Symmetry reductions of integrable equations on quad-graphs and discrete Painlevé equations

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Symmetries of Ordinary Difference Equations

Let M be an N-dimensional manifold with smooth (holomorphic) local coordinates $x^i = (x^1, x^2, \cdots, x^N)$.

A map $\Phi : M \to M$

$$x^i \to \Phi^i(x^j) \qquad i, j = 1, 2, ..., N$$

defines a system of ordinary difference equations of first order, on M.

The equations are defined by assigning labels on two successive images and pre-images of the map, which vary by unit steps *only*, i.e.

$$x^i_{n+1} = \Phi^i(x^j_n)$$

- A solution of such an equation, is a sequence of points

$$x_0 \quad x_1 \quad \ldots \quad x_n \quad \ldots$$

where

$$x_n = \Phi \circ \Phi \circ \cdots \circ \Phi(x_0) \qquad n\text{-times}$$

(S. Maeda *Math. Japonica* **25** No.4 (1980), 405-420).

• A smooth (holomorphic) function $f : M \to \mathbb{R}\,(\mathbb{C})$ is said to be an *invariant*, or integral, iff

$$\Phi^* f = f$$

holds $\forall x \in M$, where $\Phi^* f$ is the function defined by

$$(\Phi^* f)(x) = f(\Phi(x)).$$

• A group G which acts on M, is said to be a *symmetry group* of Φ, iff every element g of G transforms any solution of Φ to another solution.

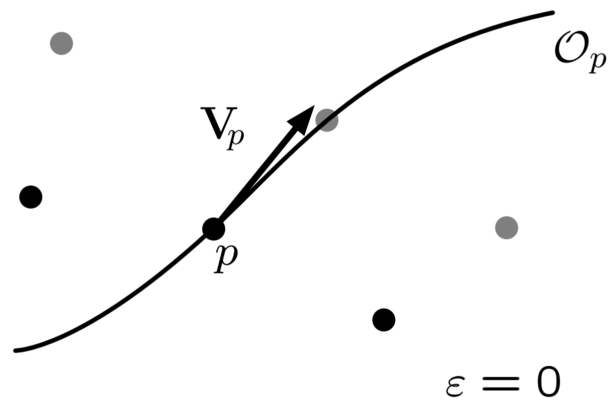
$$\begin{array}{ccccc}
x_0 & \xrightarrow{g} & g(x_0) & & \\
\Phi\big\downarrow & & \big\downarrow\Phi & & \\
x_1 & \xrightarrow{g} & g(\Phi(x_0)) = \Phi(g(x_0)) & & g \circ \Phi = \Phi \circ g \\
\vdots & & \vdots & & \\
x_n & \xrightarrow{g} & g(x_n) & &
\end{array}$$

• This definition implies that the symmetry group acts on the domain of the dependent variables.

One-parameter groups of transformations

Let G be a one-parameter group of transformations acting on the domain of the dependent variables,

$$G : x_n^i \mapsto \Psi^i(n, x_n^j; \varepsilon)\,, \quad \varepsilon \in \mathbb{C}\,.$$



The infinitesimal generator of the group action of G on f is the vector field

$$\mathbf{v} = \sum_{i=1}^{N} Q^i(n, x_n^j)\,\frac{\partial}{\partial x_n^i}\,, \qquad Q^i(n, x) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Psi^i(n, x_n^j; \varepsilon)\Big|_{\varepsilon=0}$$

There is a one-to-one correspondence between connected groups of transformations and their associated infinitesimal generators since the group action is reconstructed by the flow of the vector field $\mathbf{v}$ by exponentiation

$$\Psi^i(n, x_n^j; \varepsilon) = \exp(\varepsilon\,\mathbf{v})x_n^i$$

The prolongation of the group action on the space J with coordinates $(x, x_{n+1} \ldots)$ is specified by

$$\begin{array}{rcl} x_{n+1} & \mapsto & \Psi(n+1, x_{n+1}; \varepsilon) \\ & \vdots & \\ x_{n+k} & \mapsto & \Psi(n+k, x_{n+k}; \varepsilon) \\ & \vdots & \end{array}$$

The prolongation of the infinitesimal action of G on J is generated by the prolonged vector field

$$\widehat{\mathbf{v}} = \sum_{i=1}^{N} Q^i(n, x_n^j)\,\frac{\partial}{\partial x_n^i} + \cdots \sum_{i=1}^{N} Q^i(n+k, x_{n+k}^j)\,\frac{\partial}{\partial x_{n+k}^i} + \cdots$$

Definition: G is a *symmetry* of equation $x_{n+1}^i = \Phi^i(x_n^j)$, if it transforms any solution of $x_{n+1}^i = \Phi^i(x_n^j)$, to another solution of the same equation.

Infinitesimal criterion

For connected groups of transformations, a necessary and sufficient condition for $\mathbf{v}$ to be a symmetry generator is

$$\widehat{\mathbf{v}}(x_{n+1} - \Phi(x_n)) = 0 \qquad \text{mod} \quad x_{n+1} = \Phi(x_n)$$

$$Q^i(n+1, \Phi^k(x_n)) = \sum_{j=1}^{N} Q^j(n, x_n^k)\frac{\partial \Phi^i}{\partial x_n^j} \qquad i, j = 1, 2, ..., N$$

Example A discrete Riccati equation

Let $M = \mathbb{C}\mathrm{P}^1$, and the mapping

$$x_{n+1} = \frac{a x_n + b}{c x_n + d} \qquad ad - bc \neq 0$$

Symmetry condition

$$\xi\left(\frac{a x_n + b}{c x_n + d}\right) = \xi(x) \frac{ad - bc}{(cx + d)^2}$$

Adopting a quadratic *Ansatz* in the arguments of ξ, we find that the Riccati discrete equation admits the symmetry generator $X = \xi(x)\partial_x$ where

$$\xi(x) = cx^2 + (d - a)x - b$$

- Integration algorithm:

Define canonical variable y

$$y = \int \frac{\mathrm{d}\,x}{\xi(x)}$$

$$y = \frac{1}{r_1 - r_2} \log\left(\frac{x - r_1}{x - r_2}\right)$$

The Riccati equation linearizes to

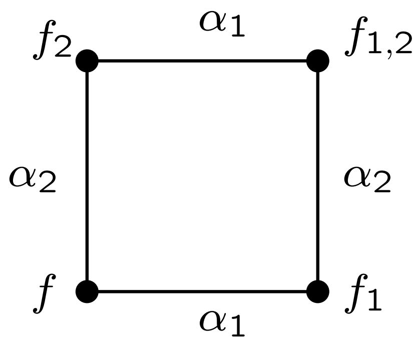
$$y_{n+1} = y_n + A$$

for suitable constant $A = A(a, b, c, d)$.

General solution

$$y = An + C$$

Equations on quad-graphs



$f : \mathbb{Z}^2 \to \mathbb{C}$ (or $\mathbb{CP}^1$), complex fields assigned on the vertices of a square, at sites (n_1, n_2).

α_1, α_2 complex parameters assigned on the edges, being equal on opposite edges.

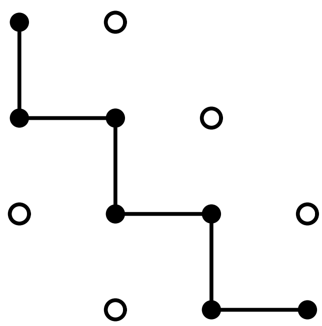
The basic building block of equations on quad-graphs consists of a relation of the form

$$\mathcal{E}(f, f_1, f_2, f_{1,2}; \alpha_1, \alpha_2) = 0\,,$$

between the values of four fields residing on the vertices of each elementary quadrilateral for which we use the shorthand notation:

$$f := f(n_1, n_2), \quad f_1 := f(n_1{+}1, n_2), \quad f_{1,2} := f(n_1{+}1, n_2{+}1)\,\ldots.$$

Solution of an elementary Cauchy problem on a staircase



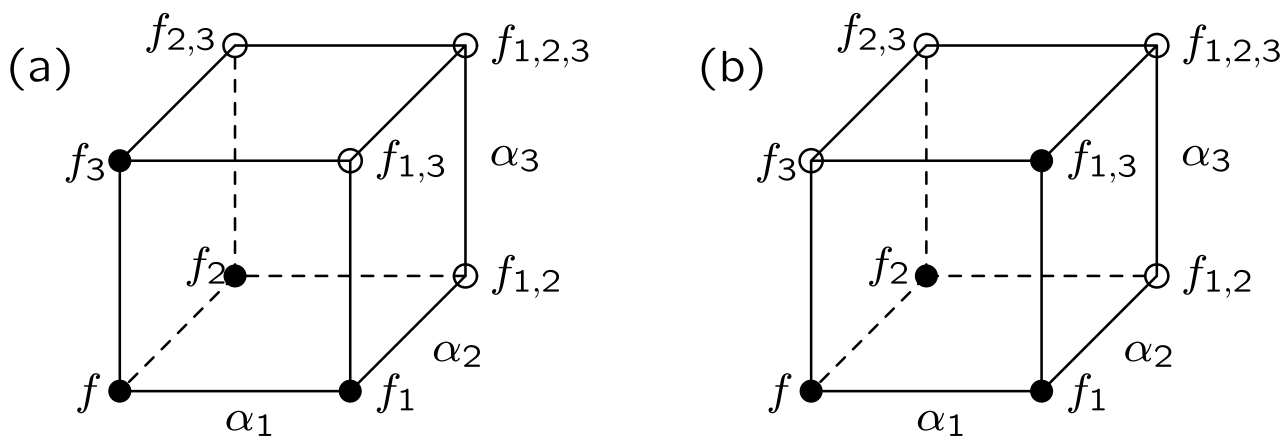
Integrability - The consistency property

The overdetermined system of equations

$$\mathcal{E}(f, f_i, f_j, f_{i,j}; a_i, a_j) = 0\,, \qquad 1 \le i < j \le n\,,$$

can be imposed on all faces of a 3-cube in a consistent way.

The property can be verified by considering an elementary initial value problem. E.g. for $n = 3$



Example The potential discrete KdV

$$(f_{1,2} - f)(f_1 - f_2) = \alpha_1 - \alpha_2\,,$$

with given initial data configuration as in figure (a). For this particular equation one finds that the value $f_{1,2,3}$ is

$$f_{1,2,3} = \frac{(\alpha_1 - \alpha_2) f_1\, f_2 + (\alpha_3 - \alpha_1) f_1\, f_3 + (\alpha_2 - \alpha_3) f_2\, f_3}{(\alpha_2 - \alpha_1) f_3 + (\alpha_1 - \alpha_3) f_2 + (\alpha_3 - \alpha_2) f_1}\,.$$

Integrable - 3D consistent lattice equations

$$\mathcal{E}_1 : \qquad (f_{1,2} - f)(f_1 - f_2) - \alpha_1 + \alpha_2 = 0\,,$$

$$\mathcal{E}_2 : \qquad \alpha_1(f\, f_1 + f_2\, f_{1,2}) - \alpha_2(f\, f_2 + f_1\, f_{1,2}) + \delta({\alpha_1}^2 - {\alpha_2}^2) = 0\,,$$

$$\mathcal{E}_3 : \qquad \frac{f_1 - \alpha_1 f}{f_2 - \alpha_2 f}\, \frac{f_2 - \alpha_1 f_{1,2}}{f_1 - \alpha_2 f_{1,2}} = \frac{1 - {\alpha_1}^2}{1 - {\alpha_2}^2}\,.$$

• A classification of 3-D consistent equations on quad-graphs has been obtained recently (V. Adler, A. Bobenko and Yu. Suris 2003)

$$\mathcal{E}_4 : \qquad \begin{aligned} f_{1,2} &= g\, \frac{\alpha_1\, f_2 - \alpha_2\, f_1}{\alpha_1\, g_1 - \alpha_2\, g_2}\,, \\ g_{1,2} &= \frac{g}{f}\, \frac{\alpha_1\, f_1\, g_2 - \alpha_2\, f_2\, g_1}{\alpha_1\, g_1 - \alpha_2\, g_2} \end{aligned}$$

$$\mathcal{E}_5 : \qquad (\boldsymbol{f}_{1,2} - \boldsymbol{f}) = \frac{\alpha_1 - \alpha_2}{|\boldsymbol{f}_1 - \boldsymbol{f}_2|^2}(\boldsymbol{f}_1 - \boldsymbol{f}_2) \qquad \boldsymbol{f} : \mathbb{Z}^2 \mapsto \mathbb{C}^r$$

Lie point symmetry groups of quadrilateral equations

$$\mathcal{E}(f, f_1, f_2, f_{1,2}; \alpha_1, \alpha_2) = 0$$

Let G be a one-parameter group of transformations acting on the domain of the dependent variables,

$$G : f \mapsto \Psi(n_1, n_2, f; \varepsilon)\,, \quad \varepsilon \in \mathbb{C}\,.$$

The infinitesimal generator of the group action of G on f is the vector field

$$\mathbf{v} = Q(n_1, n_2, f)\,\partial_f\,, \qquad Q(n_1, n_2, f) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Psi(n_1, n_2, f; \varepsilon)\Big|_{\varepsilon=0}$$

$$\begin{array}{rcl} f_1 & \mapsto & \Psi(n_1+1, n_2, f_1; \varepsilon) \\ f_2 & \mapsto & \Psi(n_1, n_2+1, f_2; \varepsilon) \\ f_{1,2} & \mapsto & \Psi(n_1+1, n_2+1, f_2; \varepsilon) \\ & \vdots & \end{array}$$

Discrete prolongation of the vector field $\mathbf{v}$ on the lattice jet space J with coordinates $(f, f_1, f_2, f_{1,2})$

$$\widehat{\mathbf{v}} = Q\,\partial_f + Q_1\,\partial_{f_1} + Q_2\,\partial_{f_2} + Q_{1,2}\,\partial_{f_{1,2}}\,,$$

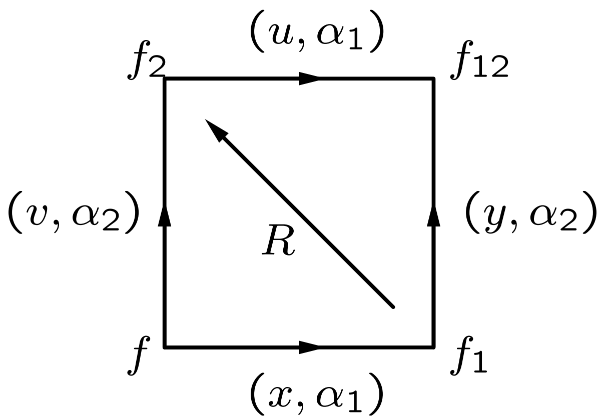
Definition: G is a *symmetry* of the lattice equation $\mathcal{E} = 0$, if it transforms any solution of $\mathcal{E} = 0$, to another solution of the same equation.

Infinitesimal criterion :

$$\widehat{\mathbf{v}}\big(\mathcal{E}(f, f_1, f_2, f_{1,2}; \alpha_1, \alpha_2)\big) = 0\,,$$

Invariants of symmetry groups

Definition: A function $I : J \to \mathbb{C}$ is a *lattice invariant* of the transformation group G, if I is not affected under the action of G.



Infinitesimal invariance condition

$$\widehat{\mathbf{v}}(I) = 0\,.$$

$$\frac{\mathrm{d}\,Q}{\mathrm{d}\,f} = \frac{\mathrm{d}\,Q_1}{\mathrm{d}\,f_1} = \frac{\mathrm{d}\,Q_2}{\mathrm{d}\,f_2} = \frac{\mathrm{d}\,Q_{1,2}}{\mathrm{d}\,f_{1,2}}$$

Lattice invariants assigned to the edges

$$x = I(f, f_1), \quad y = I(f_1, f_{1,2}), \quad u = I(f_2, f_{1,2}), \quad v = I(f, f_2)\,,$$

Application: Construction of Yang-Baxter maps

Let G a one-parameter group of Lie point symmetry transformations of a quadrilateral equation

$$\mathcal{E}(f, f_1, f_2, f_{1,2}; \alpha_1, \alpha_2) = 0$$

and x, y, u, v the associated symmetry invariants assigned to the edges of the square

- Equation $\mathcal{E} = 0$ can be written in terms of the invariants

$$\mathcal{D}(x, y, u, v; \alpha_1, \alpha_2) = 0 \qquad (*)$$

- There exists a relation among them

$$\mathcal{F}(x, y, u, v; \alpha_1, \alpha_2) = 0 \qquad (**)$$

following from the fact that the space of G-orbits is three-dimensional.

Solving equations $(*)$, $(**)$ for u, v in terms of x, y, assuming that the solution is unique, we obtain a map $R(x, y) = (u, v)$.

Proposition: *If the discrete equation $\mathcal{E} = 0$ satisfies the 3D consistency property, then the map $R(x, y) = (u, v)$, satisfies the YB relation.*

(V. Papageorgiou, A.T. and A. Veselov 2006)

Symmetries of the lattice (potential) KdV

$$(f_{1,2} - f)(f_1 - f_2) = \alpha_1 - \alpha_2 \,,$$

Lie point symmetries :

$$\mathbf{v}_1 = \partial_f, \qquad \mathbf{v}_2 = (-1)^{n_1+n_2} f\,\partial_f\,, \qquad \mathbf{v}_3 = (-1)^{n_1+n_2}\,\partial_f\,.$$

- $Iso(\mathbb{R}^{1,1})$

Generalized symmetries:

$\Delta = f(n_1+1, n_2) - f(n_1-1, n_2)$

$\Lambda = f(n_1, n_2+1) - f(n_1, n_2-1)$

$$\mathbf{w}_1 = \frac{1}{\Delta}\partial_f, \qquad \mathbf{w}_2 = \frac{1}{\Lambda}\partial_f,$$

$$\mathbf{w}_3 = \left(\frac{n_1}{\Delta} + \frac{n_2}{\Lambda}\right)\partial_f$$

$$\mathbf{w}_4 = \left(\frac{n_1\,\alpha_1}{\Delta} + \frac{n_2\,\alpha_2}{\Lambda} - \frac{1}{2}f\right)\partial_f$$

Taking the commutators $[\mathbf{w}_1, \mathbf{w}_3]$ and $[\mathbf{w}_2, \mathbf{w}_3]$ we find new symmetry operators

$$\mathbf{w}_5 = [\mathbf{w}_1, \mathbf{w}_3] = \frac{1}{\Delta^2}\left(\Delta_1 + \Delta_{-1}\right)\,\partial_f$$

$$\mathbf{w}_6 = [\mathbf{w}_2, \mathbf{w}_3] = \frac{1}{\Lambda^2}\left(\Lambda_2 + \Lambda_{-2}\right)\,\partial_f$$

Invariant solutions

Definition. We say that a solution f of the lattice equation $\mathcal{E} = 0$ is invariant under the symmetry operator

$$\mathbf{v} = Q[n_1, n_2, f]\partial_f \,,$$

if it satisfies in addition the compatible constraint

$$\boxed{\mathbf{v}(f) = 0 \quad \Leftrightarrow \quad Q[n_1, n_2, f] = 0}$$

Reduction of KdV to a discrete Riccati equation

For $\mathbf{v}$ we take a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$. Symmetry constraint:

$$\frac{1}{f(n_1+1, n_2) - f(n_1-1, n_2)} = \frac{\lambda}{f(n_1, n_2+1) - f(n_1, n_2-1)}$$

Reduced equation

$$w(n_1+1, n_2) = \frac{c_1 w(n_1, n_2) + c_2}{c_3 w(n_1, n_2) + c_4}$$

where

$$w = (f(n_1{+}1, n_2{+}1) - f(n_1, n_2))(f(n_1{+}2, n_2{+}1) - f(n_1{+}1, n_2))$$

and

$$c_1 = -c_4 = \lambda(\alpha_1 - \alpha_2), \quad c_2 = (\alpha_1 - \alpha_2)^2(1+\lambda), \quad c_3 = (1-\lambda)$$

Invariant solutions of KdV under $\boxed{\mathbf{v} = \mathbf{w}_3 - \lambda \mathbf{v}_1 - \mu \mathbf{v}_3}$

$$\mathbf{v} = \left(\frac{n_1}{f_1 - f_{-1}} + \frac{n_2}{f_2 - f_{-2}} - \lambda - \mu(-1)^{n_1+n_2} \right) \partial_f$$

$$(f_{1,2} - f)(f_1 - f_2) = \alpha_1 - \alpha_2$$
$$\frac{n_1}{f_1 - f_{-1}} + \frac{n_2}{f_2 - f_{-2}} = \lambda + \mu(-1)^{n_1+n_2}$$

Lattice invariant under $\mathbf{v}_1$: $\boxed{w = f_{1,2} - f}$

$$f_1 - f_{-1} = w_{-1} + \frac{r}{w} \qquad r = \alpha_1 - \alpha_2$$

$$B := f_2 - f_{-2} \qquad B_1 = w - \frac{rw}{Bw + r}$$

Solutions of discrete KdV equation subject to the symmetry constaint $\mathbf{v}(f) = 0$ are determined by solving the following second order difference equation

$$\frac{(n_1+1)r}{w_{n_1}\, w_{n_1+1} + r} + \frac{n_1 r}{w_{n_1}\, w_{n_1-1} + r} = n_1 + n_2 + 1 + \lambda\left(\frac{r}{w_{n_1}} - w_{n_1}\right) + \mu(-1)^{n_1+n_2}\left(\frac{r}{w_{n_1}} + w_{n_1}\right)$$

- alt-dPII when $\mu = 0$

Invariant solutions of KdV under $\boxed{\mathbf{w}_3 - \lambda \mathbf{v}_2}$

The symmetry constraint

$$\frac{n_1}{f\,f_1 - f\,f_{-1}} + \frac{n_2}{f\,f_2 - f\,f_{-2}} = \lambda(-1)^{n_1+n_2}$$

Using the following symmetry invariants under $\mathbf{v}_2$

$$x = f\,f_1, \qquad y = f_1\,f_{1,2}$$

we arrive at the following system of ordinary difference equations

$$(y_{n_1-1} - x_{n_1})(x_{n_1} - y_{n_1}) = r\,x_{n_1}$$

$$n_1 \frac{x_{n_1} - y_{n_1-1}}{x_{n_1} - x_{n_1-1}} = \frac{n_1 + n_2}{2} + \frac{1}{4} + (\lambda y_{n_1-1} + c)(-1)^{n_1+n_2}$$

The system can be decoupled by solving the last equation for y_{n_1-1} and substituting the result into the first equation. The result is a second order difference equation for x_{n_1} with four free parameters (n_2, r, λ, c).

Lattice modified Boussinesq

$$f_{1,2} = g\,\frac{\alpha_1\,f_2 - \alpha_2\,f_1}{\alpha_1\,g_1 - \alpha_2\,g_2}\,, \qquad g_{1,2} = \frac{g}{f}\,\frac{\alpha_1\,f_1\,g_2 - \alpha_2\,f_2\,g_1}{\alpha_1\,g_1 - \alpha_2\,g_2}$$

$f, g : \mathbb{Z}^2 \to \mathbb{CP}^1$ (Nijhoff 1999)

Lie point symmetry transformations

$$\begin{aligned}
G^1 &: (f,g) \longmapsto (e^{\epsilon_1}\,f, f)\ ,\\
G^2 &: (f,g) \longmapsto (f, e^{\epsilon_2}\,g)\ ,\\
G^3 &: (f,g) \longmapsto \left(e^{\epsilon_3\,\omega^{n_1+n_2}}\,f, e^{-\epsilon_3\,\omega^{n_1+n_2+1}}\,g\right)\ ,\\
G^4 &: (f,g) \longmapsto \left(e^{\epsilon_4\,\omega^{2(n_1+n_2)}}\,f\,, e^{-\epsilon_4\,\omega^{2(n_1+n_2+1)}}\,g\right)\ ,
\end{aligned}$$

ω a primitive cubic root of unity. Infinitesimal generator

$$\begin{aligned}
\mathbf{v} = &\left(\lambda_1 + \lambda_3\,\omega^{n_1+n_2} + \lambda_4\,\omega^{2(n_1+n_2)}\right)\,f\,\partial_f +\\
&\left(\lambda_2 - \lambda_3\,\omega^{n_1+n_2+1} - \lambda_4\,\omega^{2(n_1+n_2+1)}\right)\,g\,\partial_g\,.
\end{aligned}$$

Generalized symmetries

$$\begin{aligned}
\mathbf{w}_1 &= (\xi^1 - 1)\,f\,\partial_f - (\zeta^1 - 1)\,g\,\partial_g\,,\\
\mathbf{w}_2 &= (\xi^2 - 1)\,f\,\partial_f - (\zeta^2 - 1)\,g\,\partial_g\,,\\
\mathbf{w}_3 &= \left(n_1\,(\xi^1 - 1) + n_2\,(\xi^2 - 1)\right)\,f\,\partial_f -\\
&\qquad \left(n_1\,(\zeta^1 - 1) + n_2\,(\zeta^2 - 1)\right)\,g\,\partial_g\,,
\end{aligned}$$

where

$$\xi^1 = \frac{3\,f_1\,g}{f_1\,g + f_{-1}\,g_1 + f\,g_{-1}}, \qquad \xi^2 = \frac{3\,f_2\,g}{f_2\,g + f_{-2}\,g_2 + f\,g_{-2}},$$

$$\zeta^1 = \frac{3\,f\,g_{-1}}{f_1\,g + f_{-1}\,g_1 + f\,g_{-1}}, \qquad \zeta^2 = \frac{3\,f\,g_{-2}}{f_2\,g + f_{-2}\,g_2 + f\,g_{-2}},$$

Group invariant solutions under $\boxed{\mathbf{w}_3 - \mathbf{v}}$

Joint invariants of the subgroup $\{G^1, G^2\}$

$$x(n_1,n_2) = \frac{f(n_1,n_2+1)}{f(n_1+1,n_2)}, \quad y(n_1,n_2) = \frac{g(n_1,n_2+1)}{g(n_1+1,n_2)}$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, r, c \longrightarrow$ complex parameters

Reduced system of ordinary difference equations:

$$\beta^1_{n_1} - n_1\,\xi_{n_1} = \frac{x_{n_1}(r^2\,y_{n_1}-1)\left(\beta^1_{n_1+1}-(n_1+1)\,\xi_{n_1+1}\right)}{(r\,x_{n_1}-1)(r\,y_{n_1}-x_{n_1})} + \frac{x_{n_1}\,\left(\beta^2_{n_1+1}-(n_1+1)\zeta_{n_1+1}-3\,c\right)}{r\,y_{n_1}-x_{n_1}},$$

$$\beta^2_{n_1} - n_1\,\zeta_{n_1} = \frac{\beta^1_{n_1+1}-(n_1+1)\,\xi_{n_1+1}}{1-r\,x_{n_1}} + \frac{r\,\left(\beta^2_{n_1+1}-(n_1+1)\zeta_{n_1+1}\right)}{r-y_{n_1}}.$$

auxiliary quantities

$$\xi_{n_1} = 3\,\left(1+\frac{x_{n_1}}{y_{n_1}}\,\frac{r\,y_{n_1-1}-x_{n_1-1}}{r\,x_{n_1-1}-1} + x_{n_1}\,\frac{r-y_{n_1-1}}{r\,x_{n_1-1}-1}\right)^{-1},$$
$$\zeta_{n_1} = 3\,\left(1+\frac{1}{x_{n_1}}\,\frac{r\,x_{n_1-1}-1}{r-y_{n_1-1}} + \frac{1}{y_{n_1}}\,\frac{r\,y_{n_1-1}-x_{n_1-1}}{r-y_{n_1-1}}\right)^{-1},$$

$$\beta^1_{n_1} = \lambda_1 + \lambda_3\,\omega^{n_1} + \lambda_4\,\omega^{2n_1} + n_1 + c\,,$$
$$\beta^2_{n_1} = -\lambda_2 + \lambda_3\,\omega^{n_1+1} + \lambda_4\,\omega^{2(n_1+1)} + n_1 + c\,,$$

A.T. and F. Nijhoff 2006

Related References

V.G. Papageorgiou, A.G. Tongas, A.P. Veselov, Yang-Baxter maps and symmetries of integrable equations on quad-graphs *J. Math. Phys.* **47** (2006) 083502.

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