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# Tropical Representation of Weyl Groups Associated with Certain Rational Varieties

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collaborated with

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Ref.) T & Takenawa: arXiv:Math.AG/0607661

## §1 Intro: Symmetry of Painlevé Equations

> Classification of 2nd order algebraic ODE with Painlevé property.

$$
\begin{array}{ccccccc}
 & & & P_{\mathrm{IV}} & & & \\
 & & \nearrow & & \searrow & & \\
P_{\mathrm{VI}} \rightarrow & P_{\mathrm{V}} & & & & P_{\mathrm{II}} & \rightarrow P_{\mathrm{I}} \\
 & & \searrow & & \nearrow & & \\
 & & & P_{\mathrm{III}} & & &
\end{array}
$$

**Ex.)** $P_{\mathrm{IV}}$

$$
\begin{cases}
f_0{}' = f_0(f_1 - f_2) + \alpha_0 \\
f_1{}' = f_1(f_2 - f_0) + \alpha_1 \\
f_2{}' = f_2(f_0 - f_1) + \alpha_2
\end{cases}
$$

$\alpha_i \in \mathbb{C}$: parameters s.t. $\alpha_0 + \alpha_1 + \alpha_2 = 1$.

$f_0 + f_1 + f_2 = t \Rightarrow$ 2nd order.

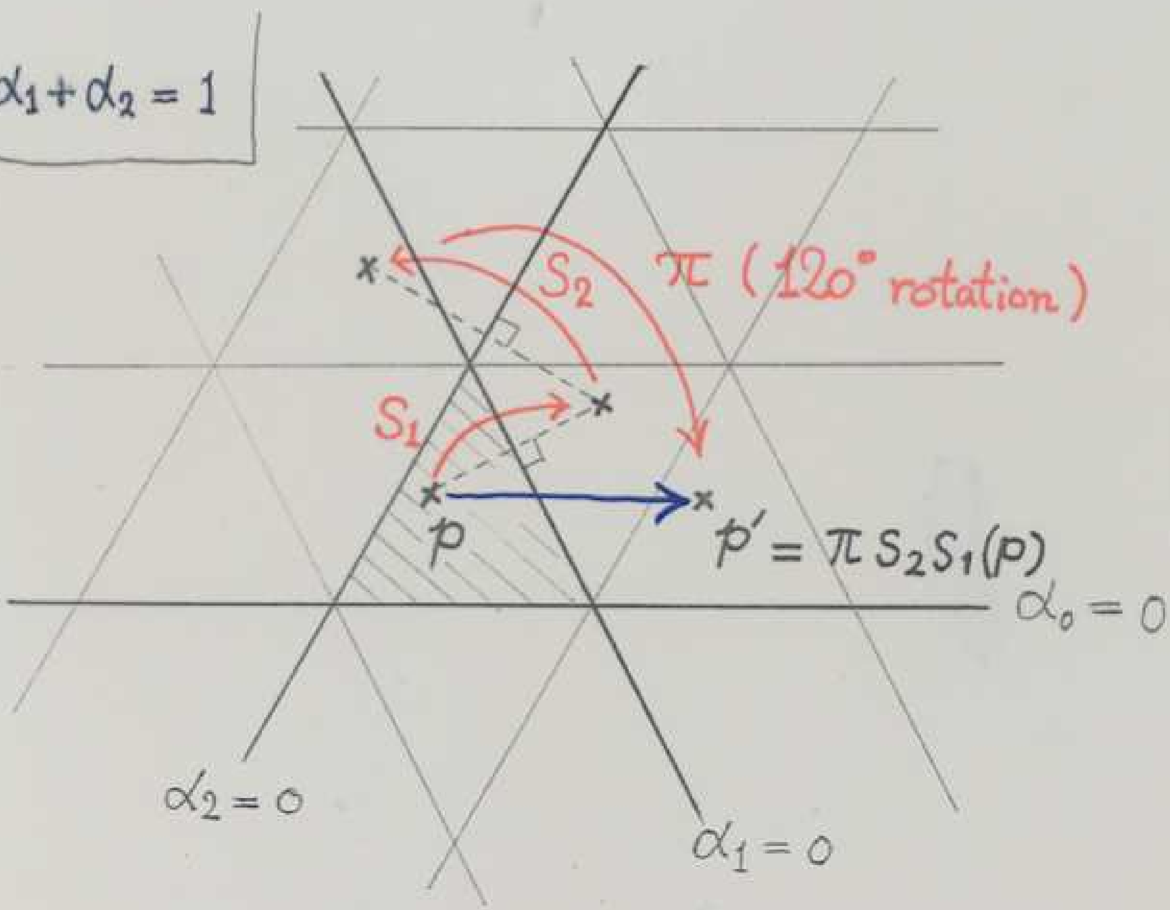
1st integral

## Affine Weyl group symmetry $\langle s_0, s_1, s_2, \pi\rangle \simeq \widetilde{W}(A_2^{(1)})$

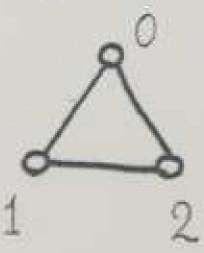
$P_{\mathrm{IV}}$ is invariant under

$$\begin{cases} \pi : f_i \mapsto f_{i+1},\ \alpha_i \mapsto \alpha_{i+1}\ (i \in \mathbb{Z}/3\mathbb{Z}) \\ s_i : f_{i\pm1} \mapsto f_{i\pm1} \pm \dfrac{\alpha_i}{f_i},\ \alpha_i \mapsto -\alpha_i,\ \alpha_{i\pm1} \mapsto \alpha_{i\pm1} + \alpha_i \end{cases}$$

$\alpha_0 + \alpha_1 + \alpha_2 = 1$

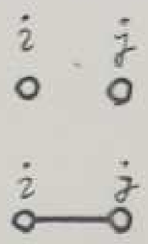



Dynkin diagram




$= A_2^{(1)}$

$S_i^2 = \mathrm{id}$




$S_i S_j = S_j S_i$

$S_i S_j S_i = S_j S_i S_j$

# §2 Discrete Painlevé Equations

Originally, a discrete Painlevé equation is

> $2^{\text{nd}}$ order ordinally difference equation with singularity confinement property that admits a continuous limit to some of Painlevé differential equations $P_{\text{I}}, \ldots, P_{\text{VI}}$.

(Ramani-Grammaticos-Hietarinta, 1991 *PRL* **67**)

Nowadays, more generally,

**Discrete Painlevé** = Lattice part of birational actions of affine Weyl groups

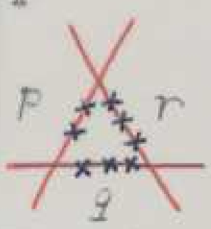
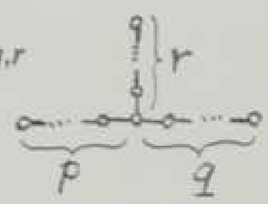
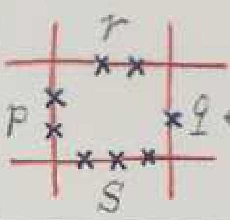
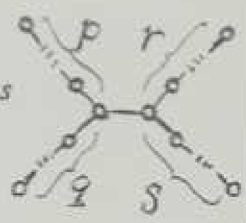
**non-autonomous:** coeff's depend on time variable $n$.

- Rational → difference
- Trigonometric → $q$-difference
- Elliptic → "elliptic-difference"

**Ex.)** d-$P_{\text{II}}$

$$x_{n+1} + x_{n-1} = \frac{(an+b)x_n + c}{1 - {x_n}^2}$$

Rational variety $X \overset{\text{birat}}{\curvearrowleft} W = W(R)$ Weyl group

| | Geometric objects | $R$: root system | Remarks |
|---|---|---|---|
| Sakai ('01 *CMP* **220**) | $\mathbb{P}^2 \longleftarrow X$<br>9 points blow up | $E_8^{(1)}$<br> | • 9 pts in general position<br>⇒ elliptic Painlevé<br><br>• Degeneration of pts set ⇒<br>discrete Painlevé's (19 types) |
| T ('06 *LMP* **77**) | $\mathbb{P}^2$<br><br>$\leftarrow X$ | $T_{p,q,r}$<br> | **Tropical representation**<br>• birational mappings<br>• subtraction-free<br>(~ ultra-discretizable) |
| | $\mathbb{P}^1 \times \mathbb{P}^1$<br><br>$\leftarrow X$ | $H_{p,q,r,s}$<br> | ***q*-difference cases** |
| T & Takenawa | $(\mathbb{P}^1)^N \longleftarrow X$ | $T_\ell^k$<br> | |

2 dim. (Sakai, T ('06 LMP 77))

higher dim. (T & Takenawa)

# §3 Tropical Weyl Group Action on Certain Rational Varieties

Rational variety $X$ $\longrightarrow$ $(\mathbb{P}^1)^N \ni f = (f_1, f_2, \ldots, f_N)$

**blow-up along subvarieties $\{C_n^i, C_n^{-j}\}$ of codim 3**

$$C_n^i = \left\{ f_{n-1} = 0, f_n = -u_n^i, f_{n+1} = \infty \right\}, \quad 1 \le i \le k_n$$

$$C_n^{-j} = \left\{ f_{n-1} = \infty, f_n = -1/v_n^{-j}, f_{n+1} = 0 \right\}, \quad 1 \le j \le \ell_n$$

$u_n^i, v_n^{-j} \in \mathbb{C}^\times$ : parameters

- **Néron-Severi bilattice** $N(X) \simeq (H^2(X, \mathbb{Z}), H_2(X, \mathbb{Z}))$

$$H^2(X, \mathbb{Z}) \simeq \text{linear span of } \{H_n, E_n^i, E_n^{-j}\} \,/\mathbb{Z}$$

$$H_2(X, \mathbb{Z}) \simeq \text{linear span of } \{h_n, e_n^i, e_n^{-j}\} \,/\mathbb{Z}$$

- **Intersection pairing** $\langle\, , \rangle : H^2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}$

$$\langle H_m, h_n \rangle = \delta_{m,n}, \quad \langle E_m^i, e_n^j \rangle = -\delta_{m,n}\delta_{i,j}, \quad \langle \text{otherwise} \rangle = 0$$

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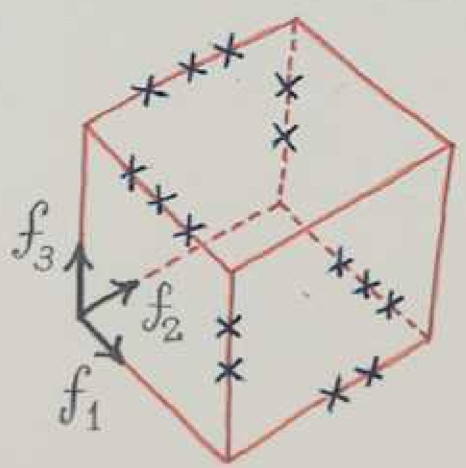
**Ex.)** $N = 3$

$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$



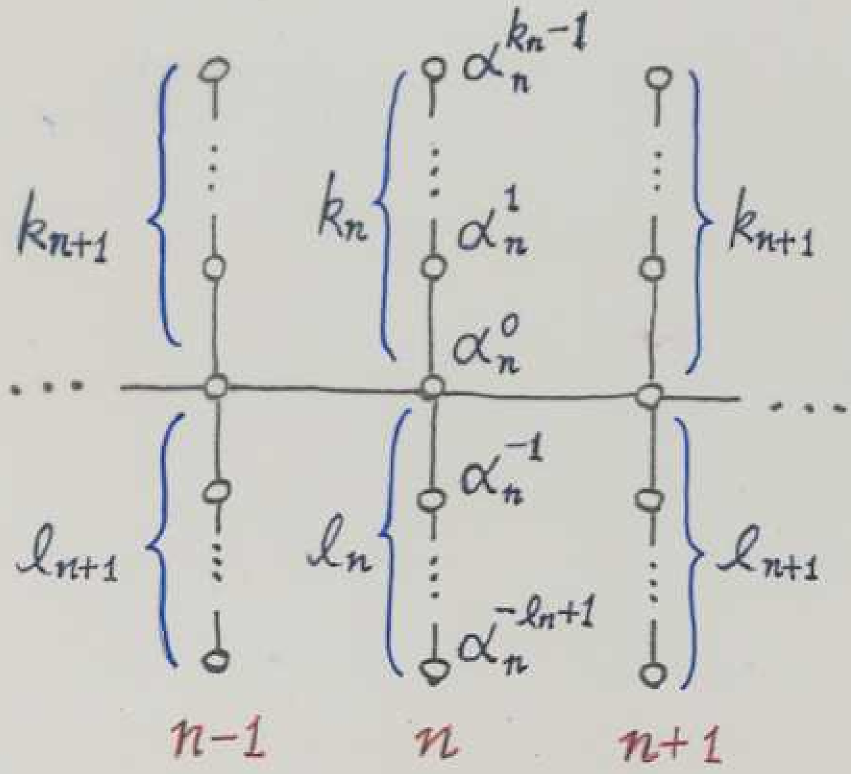
## Root lattice $Q(T^k_\ell) \subset N(X)$

- **Root and coroot bases**

$$\alpha^0_n = H_n - E^1_n - E^{-1}_n, \quad \check{\alpha}^0_n = h_{n-1} + h_{n+1} - e^1_n - e^{-1}_n$$
$$\alpha^i_n = E^i_n - E^{i+1}_n, \quad \check{\alpha}^i_n = e^i_n - e^{i+1}_n \quad (1 \le i \le k_n - 1)$$
$$\alpha^{-j}_n = E^{-j}_n - E^{-j-1}_n, \quad \check{\alpha}^{-j}_n = e^{-j}_n - e^{-j-1}_n \quad (1 \le j \le \ell_n - 1)$$

- **Dynkin diagram $T^k_\ell$**

specified by $\boldsymbol{k} = (k_1, \dots, k_N), \boldsymbol{\ell} = (\ell_1, \dots, \ell_N) \in (\mathbb{Z}_{>0})^N$




**Remark**

$T^k_\ell$ includes all of the simply-laced affine cases $A^{(1)}_n$, $D^{(1)}_n$, $E^{(1)}_n$

## Weyl group $W(T^k_\ell) = \langle s^i_n \rangle$

- **Simple reflections**

$$s^i_n(\Lambda) = \Lambda + \langle \Lambda, \check{\alpha}^i_n \rangle \alpha^i_n, \quad \Lambda \in H^2(X, \mathbb{Z})$$
$$s^i_n(\lambda) = \lambda + \langle \alpha^i_n, \lambda \rangle \check{\alpha}^i_n, \quad \lambda \in H_2(X, \mathbb{Z})$$

**Ex.)** Action of $s^0_n$ on the basis of $H^2(X, \mathbb{Z})$:

$$s^0_n(H_{n\pm1}) = H_{n\pm1} + H_n - E^1_n - E^{-1}_n, \quad s^0_n(E^{\pm1}_n) = H_n - E^{\mp1}_n$$

$$\Downarrow$$

**Thm 1.** The birational transformations $s^i_n$ defined by

$$s^0_n(f_{n-1}) = (a^0_n)^{\ell_{n-1}/(k_{n-1}+\ell_{n-1})} f_{n-1} \frac{f_n + {v_n}^{-1}}{f_n + u_n}$$
$$s^0_n(f_{n+1}) = (a^0_n)^{-k_{n+1}/(k_{n+1}+\ell_{n+1})} f_{n+1} \frac{f_n + u_n}{f_n + {v_n}^{-1}}$$

and $s^i_n(f_m) = f_m$ $(i \neq 0)$ realize $W(T^k_\ell)$. Moreover, $s^i_n$ maps $X_a$ to $X_{s^i_n(a)}$ as a pseudo isomorphism of rational varieties.

## $\tau$-Functions

- $\tau$-variables $\tau_n^i \longleftrightarrow C_n^i$ (center of blow-up)

Put $\boxed{\omega_n = \theta_n^0/(\theta_n^0 + \theta_n^\infty)}$ with $\theta_n^0 = k_{n+1} + \ell_{n-1}$, $\theta_n^\infty = k_{n-1} + \ell_{n+1}$.

**Thm 2 ($\tau$-representation).**
The birational transformations

$$s_n^0(\tau_n^1) = \frac{{v_n}^{\omega_n}\xi_{n+1}\eta_{n-1} + {v_n}^{-1+\omega_n}\xi_{n-1}\eta_{n+1}}{\tau_n^{-1}}$$

$$s_n^0(\tau_n^{-1}) = \frac{{u_n}^{-\omega_n}\xi_{n+1}\eta_{n-1} + {u_n}^{1-\omega_n}\xi_{n-1}\eta_{n+1}}{\tau_n^1}$$

$$s_n^i(\tau_n^{\{i,i+1\}}) = \tau_n^{\{i+1,i\}} \quad (1 \le i \le k_n - 1)$$

$$s_n^{-j}(\tau_n^{\{-j,-j-1\}}) = \tau_n^{\{-j-1,-j\}} \quad (1 \le j \le \ell_n - 1)$$

realize Weyl group $W(T_\ell^k)$. Here

$$\xi_n = \prod_{i=1}^{k_n} \tau_n^i \quad \text{and} \quad \eta_n = \prod_{j=1}^{\ell_n} \tau_n^{-j}.$$

**Remark**

$$f_n = \frac{\xi_{n+1}\eta_{n-1}}{\xi_{n-1}\eta_{n+1}}$$

Thm 2 $\Rightarrow$ Thm 1

## Geometric Background

*cf.*) 27 lines on cubic surface and Weyl group $W(E_6)$.

- $H^2(X,\mathbb{Z}) \supset M = \{\text{Exceptional divisors}\} \curvearrowleft W = W(T^k_\ell)$ (permutations)

$\Lambda = \sum d_n H_n - \sum \mu^i_n E^i_n \in M$ $\overset{1:1}{\longleftrightarrow}$ **Hypersurface of degree** $d = (d_1, \ldots, d_N)$ **passing through** $C^i_n$ **with multiplicity** $\mu^i_n$

**Def.** The $\tau$-function $\tau(\Lambda)$ $(\Lambda \in M)$ is a rational function in $\tau^i_n$ s.t.
(i) $\tau(E^i_n) = \tau^i_n$
(ii) $\tau(w.\Lambda) = w.\tau(\Lambda)$ for $\forall\Lambda \in M, \forall w \in W(T^k_\ell)$

*cf.*) Kajiwara-Masuda-Noumi-Ohta-Yamada '03 *J. Phys. A* **36**

**Thm 3.** $\tau(\Lambda)$ ~ defining polynomial of the corresponding hypersurface

**Remark** Regularity of $\tau(\Lambda) \longleftrightarrow \infty$-integrable systems

In some affine cases:

| | |
|---|---|
| $q$-KP | $A^{(1)}_n$ |
| $q$-UC | $A^{(1)}_{2g+1}$, $D^{(1)}_5$, $E^{(1)}_6$ |

## Summary

**Rational Varieties**

↓

**Weyl Groups**

↓

**(Higher Order) Discrete Painlevé**

Thank you very much for your kind attention!