# Asymptotics of the Painlevé Equations II

Nalini Joshi

School of Mathematics and Statistics
The University of Sydney

12 September 2006

# Outline

The Story So Far
- $P_I$

Hierarchies
- PDE Hierarchies
- Two $P_{II}$ Hierarchies
- Solutions of $P_{II}$ and its Hierarchies

Discrete Painlevé Equations
- $dP_I$

Conclusion

# Outline

The Story So Far

$P_I$

Hierarchies

PDE Hierarchies

Two $P_{II}$ Hierarchies

Solutions of $P_{II}$ and its Hierarchies

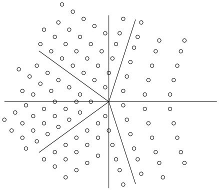
Discrete Painlevé Equations

$dP_I$

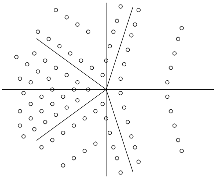
Conclusion

# $P_I$ Solutions

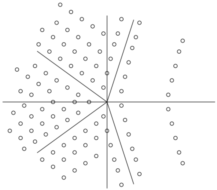
- Generic solutions:



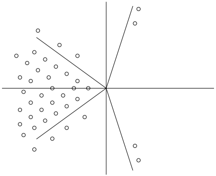
- Approaching a tritronquée soln:



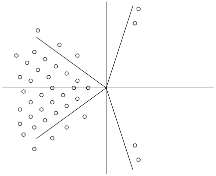
- Polefree or *tronquée* solution:



- Limit to a tritronquée solution:



# $P_I$ Tritronquée Solution



### Theorem

*There exists a unique solution $Y(x)$ of $P_I$ which has asymptotic expansion*

$$y_f(x) = -\sqrt{\frac{x}{6}}\sum_{k=0}^{\infty}\frac{a_k}{(x^{1/2})^{5k}}$$

*in the sector* $|\arg(x)| \leq 4\pi/5$.

- $Y(x)$ is real for real $x$ in its maximal interval of existence.
- Its interval of existence contains the positive semi-axis.
- $Y(x)$ lies below the parabola $\Pi_- := \{(x,y) :| \; x > 0, y = -\sqrt{x/6}\}$.
- $Y(x)$ is monotonically decaying in its interval of existence.
- Its first pole to the left is located at $x_p = -2.3841687\ldots$.
- Such results remain to be proved for most Painlevé equations.

# The Painlevé Equations

$$
\begin{aligned}
y'' &= 6y^2 + x \\
y'' &= 2y^3 + xy + \alpha \\
y'' &= \frac{y'^2}{y} - \frac{y}{x} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
y'' &= \frac{y'^2}{2y} + \frac{3y^3}{2} + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y} \\
y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2 y}(\alpha y^2 + \beta) + \frac{\gamma y}{x} \quad + \frac{\delta y(y+1)}{y-1} \\
y'' &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' \\
&\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right)
\end{aligned}
$$

# Summary

- Interesting solutions of the Painlevé equations have subtle asymptotics.
  - Generic solutions have elliptic-function-type behaviours
  - One-parameter *tronquée* solutions exist that have no poles in two adjacent sectors near $\infty$
  - A discrete set of unique *tritronquée* solutions exist with no poles in four adjacent sectors near $\infty$
- These solutions can be continued to finite regions. However, finite properties of the Painlevé equations have not received as much attention. They deserve as much scrutiny as special functions we all know.
- There still remain many, many open problems. E.g., while we can describe real solutions of the first Painlevé equation completely for $x \leq 0$ there still remain open problems for $x > 0$.

# Outline

The Story So Far
- $P_I$

Hierarchies
- PDE Hierarchies
- Two $P_{II}$ Hierarchies
- Solutions of $P_{II}$ and its Hierarchies

Discrete Painlevé Equations
- $dP_I$

Conclusion

# KdV Hierarchy

- The KdV hierarchy is given by

$$U_{t_{2n+1}} + \partial_x \mathcal{L}_{n+1}[U] = 0,$$

$n = 0, 1, 2, \ldots$, where[1] $\partial_x \mathcal{L}_{n+1} = \left(\partial_x^3 + 4U\partial_x + 2U_x\right) \mathcal{L}_n$ with

$$\begin{aligned}
\mathcal{L}_0[U] &= 1/2 \\
\mathcal{L}_1[U] &= U && \Rightarrow \; U_{t_1} + U_x = 0 \\
\mathcal{L}_2[U] &= U_{2x} + 3U^2 && \Rightarrow \; U_{t_3} + \left(U_{xx} + 3\,U^2\right)_x = 0 \\
\mathcal{L}_3[U] &= U_{4x} + 10UU_{2x} + 5U_x^2 + 10U^3 \\
&\vdots && \Rightarrow \; U_{t_5} + \left(U_{4x} + 10UU_{2x} + 5U_x^2 + 10U^3\right)_x = 0,
\end{aligned}$$

- These form a completely integrable Hamiltonian system, with an infinite set of conserved, independent Hamiltonians in involution. All solutions are known if those of one are known.

- Linear combinations of earlier equations in the hierarchy can be added to the latest equation without affecting the solutions.

---

[1] *Lenard recursion relation (see Lax, '76)*

# Linear Problems

- Drach (1919!) found the solutions of the stationary KdV hierarchy through the linear problem.
- The KdV hierarchy is the compatibility condition of

$$\phi_{xx} + \Big(\lambda + U\big(x, t_{2n+1}\big)\Big)\phi = 0$$

$$\phi_{t_{2n+1}} = \left[\partial_x\left(\sum_{k=0}^{n} \mathcal{L}_k[U](-4\lambda)^{n-k}\right) + (-4)^n a_0\right]\phi$$

$$-2\sum_{k=0}^{n} \mathcal{L}_k[U](-4\lambda)^{n-k} \cdot \phi_x$$

# The MKdV Hierarchy

The Miura map $U = W_x - W^2$ transforms the KdV to

$$(\partial_x - 2W)(W_t - 6W^2 W_x + W_{xxx}) = 0$$

Also

$$\begin{aligned}
\partial_x \mathcal{L}_{n+1} &= (\partial_x - 2W)\partial_x (\partial_x + 2W)\, \mathcal{L}_n[W_x - W^2] \\
U_{t_{2n+1}} &= (\partial_x - 2W)\partial_{t_{2n+1}} W
\end{aligned}$$

$\Rightarrow$ the KdV hierarchy becomes

$$(\partial_x - 2W)\left\{\partial_{t_{2n+1}} W + (\partial_{xx} + 2\partial_x W)\mathcal{L}_n[W_x - W^2]\right\} = 0$$

$\Rightarrow$ So the MKdV hierarchy is

$$W_{t_{2n+1}} + \partial_x (\partial_x + 2W)\, \mathcal{L}_n[W_x - W^2] = 0$$

$\star$ All integrable PDEs have hierarchies.

$\star$ Are there ODE hierarchies?

# Outline

# From MKdV to $P_{II}$

The MKdV equation $W_{t_3} - 6W^2 W_x + W_{xxx} = 0$

under

$$W(x, t_3) = \frac{V(z)}{(3t_3)^{\frac{1}{3}}}, \quad z = \frac{x}{(3t_3)^{\frac{1}{3}}}$$

reduces to

$$\underbrace{-\frac{V}{(3t_3)^{\frac{4}{3}}} - \frac{x}{(3t_3)^{\frac{5}{3}}} V'}_{W_{t_3}} - \underbrace{6\frac{V^2}{(3t_3)^{\frac{2}{3}}} \cdot \frac{V'}{(3t_3)^{\frac{2}{3}}}}_{6W^2 W_x} + \underbrace{\frac{V'''}{(3t_3)^{\frac{4}{3}}}}_{W_{xxx}} = 0$$

$\Rightarrow$

$$V''' = 6V^2 V' + zV' + V \quad \Rightarrow \quad P_{II}: \; V'' = 2V^3 + zV + \alpha_1$$

# Reduction of the MKdV Hierarchy

For the MKdV hierarchy

$$W(x,t_{2n+1}) = \frac{V(z)}{\Big((2n+1)t_{2n+1}\Big)^{\frac{1}{2n+1}}}, \quad z = \frac{x}{\Big((2n+1)t_{2n+1}\Big)^{\frac{1}{2n+1}}}$$

$\Rightarrow$

$$W_x - W^2 = \frac{V' - V^2}{\Big((2n+1)t_{2n+1}\Big)^{\frac{2}{2n+1}}}$$

$$\vdots$$

$$\mathcal{L}_k[U] = \frac{1}{\Big((2n+1)t\Big)^{\frac{2k}{2n+1}}}\mathcal{L}_k[V' - V^2]$$

$\Rightarrow$

$$\frac{1}{\Big((2n+1)t_{2n+1}\Big)^{\frac{2(n+1)}{2n+1}}}\left(-V - zV' + \frac{d}{dz}\Big(\frac{d}{dz} + 2V\Big)\mathcal{L}_n[V' - V^2]\right) = 0$$

# $\mathrm{P_{II}}$ hierarchy

Integration gives the second Painlevé hierarchy $\mathrm{P}_{\mathrm{II}}^{(n)}$

$$[V, \alpha_n] \equiv \left(\frac{d}{dz} + 2V\right)\mathcal{L}_n[V' - V^2] = zV + \alpha_n$$

Explicitly

$$\begin{aligned}
\mathrm{P}_{\mathrm{II}}^{(1)} &: \quad V'' - 2V^3 - zV - \alpha_1 = 0 \\
\mathrm{P}_{\mathrm{II}}^{(2)} &: \quad V'''' - 10V^2V'' - 10V(V')^2 + 6V^5 - zV - \alpha_2 = 0 \\
\mathrm{P}_{\mathrm{II}}^{(3)} &: \quad V^{(6)} - 14(V'')^2V^{(4)} - 56VV'V''' - 70(V')^2V'' - 42V(V'')^2 \\
&\qquad\quad +70V^4V'' + 140V^3(V')^2 - 20V^7 = zV + \alpha_3 \\
&\;\;\vdots
\end{aligned}$$

- ⋆ Is this the only hierarchy associated with $\mathrm{P_{II}}$? No!
- ⋆ What are the solutions' analytic properties?

# $P_{II}$ hierarchy from a Linear Problem

$P_{II}^{(n)}$ (from MKdV) is the compatibility problem of *(Clarkson, J. & Mazzocco, 2005)*

$$
\begin{aligned}
\frac{\partial \Psi}{\partial z} &= \begin{pmatrix} -i\lambda & V \\ V & i\lambda \end{pmatrix} \Psi \\
\lambda \frac{\partial \Psi}{\partial \lambda} &= \left\{ z \begin{pmatrix} -i\lambda & V \\ V & i\lambda \end{pmatrix} + \begin{pmatrix} \sum_{j=0}^{2n+1} A_j (i\lambda)^j & \sum_{j=0}^{2n} B_j (i\lambda)^j \\ \sum_{j=0}^{2n} C_j (i\lambda)^j & -\sum_{j=0}^{2n+1} A_j (i\lambda)^j \end{pmatrix} \right\} \Psi
\end{aligned}
$$

where

$$
\begin{aligned}
A_{2n+1} &= 4^n, \qquad A_{2k} = 0, \quad \forall\, k = 0, \dots, n, \\
A_{2k+1} &= \frac{4^{k+1}}{2} \left\{ \mathcal{L}_{n-k}\left[V' - V^2\right] - \frac{d}{dz}\left(\frac{d}{dz} + 2V\right) \mathcal{L}_{n-k-1}\left[V' - V^2\right] \right\}, \quad k = 0, \dots, n-1, \\
B_{2k+1} &= \frac{4^{k+1}}{2} \frac{d}{dz}\left(\frac{d}{dz} + 2V\right) \mathcal{L}_{n-k-1}\left[V' - V^2\right], \quad k = 0, \dots, n-1, \\
B_{2k} &= -4^k \left(\frac{d}{dz} + 2V\right) \mathcal{L}_{n-k}\left[V' - V^2\right], \quad k = 1, \dots, n, \ B_0 = -zV - \alpha_n, \\
C_{2k+1} &= -B_{2k+1}, \quad k = 0, \dots, n-1, \ C_{2k} = B_{2k}, \quad k = 0, \dots, n,
\end{aligned}
$$

⋆An extension of the Flaschka-Newell Lax pair.

# Another $P_{II}$ hierarchy

The dispersive water wave hierarchy *(J., Gordoa, Pickering, 2003)* gives rise to

$$\mathcal{R}^n u_x \quad + \quad \frac{g_{n-1}}{4}\begin{pmatrix} \left(xu^2 - (x\,u)_x\right)_x + 4v + 2xv_x \\ 2x\,u_x\,v + 4uv + 2xuv_x + 3v_x + xv_{xx} \end{pmatrix}$$
$$+\frac{g_n}{2}\begin{pmatrix} (xu)_x \\ 2v + xv_x \end{pmatrix} + g_{n+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

where

$$\mathcal{R} = \frac{1}{2}\begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2 \\ 2v + v_x\partial_x^{-1} & u + \partial_x \end{pmatrix}$$

The case $g_{n-1} = 0, g_n = 0 \Rightarrow$ a $P_{II}$ hierarchy.

# Cases

$n = 1$ gives

$$\begin{pmatrix} -u_{xx} + (u^2)_x + 2v_x \\ 2vu_x + (uv)_x + v_{xx} \end{pmatrix} + \begin{pmatrix} g_2 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow u_{xx} = 2u^3 + 4g_2 xu - 2\gamma_1\, u + 2(\delta_1 + g_2)$$

$n = 2$ gives

$$\begin{aligned} u_{xxxx} &= 2\frac{u_{xxx}u_x}{u} + \frac{3}{2}\frac{u_{xx}^2}{u} - 2\frac{u_{xx}u_x^2}{u^2} + 5u^2 u_{xx} - 8(\gamma_2 - g_3 x)\frac{u_{xx}}{u} \\ &\quad + \frac{5}{2}uu_x^2 + 8(\gamma_2 - g_3 x)\frac{u_x^2}{u^2} + 8g_3\frac{u_x}{u} - \frac{5}{2}u^5 - 12cu^3 \\ &\quad + 8(\gamma_2 - g_3 x)u^2 - 4(2c^2 + 6\delta_2 + 3g_3)u + 8\frac{(\gamma_2 - g_3 x)^2}{u} \end{aligned}$$

★This equation was also found by Kitaev (1994).

★We (Gordoa, J. & Pickering, 2006) have just discovered that this hierarchy is a Jimbo-Miwa hierarchy!

# Outline

The Story So Far

$P_I$

Hierarchies

PDE Hierarchies

Two $P_{II}$ Hierarchies

Solutions of $P_{II}$ and its Hierarchies

Discrete Painlevé Equations

$dP_I$

Conclusion

# Asymptotic Behaviours of $P_{II}$

The *Boutroux* transformation $V(z) = \sqrt{z}\, u(\xi)$, $\xi = 2z^{3/2}/3 \Rightarrow$

$$\boxed{u_{\xi\xi} = 2u^3 + u - \frac{u_\xi - 2\alpha_1/3}{\xi} + \frac{1}{9}\frac{u}{\xi^2}}$$

The transformation $V(z) = {\alpha_n}^{1/3}\, v(\zeta)$, $z = {\alpha_n}^{-1/3}\,\zeta \Rightarrow$

$$\boxed{v_{\zeta\zeta} = 2v^3 + 1 + \frac{\zeta\, v}{\alpha_1}}$$

Generic (2-parameter) solutions are given to leading-order by Jacobian elliptic functions. There are also 1-parameter (*tronquée* ) solutions and 0-parameter (*tritronquée*) solutions asymptotic to algebraic power series.

★ Do such behaviours exist for each member of $P_{II}^{(n)}$?

★ What about the more generic behaviours?

# Tronquée and Tri-tronquée Solns of the $P_{II}$ Hierarchy

## Theorem

*For each integer $n \geq 1$, $\exists\, x_0$, $|x_0| > 1$, s.t. $P_{II}^{(n)}$ (from MKdV) has solutions asymptotic to*

$$V_{f,\infty} = \left(\frac{(-1)^n x}{2c_n}\right)^{\frac{1}{2n}} \sum_{k=0}^{\infty} a_k^{(n)} \left(\frac{2n}{2n+1} x^{\frac{2n+1}{2n}}\right)^{-k}, \; a_0^{(n)} = 1 \qquad (1)$$

$$V_{f,0} = \left(\frac{-2\,\alpha_n}{(2n+1)\,x}\right) \sum_{k=0}^{\infty} b_k^{(n)} \left(\frac{2n}{2n+1} x^{\frac{2n+1}{2n}}\right)^{-k}, \; b_0^{(n)} = 1 \qquad (2)$$

*where*

$$c_n := \frac{2^{2n-1}\Gamma(n+1/2)}{\Gamma(n+1)\Gamma(1/2)}$$

*in the sectors*

$$\mathcal{S}_n \;=\; \left\{x \in \mathbb{C} \;\middle|\; |x| > |x_0|, |\arg(x - x_0)| < n\pi/(2n+1)\right\}$$

*(J. & Mazzocco, 2003)* There are *unique* solutions in wider sectors.

# The Large-Parameter Limit of the $\mathrm{P_{II}}$ Hierarchy

Under $V(z) = {\alpha_n}^{1/(2n+1)}\, v(\zeta)$, $z = {\alpha_n}^{-1/(2n+1)}\,\zeta$, $\mathrm{P}_{\mathrm{II}}^{(n)}$ becomes

$$\left(\frac{d}{d\zeta} + 2v\right)\mathcal{L}_n\left\{v_\zeta - v^2\right\} = 1 + \frac{\zeta v}{\alpha_n}$$

$\Rightarrow$ as $\alpha_n \to \infty$

$$\left(\frac{d}{d\zeta} + 2v\right)\mathcal{L}_n\left\{v_\zeta - v^2\right\} = 1$$

a first integral of the stationary version of the MKdV hierarchy.

$\star$ We will use an invertible mapping between $\mathrm{P}_{\mathrm{II}}^{(n)}$ and another hierarchy called $\mathrm{P}_{34}^{(n)}$ to find solutions of this leading order equation.

# $\mathrm{P}_{\mathrm{II}}^{(n)} \Leftrightarrow \mathrm{P}_{34}^{(n)}$

Suppose integer $n \geq 1$ is given. If $\mathcal{L}_n\{U\} - z/2 \neq 0$, then

$$U = V_z - V^2 \tag{3a}$$

$$V = -\frac{1}{2\,\mathcal{L}_n\{U\} - z}\left(\frac{d}{dz}\left(\mathcal{L}_n\{U\}\right) - \alpha_n\right) \tag{3b}$$

maps between the solutions $V(z)$ of $\mathrm{P}_{\mathrm{II}}^{(n)}$ and solutions $U(z)$ of

$$\mathrm{P}_{34}^{(n)}: \quad (2\,\mathcal{L}_n\{U\} - z)\,\frac{d^2}{dz^2}\left(\mathcal{L}_n\{U\}\right) - \left(\frac{d}{dz}\mathcal{L}_n\{U\}\right)^2 + \frac{d}{dz}\mathcal{L}_n\{U\} + (2\,\mathcal{L}_n\{U\} - z)^2\; U - \alpha_n\,(1 - \alpha_n) = 0$$

$$\Rightarrow \quad \frac{d}{dz}\mathcal{L}_{n+1}\{U\} = 2\,U + z\,U_z$$

*(Clarkson, J. & Pickering, 1999)*

★ This is a similarity reduction of the KdV hierarchy obtained via $U(x, t_{2n+1}) = U(z)/[(2n+1)t_{2n+1}]^{2/(2n+1)}$, $z = x/[(2n+1)t_{2n+1}]^{1/(2n+1)}$.

# Large Parameter Limit Cont'd

Under $V(z) = {\alpha_n}^{1/(2n+1)}\, v(\zeta)$, $z = {\alpha_n}^{-1/(2n+1)}\, \zeta$, the mapping between $\mathrm{P}_{\mathrm{II}}^{(n)}$ and $\mathrm{P}_{34}^{(n)}$ becomes

$$
\begin{aligned}
u &= v_\zeta - v^2 \\
v &= -\frac{1}{2\,\mathcal{L}_n\{u\} - \zeta/\alpha_n}\left(\frac{d}{dz}\left(\mathcal{L}_n\{u\}\right) - 1\right)
\end{aligned}
$$

Moreover, the differentiated $\mathrm{P}_{34}^{(n)}$ hierarchy becomes

$$
\frac{d}{d\zeta}\mathcal{L}_{n+1}\{u\} = \frac{1}{\alpha_n}\left(2\,u + \zeta\, u_\zeta\right) \;\Rightarrow\; \frac{d}{d\zeta}\mathcal{L}_{n+1}\{u\} = 0,\; \alpha_n \to \infty
$$

Any solution $u(\zeta)$ of this stationary KdV hierarchy yields a solution $v(\zeta)$ of $\mathrm{P}_{\mathrm{II}}^{(n)}$ as $\alpha_n \to \infty$ through

$$
v = -\frac{1}{2\,\mathcal{L}_n\{u\}}\left(\frac{d}{dz}\left(\mathcal{L}_n\{u\}\right) - 1\right).
$$

# Drach's Approach

Consider two independent solutions $\eta_1$, $\eta_2$ of

$$\eta_{xx} + (\lambda + U(x,\tau))\eta = 0$$

which forms the first half of the KdV's linear problem. The Wronskian $\omega := {\eta_1}'\,\eta_2 - \eta_1\,{\eta_2}'$ is constant in $x$. The product $R(x,\lambda) := \eta_1\,\eta_2$ satisfies

$$\begin{aligned} R'' &= -2(\lambda + U(x,\tau))R + 2{\eta_1}'\,{\eta_2}' \\ &= -2(\lambda + U(x,\tau))R + \frac{1}{2\,R}\left(({\eta_1}'\,\eta_2 + \eta_1\,{\eta_2}')^2 - ({\eta_1}'\,\eta_2 - \eta_1\,{\eta_2}')^2\right) \\ &= -2(\lambda + U(x,\tau))R + \frac{1}{2\,R}\left({R'}^2 - \omega^2\right) \end{aligned}$$

where primes denote differentiation in $x$. I.e., $R$ satisfies

$$2RR'' + 4(\lambda + U(x,\tau))R^2 - {R'}^2 + \omega^2 = 0.$$

The solutions of the KdV that we seek correspond to products $R$ that are polynomial in $\lambda$. The $R$-equation above is called the *Bank-Laine* equation in complex differential equations theory.

# Drach's Approach Continued

We write

$$R(x;\lambda) = \prod_{k=1}^{n} \big(\gamma_k(x) - \lambda\big), \qquad \Omega(\lambda) := \omega^2 = -4 \prod_{i=1}^{2n+1} \big(\lambda - \lambda_i\big),$$

Evaluating $2RR'' + 4(\lambda + U(x,\tau))R^2 - {R'}^2 + \omega^2 = 0$ at each zero of $R$ $\Rightarrow$

$$\gamma_j{}'(x)^2 \prod_{k=1,k\neq j}^{n} \big(\gamma_k(x) - \gamma_j(x)\big)^2 = -4 \prod_{i=1}^{2n+1} \big(\gamma_j(x) - \lambda_i\big),\ 1 \leq j \leq n.$$

Also, the coefficients of $\lambda^{2n}$ give

$$U(x,\tau) = 2\sum_{k=1}^{n} \gamma_k(x,\tau) - \sum_{i=1}^{2n+1} \lambda_i.$$

# Case $n = 2$

The zeroes $\gamma_j$ of $R$ satisfy

$$\gamma_1'(x)^2 = \frac{\Omega(\gamma_1(x))}{(\gamma_1(x) - \gamma_2(x))^2}, \qquad \gamma_2'(x)^2 = \frac{\Omega(\gamma_2(x))}{(\gamma_1(x) - \gamma_2(x))^2}$$

where $\Omega(\gamma_k(x)) = -4 \prod_{j=1}^{5}(\lambda_j - \gamma_k(x))$. Therefore

$$\int^{\gamma_1} \frac{ds}{\sqrt{\Omega(s)}} + \int^{\gamma_2} \frac{ds}{\sqrt{\Omega(s)}} = c_0, \qquad \int^{\gamma_1} \frac{s\,ds}{\sqrt{\Omega(s)}} + \int^{\gamma_2} \frac{s\,ds}{\sqrt{\Omega(s)}} = x + c_1$$

where $c_0$ and $c_1$ are arbitrary constants. The solution

$$U(x) = 2\left(\gamma_1(x) + \gamma_2(x)\right) - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)$$

is meromorphic. Note that differentiation gives

$$\frac{\gamma_1'(x)}{\sqrt{\Omega(\gamma_1(x))}} + \frac{\gamma_2'(x)}{\sqrt{\Omega(\gamma_2(x))}} = 0, \quad \frac{\gamma_1(x)\,\gamma_1'(x)}{\sqrt{\Omega(\gamma_1(x))}} + \frac{\gamma_2(x)\,\gamma_2'(x)}{\sqrt{\Omega(\gamma_2(x))}} = 1 \;\Rightarrow\; \frac{(\gamma_1(x) - \gamma_2(x))\,\gamma_k'(x)}{\sqrt{\Omega(\gamma_k(x))}} = 1$$

# Drach's Approach for the Jimbo-Miwa $P_{II}$ Hierarchy

For any $2 \times 2$ Lax pair

$$\begin{aligned} \frac{\partial \eta}{\partial \lambda} &= A\,\eta, \quad \frac{\partial \eta}{\partial x} = B\,\eta \\ \eta_1(x,\lambda) &= \sqrt{B_{12}}\,q(x,\lambda) \;\Rightarrow\; q_{xx} + g(x,\lambda)q = 0 \end{aligned}$$

Consider two independent solutions $q$, $\widetilde{q}$ of the latter equation. Define the wronskian $\omega = q\,\widetilde{q}' - q'\widetilde{q}$. Then the product $P(x,\lambda) = q\,\widetilde{q}$ satisfies

$$P'' = -2g(x,\lambda)P + \frac{1}{2\,P}\left(P'^2 - \omega^2\right)$$

where primes denote differentiation in $x$.

# Case $n = 1$

We have

$$q_{xx} = \frac{1}{4}\left(3u(x)^2 + 2\lambda u(x) + \lambda^2 + x\right) q \tag{4a}$$

$$q_\lambda = -2\big(u(x) - \lambda\big) q_x + u'(x) q \tag{4b}$$

For large $\alpha$, take

$$u(x) = \alpha^{1/3} v(z), \quad z = \alpha^{1/3} x, \quad \lambda = \alpha^{1/3}\mu \;\Rightarrow\; \omega = \alpha^{1/3}\tilde{\omega}$$

which gives the $P$-equations

$$P\,P_{zz} = \frac{1}{2}P_z^2 + \frac{1}{2}\left(3v^2 + 2\mu v + \mu^2 + \frac{z}{\alpha}\right)P^2 - \frac{1}{2}\tilde{\omega}^2, \quad v_z\,P - (v-\mu)\,P_z = \frac{P_\mu}{2\alpha}$$

$\Rightarrow\; P \sim c\,(v(z) - \mu)$, as $\alpha \to \infty$. Writing $\tilde{\omega}^2 = 2c\prod_{j=1}^{4}(\mu - \mu_j)$ we get

$$\sum_{j=1}^{4}\mu_j = 0, \; \sum_{\substack{i,j=1\\ i\neq j}}^{4}\mu_i\mu_j = 0, \; \sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4}\mu_i\mu_j\mu_k = -2, \quad {v_z}^2 = v^4 + z\,v^2 + 2\,v + \mu_1\mu_2\mu_3\mu_4$$

# Drach's Approach, Continued

For $n = 1$, we get Jacobi elliptic functions directly, as opposed to a Miura transformation of Weierstrass $\wp$ functions.

For $n \geq 2$, the solutions of $\mathrm{P}_{\mathrm{II}}^{(n)}$ in the limit $\alpha \to \infty$ correspond to $P$ being higher-degree polynomials in $\lambda$, which give hyperelliptic functions.

So this approach also works for the Jimbo-Miwa $\mathrm{P}_{\mathrm{II}}$ hierarchy. But to know whether this defines different transcendental functions, we need to work out how their asymptotic behaviours in terms of hyperelliptic functions differ. We need a Jacobi-based hyperelliptic function theory.

# Outline

# dP1: $y_{n+1} + y_n + y_{n-1} = (\alpha\, n + \beta)/y_n + \gamma$

- Boutroux transformation: $y_n = \sqrt{n}\, u_n \Rightarrow$

$$u_{n+1} + u_n + u_{n-1} = \frac{\alpha}{u_n} + \frac{\gamma}{\sqrt{n}} + \mathcal{O}(1/n)$$

- Using $u_{n+1} = \overline{u}$, $u_n = u$, $u_{n-1} = \underline{u}$, we get to leading-order

$$\overline{u} + u + \underline{u} = \frac{\alpha}{u}$$

which can be "integrated" by using the integrating factor $u(\overline{u} - \underline{u})$:

$$\overline{u}^2\, u + \overline{u}\, u^2 - \alpha(\overline{u} + u) = E$$

This is solved by Jacobi elliptic functions through a special case of their addition formula.

# dP1 continued

- For the full equation, $E$ is not constant but varies slowly with $n$. Along a chain of points $\Phi$ where $u$ takes on the same value, and $\omega$ denotes a period of the elliptic function,

$$E(\Phi + \omega) - E(\Phi) \sim -\frac{1}{2\Phi}\tilde{\omega},$$

  where $\tilde{\omega}$ denotes a discrete analogue of an elliptic integral of the first kind, just like the continuous case.

- However, no asymptotic results have been found for q-Painlevé equations, such as

$$\overline{y}\,\underline{y} = \alpha q^n + \frac{1}{y}.$$

# Open Problems

- Very few analytic results are known for solutions of ODE hierarchies. We have deduced some behaviours that extend *tritronquée* solutions and elliptic-function-type behaviours. *Are known asymptotic behaviours complete?*
- An important open problem is to show the existence and uniqueness of bounded solutions on the real line. *No results for members of most hierarchies of order* $> 2$.
- Miwa's theorem asserts the Painlevé property of hierarchies. *Is there a direct proof? What complex growth or order do the solutions have?*
- For most discrete Painlevé equations, asymptotic problems remain wide open.

⋆ www.maths.usyd.edu.au/u/nalini/papers/