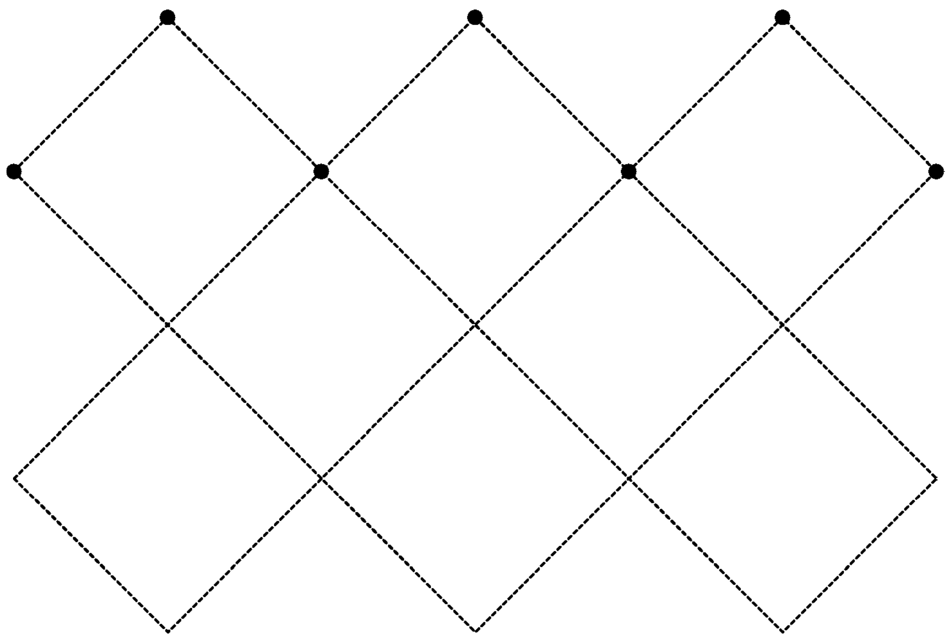
$$a_1(f\,f_1 + f_2\,f_{12}) - a_2(f\,f_2 + f_1\,f_{12}) = \delta({a_1}^2 - {a_2}^2)\,,$$

$$\frac{f_1 - \alpha_1 f}{f_2 - \alpha_2 f}\,\frac{f_2 - \alpha_1 f_{12}}{f_1 - \alpha_2 f_{12}} = \frac{1 - {\alpha_1}^2}{1 - {\alpha_2}^2}\,.$$

An Initial Value Problem

$$(f_{12} - f)(f_1 - f_2) = \alpha_1 - \alpha_2$$

(P, Nijhoff, Capel 1990)



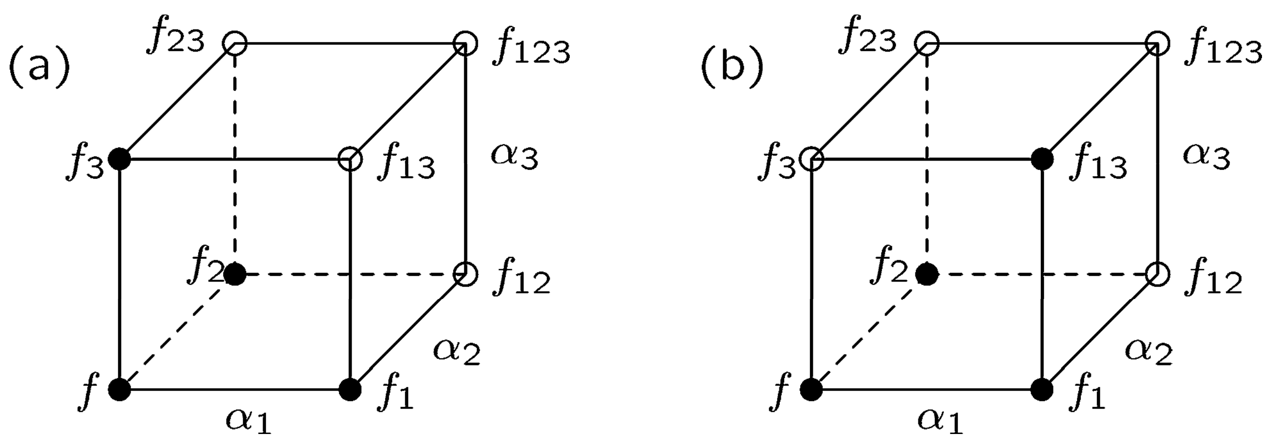
The consistency property

The overdetermined system of equations

$$\mathcal{E}(f, f_i, f_j, f_{ij}; a_i, a_j) = 0\,, \qquad 1 \le i < j \le n\,,$$

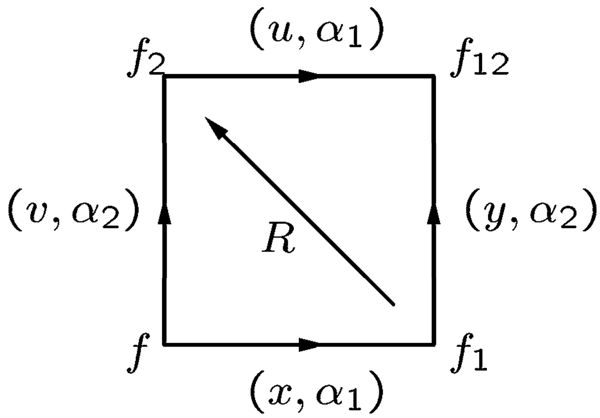
possesses a non empty set of solutions.

The property can be verified by considering an elementary IVP on the hypercube. E.g. for $n = 3$



YB maps from discrete pKdV

$$(f_{12}-f)(f_1-f_2)=\alpha_1-\alpha_2\,,$$



Variables assigned to the edges (differences):

$$x=f_1-f,\quad y=f_{12}-f_1,\quad u=f_{12}-f_2,\quad v=f_2-f,$$

- $(x+y)(x-v)=\alpha_1-\alpha_2$
- $x+y=u+v$

$$\boxed{u=y+\frac{\alpha_1-\alpha_2}{x+y},\qquad v=x-\frac{\alpha_1-\alpha_2}{x+y}}$$

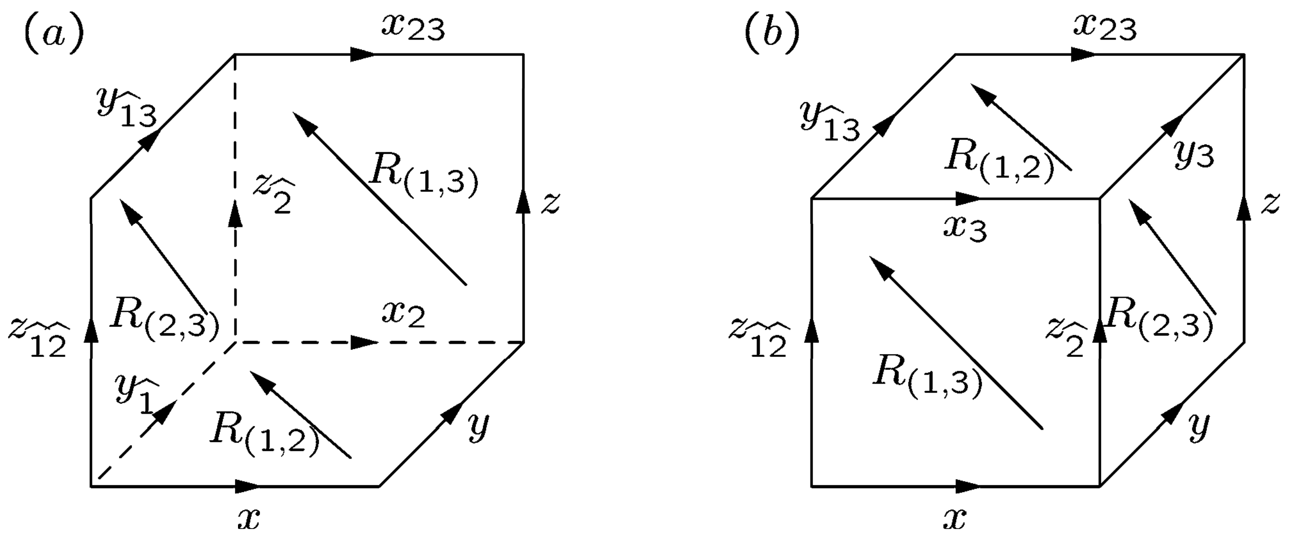
Variables assigned to the edges (products):

$$x=f\,f_1,\quad y=f_1\,f_{12},\quad u=f_2f_{12},\quad v=f\,f_2\,,$$

- $y+v-x-u=\alpha_1-\alpha_2$
- $x\,u=y\,v$

$$\boxed{u=y\left(1+\frac{\alpha_1-\alpha_2}{x-y}\right),\quad v=x\left(1+\frac{\alpha_1-\alpha_2}{x-y}\right)}$$

The consistency property and the QYBE



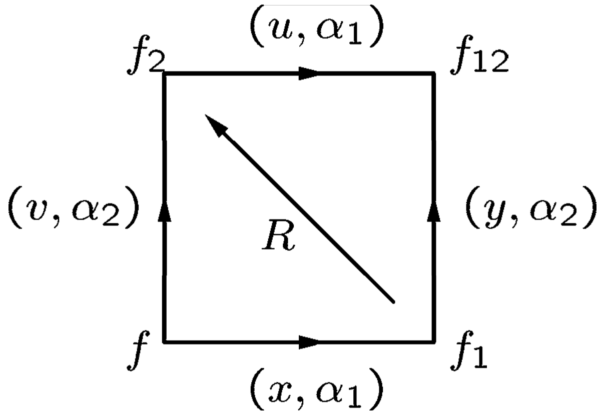
The consistency property guarantees that the composite maps

$$(a): \quad (x,y,z) \xrightarrow{R_{(1,2)}} (x_2, y_{\widehat{1}}, z) \xrightarrow{R_{(1,3)}} (x_{23}, y_{\widehat{1}}, z_{\widehat{2}}) \xrightarrow{R_{(2,3)}} (x_{23}, y_{\widehat{13}}, z_{\widehat{\widehat{12}}})$$

$$(b): \quad (x,y,z) \xrightarrow{R_{(2,3)}} (x, y_3, z_{\widehat{2}}) \xrightarrow{R_{(1,3)}} (x_3, y_3, z_{\widehat{\widehat{12}}}) \xrightarrow{R_{(1,2)}} (x_{23}, y_{\widehat{13}}, z_{\widehat{\widehat{12}}})$$

appearing in QYBE, applied on (x,y,z) give identical values for $(x_{23}, y_{\widehat{13}}, z_{\widehat{\widehat{12}}})$.

Method for constructing YB maps from consistent discrete equations



$$\mathcal{E}(f, f_1, f_2, f_{12}; \alpha_1, \alpha_2) = 0\,,$$

$$x = G(f, f_1; a_1), \quad y = G(f_1, f_{12}; a_2),$$
$$u = G(f_2, f_{12}; a_1), \quad v = G(f, f_2; a_2),$$

• Find the geometrical (Lie point) symmetries of the lattice equation

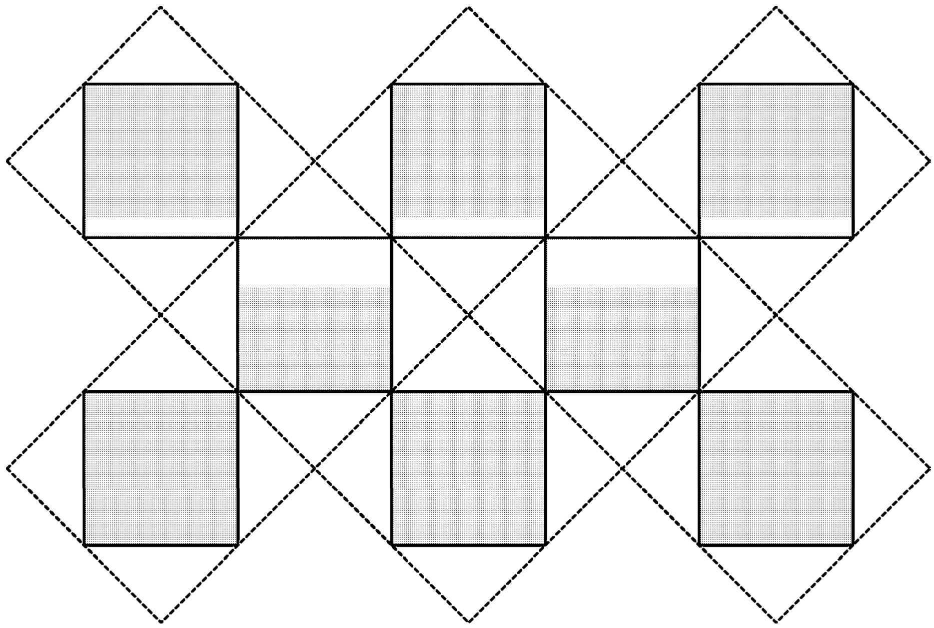
• Construct the corresponding symmetry invariants (x, y, u, v) (YB variables) assigned on the edges

• Write lattice equation in terms of the edge invariants (since there is a functional relation between them $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}v = 0$)

• Solve the resulting system for u, v in terms of x, y

• Use an orientation of the quadrilateral edges to those of the 3-cube, adapted to the Yang-Baxter relation.

Integrable difference equations as Coupled Yang-Baxter maps



The Harrison map

$$\frac{f_1 - \alpha_1 f}{f_2 - \alpha_2 f}\,\frac{f_2 - \alpha_1 f_{12}}{f_1 - \alpha_2 f_{12}} = \frac{1 - {\alpha_1}^2}{1 - {\alpha_2}^2}$$

Scaling invariance

$$f \mapsto e^{\varepsilon} f$$

YB variables by using the scaling symmetry

$$x = \frac{f_1}{\alpha_1\, f}, \quad y = \frac{f_{12}}{\alpha_2\, f_1} \quad u = \frac{f_{12}}{\alpha_1 f_2} \quad v = \frac{f_2}{\alpha_2 f}$$

Functional dependence

$$x\,y = u\,v,$$

Lattice equation written in terms of the invariants

$$\frac{1 - x^{-1}}{1 - v^{-1}} = \frac{1 - \gamma_1}{1 - \gamma_2}\,\frac{1 - \gamma_2\, y}{1 - \gamma_1\, u}$$

where $\gamma_i = {\alpha_i}^2$

YB map

$$\boxed{u = y\,Q, \qquad v = x\,Q^{-1}}$$

$$Q = \frac{(1-\gamma_2) + (\gamma_2 - \gamma_1)\,x + \gamma_2(\gamma_1 - 1)\,x\,y}{(1-\gamma_1) + (\gamma_1 - \gamma_2)\,y + \gamma_1(\gamma_2 - 1)\,x\,y}$$

Consistency of dpKdV around a 3-cube and the F_{III} map

The dpKdV equation is imposed on each face of an elementary cube

Invariants

$$x = f_1 - f_3\,, \quad y = f_{1,2} - f_{1,3}\,, \quad u = f_{1,2} - f_{2,3}\,, \quad v = f_2 - f_3\,,$$

assigned on four faces of the 3-cube. Using the dpKdV equations

$$f_{1,2} - f = \frac{\alpha_1 - \alpha_2}{f_1 - f_2}\,, \quad f_{1,3} - f = \frac{\alpha_1 - \alpha_3}{f_1 - f_3}\,, \quad f_{2,3} - f = \frac{\alpha_2 - \alpha_3}{f_2 - f_3}\,,$$

we find that

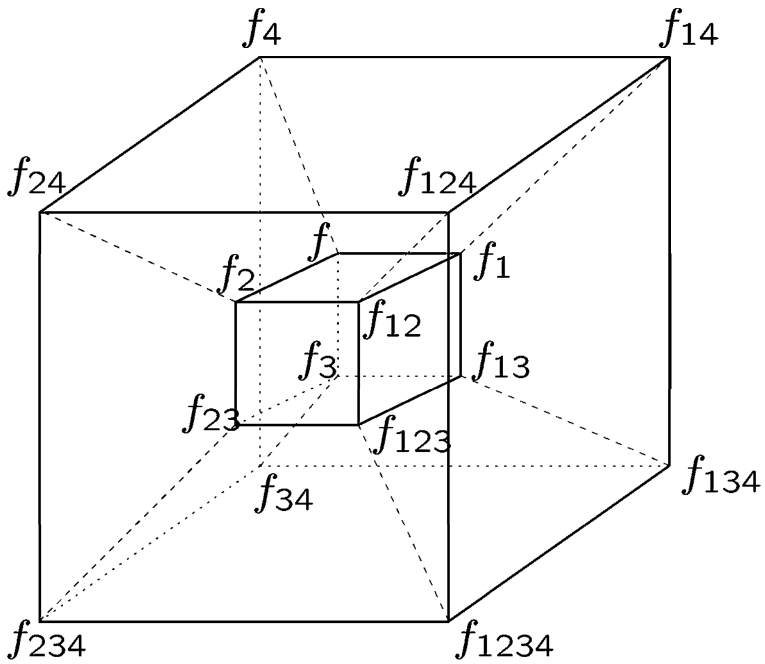
$$u\,v = x\,y\,, \qquad u - \frac{\beta_1}{x} = y - \frac{\beta_2}{v}\,,$$

where $\beta_1 = \alpha_1 - \alpha_3$, $\beta_2 = \alpha_2 - \alpha_3$. Solving for (u, v) in terms of (x, y) we obtain the (total positive) map

$$u = yP\,, \qquad v = xP^{-1}\,, \qquad P = \frac{\beta_1 + x\,y}{\beta_2 + x\,y}\,,$$

which satisfies the YB relation as it can be checked by direct calculations. This fact is also related to the higher dimensional consistency of dpKdV on $\mathbb{Z}^4$ as it is explained in the following.

The Harrison map from dpKdV on a 4-cube



$$(f_{ij} - f)(f_i - f_j) = \alpha_i - \alpha_j \qquad 1 \leq i < j \leq 4$$

$$x = \frac{f_1 - f_3}{f_2 - f_3}, \qquad v = \frac{f_1 - f_4}{f_2 - f_4},$$

$$u = x_4, \qquad y = v_3$$

$$\gamma_1 = \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}, \qquad \gamma_2 = \frac{\alpha_2 - \alpha_4}{\alpha_1 - \alpha_4}$$

$$\boxed{u = y\, Q, \qquad v = x\, Q^{-1}}$$

$$Q = \frac{(1-\gamma_2) + (\gamma_2 - \gamma_1)\, x + \gamma_2(\gamma_1 - 1)\, x\, y}{(1-\gamma_1) + (\gamma_1 - \gamma_2)\, y + \gamma_1(\gamma_2 - 1)\, x\, y}$$

YB map from the discrete modified Boussinesq system

The discrete modified Boussinesq (dmBSQ) equations (Tongas, Nijhoff 2005)
two fields $f, g : \mathbb{Z}^2 \to \mathbb{CP}^1$

$$f_{1,2} = g\,\frac{\alpha_1\, f_2 - \alpha_2\, f_1}{\alpha_1\, g_1 - \alpha_2\, g_2}\,, \qquad g_{1,2} = \frac{g}{f}\,\frac{\alpha_1\, f_1\, g_2 - \alpha_2\, f_2\, g_1}{\alpha_1\, g_1 - \alpha_2\, g_2}\,.$$

$3D$ consistency

$$f_{1,2,3} = f\,\frac{\underset{ijk}{\boldsymbol{\sigma}}\, \alpha_i\alpha_j f_k(\alpha_i g_i - \alpha_j g_j)}{\underset{ij}{\boldsymbol{\sigma}}\, \alpha_i\alpha_j(\alpha_i f_i g_j - \alpha_j f_j g_i)}\,,$$

$$g_{1,2,3} = g\,\frac{\underset{ijk}{\boldsymbol{\sigma}}\, \alpha_i\alpha_j g_k(\alpha_i f_j - \alpha_j f_i)}{\underset{ij}{\boldsymbol{\sigma}}\, \alpha_i\alpha_j(\alpha_i f_i g_j - \alpha_j f_j g_i)}\,,$$

(cyclic sum $\underset{ijk}{\boldsymbol{\sigma}}$ is over the subscripts $(i, j, k) = (1, 2, 3), (3, 1, 2), (2, 3$
and similarly the cyclic sum $\underset{ij}{\boldsymbol{\sigma}}$ is over $(i, j) = (1, 2), (2, 3), (3, 1)$).

Symmetry generators

$$\mathbf{v}_1 = f\,\partial_f\,, \qquad \mathbf{v}_2 = g\,\partial_g\,,$$

YB variables (joint lattice invariants)

$$x^1 = \frac{f_1}{f}, \quad y^1 = \frac{f_{1,2}}{f_1}, \quad u^1 = \frac{f_{1,2}}{f_2}, \quad v^1 = \frac{f_2}{f}, \qquad (1)$$
$$x^2 = \frac{g_1}{g}, \quad y^2 = \frac{g_{1,2}}{g_1}, \quad u^2 = \frac{g_{1,2}}{g_2}, \quad v^2 = \frac{g_2}{g}. \qquad (2)$$

Functional dependence

$$x^1 y^1 = u^1 v^1, \qquad x^2 y^2 = u^2 v^2 .$$

Lattice equations in terms of the invariants

$$u^1 v^1 = \frac{\alpha_1 v^1 - \alpha_2 x^1}{\alpha_1 x^2 - \alpha_2 v^2}, \qquad u^2 v^2 = \frac{\alpha_1 x^1 v^2 - \alpha_2 v^1 x^2}{\alpha_1 x^2 - \alpha_2 v^2}.$$

Reversible YB map

$$u^1 = y^1 A, \qquad v^1 = x^1 A^{-1}$$
$$u^2 = y^2 B, \qquad v^2 = x^2 B^{-1}$$

$$A = \frac{{\alpha_1}^2 x^1 + {\alpha_2}^2 x^1 x^2 y^1 + \alpha_1 \alpha_2 x^2 y^2}{\alpha_1 \alpha_2 x^1 + {\alpha_1}^2 x^1 x^2 y^1 + {\alpha_2}^2 x^2 y^2}$$

,

$$B = \frac{{\alpha_1}^2 x^1 + {\alpha_2}^2 x^1 x^2 y^1 + \alpha_1 \alpha_2 x^2 y^2}{{\alpha_2}^2 x^1 + \alpha_1 \alpha_2 x^1 x^2 y^1 + {\alpha_1}^2 x^2 y^2}$$

YB map from the discrete potential Boussinesq system

$f, g, h : \mathbb{Z}^2 \to \mathbb{CP}^1$,

$$\begin{aligned} h_1 &= f\, f_1 - g\,, \\ h_2 &= f\, f_2 - g\,, \\ h &= f\, f_{1,2} - g_{1,2} - \frac{\alpha_1 - \alpha_2}{f_1 - f_2}\,. \end{aligned}$$

Initial value problem: initial values (f, g, h), (f_1, g_1), (f_2, g_2), only. From these data the values (h_1, h_2) and $(f_{1,2}, g_{1,2}, h_{1,2})$ are determined uniquely.

$$f_{1,2} = \frac{g_1 - g_2}{f_1 - f_2}\,,$$

and subsequently the values $h_{1,2}$ and $g_{1,2}$ are determined

Symmetry generators

$$\mathbf{v}_1 = \partial_f + f\,\partial_g + f\,\partial_h\,, \quad \mathbf{v}_2 = \partial_g - \partial_h\,.$$

Symmetry transformations

$$\begin{aligned} G^1 &: (f, g, h) \mapsto (f + \varepsilon_1, g + \varepsilon_1 f + \frac{{\varepsilon_1}^2}{2}, h + \varepsilon_1 f + \frac{{\varepsilon_1}^2}{2})\,, \\ G^2 &: (f, g, h) \mapsto (f, g + \varepsilon_2, h - \varepsilon_2)\,, \end{aligned}$$

respectively.

YB variables the following invariants

$$
\begin{array}{ll}
x^1 = f_1 - f\,, & y^1 = f_{1,2} - f_1\,,\\
x^2 = g_1 - g - f(f_1 - f)\,, & y^2 = g_{1,2} - g_1 - f_1(f_{1,2} - f_1)\,,\\
x^3 = h_1 - h - f(f_1 - f)\,, & y^3 = h_{1,2} - g_1 - f_1(f_{1,2} - f_1)\,,
\end{array}
$$

$$
\begin{array}{ll}
u^1 = f_{1,2} - f_2\,, & v^1 = f_2 - f\,,\\
u^2 = g_{1,2} - g_2 - f_2(f_{1,2} - f_2)\,, & v^2 = g_2 - g - f(f_2 - f)\,,\\
u^3 = h_{1,2} - h_2 - f_2(f_{1,2} - f_2)\,, & v^3 = h_2 - h - f(f_2 - f)\,.
\end{array}
$$

Functionally related

$$
\begin{aligned}
u^1 + v^1 &= x^1 + y^1\,,\\
u^2 + v^2 &= x^2 + y^2 + x^1y^1 - u^1v^1\,,\\
u^3 + v^3 &= x^3 + y^3 + x^1y^1 - u^1v^1\,.
\end{aligned}
$$

Lattice equations in terms of the invariants

$$
\begin{aligned}
x^1 &= -y^1 + \frac{x^2}{x^1 - v^1} - \frac{v^2}{x^1 - v^1}\,,\\
x^3 &= x^2 + y^2 + x^1y^1 + \frac{\alpha_1 - \alpha_2}{x^1 - v^1}\,,\\
v^3 &= u^2 + v^2 + u^1v^1 + \frac{\alpha_1 - \alpha_2}{x^1 - v^1}\,,
\end{aligned}
$$

YB map

$$
\begin{aligned}
u^1 &= y^1 - (\alpha_1 - \alpha_2)\Gamma^{-1}\,,\\
v^1 &= x^1 + (\alpha_1 - \alpha_2)\Gamma^{-1}\,,\\
u^2 &= y^2 + (\alpha_1 - \alpha_2)\big(\alpha_1 - \alpha_2 - 2y^1\Gamma\big)\Gamma^{-2}\\
v^2 &= x^2 + (\alpha_1 - \alpha_2)(x^1 + y^1)\Gamma^{-1}\,,\\
u^3 &= y^3 + (\alpha_1 - \alpha_2)\big(\alpha_1 - \alpha_2 + (x^1 - y^1)\Gamma\big)\Gamma^{-2}\\
v^3 &= x^3\,,
\end{aligned}
$$

(2)

where $\Gamma = x^2 - x^3 + x^1y^1 + y^2$.

YB map from discrete Calapso equation and nonlinear σ–model.

From study of discrete isothermic surfaces (Schief 2001): Vector generalization of the dpKdV

$$\boldsymbol{f} : \mathbb{Z}^2 \mapsto \mathbb{C}^n$$

$$(\boldsymbol{f}_{1,2} - \boldsymbol{f}) = \frac{\alpha_1 - \alpha_2}{|\boldsymbol{f}_1 - \boldsymbol{f}_2|^2}(\boldsymbol{f}_1 - \boldsymbol{f}_2)\,,$$

discrete Calapso equation. Three dimensional consistency since $\boldsymbol{f}_{1,2,3}$ is given by

$$\boldsymbol{f}_{1,2,3} = \frac{\lambda\,|\boldsymbol{f}_2 - \boldsymbol{f}_3|^2\boldsymbol{f}_1 - \mu\,|\boldsymbol{f}_1 - \boldsymbol{f}_3|^2\boldsymbol{f}_2 + \nu\,|\boldsymbol{f}_1 - \boldsymbol{f}_2|^2\boldsymbol{f}_3}{\lambda\,|\boldsymbol{f}_2 - \boldsymbol{f}_3|^2 - \mu\,|\boldsymbol{f}_1 - \boldsymbol{f}_3|^2 + \nu\,|\boldsymbol{f}_1 - \boldsymbol{f}_2|^2}\,,$$

$\lambda = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$, $\mu = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)$,
$\nu = (\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)$.

Translational invariance suggests the following YB variables

$$\boldsymbol{x} = \boldsymbol{f}_1 - \boldsymbol{f}, \quad \boldsymbol{y} = \boldsymbol{f}_{1,2} - \boldsymbol{f}_1, \quad \boldsymbol{u} = \boldsymbol{f}_{1,2} - \boldsymbol{f}_2, \quad \boldsymbol{v} = \boldsymbol{f}_2 - \boldsymbol{f},$$

on the edges of a square.

Reversible YB map

$$\boldsymbol{u} = \boldsymbol{y} + \frac{\alpha_1 - \alpha_2}{|\boldsymbol{x} + \boldsymbol{y}|^2}(\boldsymbol{x} + \boldsymbol{y}), \qquad \boldsymbol{v} = \boldsymbol{x} - \frac{\alpha_1 - \alpha_2}{|\boldsymbol{x} + \boldsymbol{y}|^2}(\boldsymbol{x} + \boldsymbol{y})\,.$$

Discrete Calapso equation can be specialized to an integrable discrete version of the $O(n+2)$ nonlinear σ- model by imposing the constraint

$$|\boldsymbol{f}|^2 = 1\,,$$

Since the shifted values of $\boldsymbol{f}$ with respect to any lattice directions should also satisfy constraint , the equation is compatible with this constraint whenever

$$2\boldsymbol{f}\cdot\boldsymbol{f}_2 - 2\boldsymbol{f}\cdot\boldsymbol{f}_1 = \alpha_1 - \alpha_2\,.$$

This requirement can be satisfied by taking

$$-2\boldsymbol{f}\cdot\boldsymbol{f}_1 = \alpha_1\,, \qquad -2\boldsymbol{f}\cdot\boldsymbol{f}_2 = \alpha_2\,.$$

i.e.

$$|\boldsymbol{x}|^2 = 2+\alpha_1\,, \qquad |\boldsymbol{y}|^2 = 2+\alpha_2\,.$$

The map obtains the form

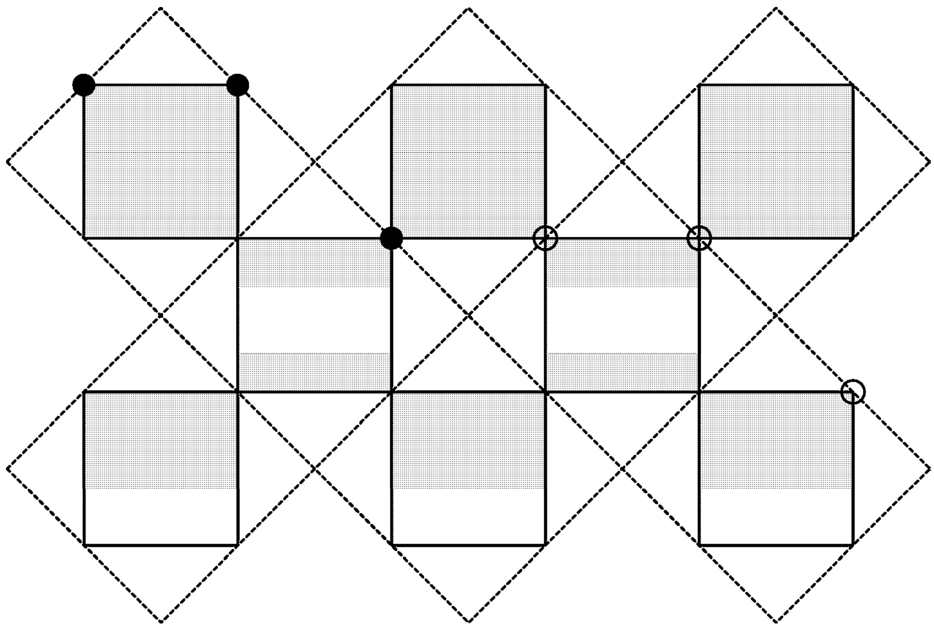
$$\boldsymbol{u} = \boldsymbol{y} + \frac{|\boldsymbol{x}|^2 - |\boldsymbol{y}|^2}{|\boldsymbol{x}+\boldsymbol{y}|^2}(\boldsymbol{x}+\boldsymbol{y}), \qquad \boldsymbol{v} = \boldsymbol{x} - \frac{|\boldsymbol{x}|^2 - |\boldsymbol{y}|^2}{|\boldsymbol{x}+\boldsymbol{y}|^2}(\boldsymbol{x}+\boldsymbol{y})\,.$$

and it is verified that

$$|\boldsymbol{u}|^2 = |\boldsymbol{x}|^2\,, \qquad |\boldsymbol{v}|^2 = |\boldsymbol{y}|^2\,.$$

The map $R : (\boldsymbol{x},\boldsymbol{y}) \mapsto (\boldsymbol{u},\boldsymbol{v})$ is a reversible YB map. Up to a permutation this map was first considered by Adler 1995 in the geometric problem of integrable deformations of a polygon.

Reduction to discrete Painlevé equations



With periodic boundary conditions and letting the edge parametres to vary consistently we get a d-P_1 equation

$$w_{n+1} + w_n + w_{n-1} = \frac{z_n}{w_n} + c$$

where $z_n = \alpha n + \beta + \gamma(-1)^n$.