non trivial

Polynomial $F(Y, Y') \in K[Y, Y']$ satisfying (*) is called an invariant divisor of the differential operator $\mathcal{D}$.

So the above result shows, or we have to show that for every diff. field ext. $K \supset \mathbb{C}(x)$ there is no non-trivial invariant divisor for $\mathcal{D}$.

We define weight on $K[Y, Y']$

$$w(Y) = 2, \quad w(Y') = 3, \quad w(K) = 0.$$

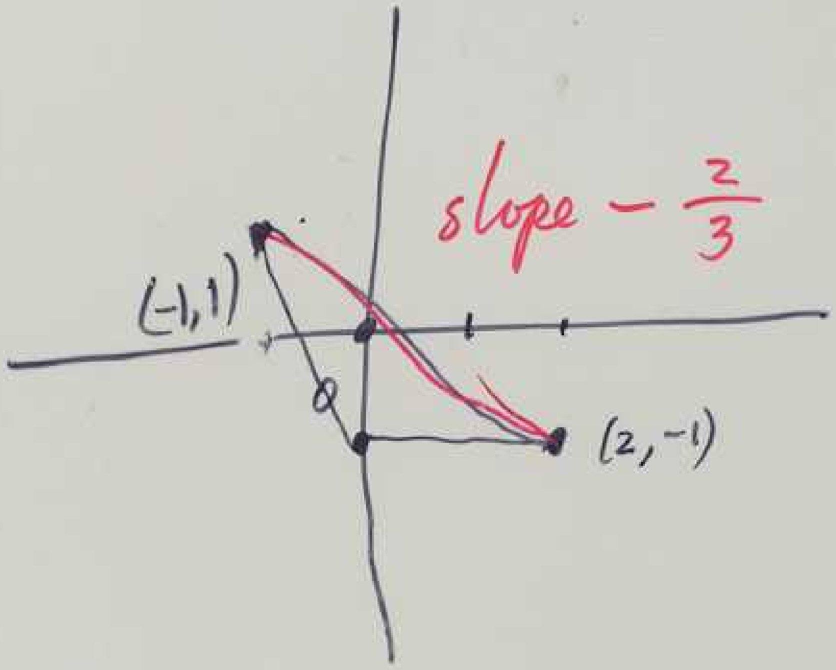
Weight decomposition of the diff. operator

$$\mathcal{D} = \underset{0}{\delta} + \underbrace{Y' \partial/\partial Y + 6Y^2 \partial/\partial Y'}_{1} + \underset{-3}{x \, \partial/\partial Y'}$$

$\mathcal{D} F = G F$ Compare homogeneous parts.

Where does the weight come from?

Newton polygon of the diff. op. $\mathcal{D}$



δ	$(0,0)$	$xY^aY'^b \mapsto xY^aY'^b$
$Y'\frac{\partial}{\partial Y}$	$(-1,1)$	$Y^aY'^b \mapsto Y^{a-1}Y'^{b+1}$
$6Y^2\frac{\partial}{\partial Y'}$	$(2,-1)$	
$x\frac{\partial}{\partial Y'}$	$(0,-1)$	

The Lemma shows that $\forall$ sol. $y'' = 6y + x$ is irreducible except for algebraic solutions.

Proposition (Painlevé) P_I has no algebraic solution.

For other Painlevé equations.

Write the Newton polygons of the corresponding operators.

$\Downarrow$

gives you good weight (not nec. unique)

(1) Determine invariant divisors.

(a) Generally no non-trivial inv. divisors.

(b) For a certain value of parameters, $\exists$ non-trivial invariant divisors. In this case determine all the invariant divisors, which determines transcendental classical solutions of the Painlevé equations

Riccati solutions, hypergeometric, Hermite-Weber ...

(2) Determine the algebraic solutions.

Except for (2.) the determination of algebraic sol. for P_{VI}, the programme is achieved.

Remark We can add to the permissible operations

(P6) f is a meromorphic function on a complex domain. If $F(y, y') = 0$, then y is known. Here the coefficients of the polynomial $F(U, V)$ are known functions.

§3 Picard-Vessiot Theory

Galois Theory of LODE.

Let $\mathcal{D}$ be a differential field of char. 0.

Let $K \subset \mathcal{D}$ be a differential subfield

$A \in M_m(K)$

$\det Y \neq 0$

$Y \in GL_m(\mathcal{D}) \qquad Y' = AY, \quad Y = (y_{ij})_{1 \leq i, j \leq m}$

$L := K(y_{ij})$ which is a diff. subfield of $\mathcal{D}$.

Assume $\underline{C_L = C_K}$. Then we say L/K is a Picard-Vessiot extension.

$\overline{C}_K = C_K$

Let us set $G =$ Diff. field $\mathrm{Aut}_K L = \mathrm{Aut}_K L$

Let $\varphi \in \mathrm{Aut}_K L$.

Then $\varphi(Y') = \varphi(AY)$

$\varphi(Y)' = \varphi(A)\varphi(Y)$

$\varphi(Y)' = A\varphi(Y)$

So that $\exists C \in GL(C_L) = GL(C_K)$

$\varphi(Y) = Y C_\varphi$

$$\mathrm{Aut}_K L \to GL(C_K), \quad \varphi \mapsto C_\varphi$$

is an injective morphism of groups

Proof $\varphi, \psi \in \mathrm{Aut}_K L$

$\varphi(Y) = Y C_\varphi, \ \psi(Y) = Y C_\psi.$

$\psi\varphi(Y) = \psi(Y C_\varphi) = \psi(Y)\psi(C_\varphi) = Y C_\psi C_\varphi$

$C_{\psi\varphi} = C_\psi C_\varphi.$

Fact 1. Image of $\mathrm{Aut}_K L$ is a closed algebraic subgroup of $GL(C_K)$.

Let us set $G = \mathrm{Aut}_K L$ that is an algebraic group over C_K. $G_K = G \otimes_{\mathrm{Spec}\, C_K} \mathrm{Spec}\, K$ operates on $X = \mathrm{Spec}\, K[y_{ij}, \frac{1}{\det(y_{ij})}]_{1 \le i, j \le n}$.

Fact 2 $(G_{\bar{K}}, X)$ is a torsor.

Namely

$$X \times_K X \xleftarrow{\sim} G_{\bar{K}} \times X.$$
$$(gx, x) \leftarrow\!\shortmid (g, x)$$

Proposition $K = \{x \in L \mid gx = x \text{ for } \forall g \in G\}$

Theorem Galois Correspondence

There exists a 1:1 correspondence between the elements of the following 2 sets.

(1) Differential intermediate fields of L/K

(2) Closed subgroups of G.

$$\begin{array}{ccc} L & & 1 \\ | & & | \\ M & & H \\ | & & | \\ K & & G \end{array}$$

$M \mapsto G(M) = \{g \in G \mid gx = x \text{ for } \forall x \in M\}$

$H \to F(H) = \{x \in L \mid hx = x \text{ for } \forall h \in H\}$

Moreover for every differential intermediate field M, L/M is a Picard-Vessiot extension with Galois group

$$G(M) = \{ g \in G \mid gx = x \text{ for all } x \in M \}.$$

(*) If H is a closed normal subgroup of G defined over C, then $F(H)/K$ is a Picard-Vessiot extension with Galois group

$$G/H.$$

$$1 \to H \to G \to G/H = \mathrm{Gal}(F(H)/K) \to 1.$$

Conversely for a differential intermediate field

$$K \subset M \subset L$$

such that M/K is Picard-Vessiot extension, the corresponding group $G(M)$ is a normal closed subgroup of G and we have an isomorphism

$$G/G(M) \xrightarrow{\sim} \mathrm{Gal}(M/K).$$

Theorem follows from
the preceding Proposition by standard argument in algebraic geometry.

(*) is the most involved.

Examples (1) $L = K(y)$ with $y' = ay$, $a \in K$.

Then $\mathrm{Gal}(L/K)$ is a closed subgroup of $\mathbb{G}_m(C) = \{z \in C \mid z \neq 0\}$

$$\mathbb{G}_m(C) = GL_1(C)$$

$$\mathrm{Gal}(L/K) \simeq \begin{cases} \mathbb{G}_m, \\ \mathbb{Z}/\mathbb{Z}_\ell \simeq \{z \in C^* \mid z^\ell = 1\}, \end{cases}$$

$\ell = 1, 2, \dots$.

The first case occurs if and only if y is transcendental over K. In this case

$$\begin{array}{ccc} K(y) & & 1 \\ | & & | \\ K(y^\ell) & \leftrightarrow & \{z \in C^* \mid z^\ell = 1\} \\ | & & | \\ K & & \mathbb{G}_m = C^* \end{array}$$

(2) $G = \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \subset GL_2(C) \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(C),\ a = d = 1,\ c = 0\right\}$

$$\begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} \in G, \quad \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & z_1 + z_2 \\ 0 & 1 \end{bmatrix}$$

The additive group of $\mathbb{C} = \mathbb{G}_a$

$\mathbb{G}_a \simeq G$

$L = K(y)$ such that $0 \neq \exists a \in K$

$(*) \quad y' = a$

$\underline{C_K = C_L}$

We may assume $y \notin K$.

$(*)$ is not a homogeneous linear equation!

We consider instead

$(**) \quad a y'' - a' y' = 0$

or

$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & \frac{a'}{a} \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$$

$$\begin{bmatrix} 1 & y \\ 0 & y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & \frac{a'}{a} \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & y' \end{bmatrix}$$

$$\begin{vmatrix} 1 & y \\ 0 & y' \end{vmatrix} = y' \neq 0 \qquad C_L = C_K, \quad y \notin K.$$

$$\eta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} y \\ y' \end{bmatrix}$$

$K(y) = K(\eta_1, \eta_2)$.

$C_L = C_K$

So $K(y)/K$ is a Picard-Vessiot extension.

Let $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL_2(C)$ be an element of $Gal(L/K)$.

$$\begin{bmatrix} 1 & y \\ 0 & y' \end{bmatrix} \mapsto \begin{bmatrix} 1 & y \\ 0 & y' \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$ gives an element of $Gal(L/K)$

$$\begin{bmatrix} 1 & \varphi(y) \\ 0 & \varphi(y') \end{bmatrix} = \begin{bmatrix} \alpha + \gamma y & \beta + y\delta \\ y'\gamma & y'\delta \end{bmatrix}$$

$\varphi(y') = \varphi(a) = a$, $\quad y'\delta = a\delta$

$\gamma = 0, \ \alpha = 1, \ \delta = 1$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$$

$$\varphi(y) = y + \beta$$

$Gal(L/K) \subset G$.

y is transcendetal $/K$.
Otherwise, $Gal(L/K)$ is a finite group
but $G = G_a$ contains no non-trivial finite group.

Now $1 = \text{tr.d.}[L:K] = \dim Gal(L/K) \leq \dim G \leq 1$

So $Gal(L/K) = G$.

(3) Bessel equation

$$y'' + x^{-1} y' + (1 - \nu^2 x^{-2}) y = 0$$

$\nu \in \mathbb{C}$, $\nu \notin \frac{1}{2} + \mathbb{Z}$.

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 + \nu^2 x^{-2} & -x^{-1} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

$K = \mathbb{C}(x)$ y_{ij}'s are holomorphic over a small fixed disc.

$L = K(y_{11}, y_{12}, y_{21}, y_{22})$ so $C_L = C_K$

Lemma $\exists c$ s.t. $\det Y = cx^{-1}$

Proof $(\det Y)^{-1}(\det Y)' = \operatorname{tr} A = -x^{-1}$.

$\det Y$ is a solution of

$$y' + x^{-1}y = 0.$$

x^{-1} also satisfies this equation

$$\det Y = c x^{-1} \quad \text{for a some } c \in \mathbb{C}.$$

$$Gal(L/K) \subset \{ g \in GL_2(\mathbb{C}) \mid |Yg| = cx^{-1} = |Y| \}$$

i.e. $|g| = 1$

$$= SL_2(\mathbb{C})$$

How to show $G = SL_2(\mathbb{C})$.

Sketch

Assume $\dim G \leq 2$

$G^\circ \subset SL_2$ is classified modulo conjugation

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^* \right\}, \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a, b \in \mathbb{C},\ a \neq 0 \right\}$$

$\sum$ residue of $\phi\, dx = 0$

$$-\frac{1}{2} + n \pm \nu = 0 \qquad \text{with } n \in \mathbb{N}$$

$$\nu \in \frac{1}{2} + \mathbb{Z}$$

Contradiction.

Applications of Picard-Vessiot theory

Definition k = diff. field $\overline{C_k} = C_k$.

$K \supset k$ is Liouvillian if

$C_K = C_k$ and

$$\exists\ k = K_0 \subset K_1 \subset \cdots \subset K_m = K$$

s.t. $K_i = K_{i-1}(t_i)$ for $i = 1, \cdots, m$,

where either

1. $t_i' \in K_{i-1}$, 2. $t_i'/t_i \in K_{i-1}$ or
3. t_i is algebraic over K_{i-1}.

Theorem Let $K \supset k$ be a Picard-Vessiot ext. with diff. Galois group G. The following conditions are equivalent.

(1) G^0 is a solvable group.

(2) K/k is a Liouvillian ext.

(3) K is contained in a Liouvillian ext. of k.

Remark Liouvillian ext. is not necessarilly contained in a strongly normal extension.

One of the most important applications of Kolchin theory seems the proof of the irreducibility of the Painlevé equations.

So $\sum\limits_{P\in\mathbb{P}^1} \operatorname{Res} \varphi\, dx = 0$.

$$\sum 1 + \sum \pm u - \frac{1}{2} = 0$$

$$\pm u - \frac{1}{2} \in \mathbb{Z}$$

$$\pm u \in \mathbb{Z} + \frac{1}{2}$$

or $$u \in \mathbb{Z} + \frac{1}{2}.$$

§4 Existence of Picard-Vessiot extension

Given a differential field K and a diff. equation

$$Y' = AY$$

$A \in M_n(K)$.

Question $\exists$? A diff. field ext. L/K satisfying the following conditions

(1) $\exists\, Y \in GL_n(L)$ s.t. $Y' = AY$.

(2) $L = K(Y)$.

(3) $C_L = C_K$.

Yes. there exits such a diff. field extension.
It is unique up to isomorphism.

The extension L/K is called the Picard-Vessiot extension of the diff. equation

$$Y' = AY.$$

§5 Tannakian category

Theory of Tannakian category consists of two parts.

(1) Let C be an algebraically closed field and G a linear algebraic group (affine group scheme) over C.

- $(G \subset GL_n(C)$ subgroup scheme for $\exists n)$.

Let Rep_G denote the category of finite dimensional G-modules (G-modules that are finite dimensional C-vector space). We have a forgetful functor

$$\omega : Rep_G \to Vect_C = \text{cat of finite dimensional } C\text{-vector spaces}$$

Example (1) Let X be a Riemann surface

$$\mathcal{C} = \{(E, \Delta) \mid E = \text{vector bundle},\ \Delta : E \to E \otimes_{\mathcal{O}_X} \Omega^1_X\}$$

connection

$x \in X$ a point of X

fiber functor

$$\omega : \mathcal{C} \to Vect_{\mathbb{C}}$$
$$(E, \Delta) \longmapsto E_x = \text{the fiber of } E \text{ at } x.$$

(2) Let G be a group that has finite dim rep. }

$\mathcal{C}$ = Category of finite dimensional G-modules

$$\omega : \mathcal{C} \to Vect_{\mathbb{C}}$$

Forgetful functor

Theorem. We can recover the group scheme G from the functor ω

$$G \simeq \mathrm{Aut}^{\omega}.$$

Rep_G is an Abelian category, with $\otimes$, for a G-module M, $\exists$ G-module M^* as well as $\mathrm{Vect}_{\mathbb{C}}$.

ω: commute with $\oplus$, $\otimes$, $*$.

(2) What category is equivalent to Rep_G?

Def. Tannakian category $\mathcal{C}$

Abelian category with $\otimes$, M^*, 1 $(1 \otimes_M \simeq M, \cdots)$

$\omega : \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$

nice functor commuting $\oplus$, $\otimes$, taking $*$, ...

Theorem. Let $\mathcal{C}$ be a Tannakian category. Then there exist an affine group scheme G such that

$$\mathcal{C} \simeq \mathrm{Rep}_G.$$

Any how $\exists$ eigenvector $\in \eta = {}^t\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \in \mathbb{C}\,{}^t\begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} + \mathbb{C}\,{}^t\begin{bmatrix} y_{21} \\ y_{22} \end{bmatrix}$

of G^0_g s.t. $\eta_2 \neq 0$. Then η_2'/η_2 is G^0_g inv.

η_2'/η_2 is algebraic $/\mathbb{C}(x)$. Namely η_2'/η_2 is

an algebraic function.

$\phi := \eta_2'/\eta_2$, then ϕ satisfies the Riccati equation

$$y' + y^2 + x^{-1}y + 1 - \nu^2 x^{-2} = 0.$$

Let R be the Riemann surface of ϕ. Conclude $R = \mathbb{P}^1$

R $\quad$ ϕ is a meromorphic function

$\downarrow$ $\quad$ Calculate the residues of 1-form $\phi\, dx$

$\mathbb{P}^1$

(1) At $x = \infty$, $\operatorname{Res} \phi\, dx = -\frac{1}{2}$

(2) $x = c \in \mathbb{C}$, $c \neq 0$

$\phi\, dx$ has a simple pole with residue 1 if it has a pole.

(3) at $x = 0$, $\phi\, dx$ has residue $\pm \nu$.

(3) k = diff. field $\overline{C_k} = C$

Diff. operator $\delta : k \to k$

ring of diff. operators $k[\delta]$

$k[\delta]$ is a non-commutative ring

$\delta a = \delta a + a\delta$ for $a \in k$

Category of finite dimensional $k[\delta]$-modules
(as k-vector space)

Tannakian category
trivial except for the fiber functor
$\omega : C \to Vect_{C_k}$ (Deligne)

(*) $Y' = AY \qquad A \in M_m(K)$

$Y = (y_1, y_2, \cdots, y_m)$

$M = \sum K y_i$ is a differential module or $k[\partial]$-module

Then there exists a C-linear equiv.
of Cat.

$$\{\{M\}\} \longrightarrow Rep_G$$

Here G is the Galois group of (*)
(of the P.-V. ext. that (*) defines)

$\{\{M\}\}$ is the tannakian ${}^{\text{sub}}$category of C
generated by M.