Special solutions of the Painlevé equations: P1 to P5 Part I

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The Painlevé equations and Monodromy problem

0. Introduction

A. R. Forsyth, "Theory of differential equations II, III" 1900
E. T. Whittaker, "A Course of Modern Analysis" 1902.
Whittaker-Watson, "A Course of Modern Analysis" 1915, 1920, 1927.
G. N. Watson, "A treatise on the theory of Bessel functions" 1922, 1944.

1697 John Bernoulli considered

$$\frac{dy}{dx} = y^2 + x^2$$

1703 James Bernoulli solved

$$y = \frac{\frac{x^3}{3} - \frac{x^7}{3\cdot4\cdot7} + \frac{x^{11}}{3\cdot4\cdot7\cdot8\cdot11} + \cdots}{1 - \frac{x^4}{3\cdot4} - \frac{x^8}{3\cdot4\cdot7\cdot8} + \frac{x^{12}}{3\cdot4\cdot7\cdot8\cdot11\cdot12} + \cdots}$$

At the age of Bernoulli, differential equations should be solved in closed form. The Riccati equation may be a surprise?

Chapter IV in Watson:

$$(*)\ \frac{dy}{dx} = ay^2 + bx^n \qquad \text{is solved by } J_{-\frac{2}{n+2}}(x)$$

♠ If and only if $n = -\dfrac{4m}{2m \pm 1}$ $(m = 0, 1, 2, \ldots)$, (*) is solved by elementary functions (Joseph Liouville: Roger's father).

♠ Daniel Bernoulli found Bäcklund transformation

$$X = \frac{x^{n+1}}{n+1}, \quad Y = -\frac{1}{y}$$

Then

$$\frac{dY}{dX} = aY^2 + b(n+1)^N X^N, \quad N = -\frac{n}{n+1}$$

♡ Nalini: Asymptotic analysis, zeros of Bessel are motivation.

♡ Watson's book contains many idea, such that Bäcklund transformation, irreducibility.

1. Special solutions

Q1. What is special solutions of the Painlevé equations?

Q2. Why do we study special solutions?

A1. The answer depends on definition of "special".
Umemura gave a definition of "classical functions".

♣ Umemura's classical functions [8]

- **algebraic functions**
- **solutions of linear differential equations**
- **abelian functions** (which do not appear in Painlevé)

1.1 Differential Galois theory

If you read a book on Differential Galois theory, you will find

- **long and boring introduction**
- **meaningful** (but only small part)
- **ununderstandable** (Drach's theory...)

successful DGT

Picard-Vessiot (linear equations) [3]

Kolchin-**Umemura** (linear + abelian)

Other way:

Cartan-Kuranishi-...: involutive systems

2. Definition of algebraic solutions

The algebraic solutions of the Painlevé equation:
1) $y(t)$ is an algebraic function as t
2) $y(t)$ satisfy the Painlevé equations for suitable coordinates.

(1) is evident. $\exists f$: polynomial, s.t. $f(y(t), t) = 0$.
(2) "Suitable coordinates" are coordinates on the initial value space.

Theorem(Takano [6]) The initial value space is a finite union of the affine space $\mathbb{C}^2$ except P3($D_8^{(1)}$). On each copy of $\mathbb{C}^2$, there exist a unique polynomial Hamiltonian, which gives the Painlevé equation. The patching between two charts are symplectic and rational.

It may happen that there exist a special solution OUTSIDE the standard affine space.

P6:

$$\begin{cases} t(t-1)q' = 2q(q-1)(q-t)p - \alpha_3 q(q-t) \\ \qquad -(\alpha_0-1)q(q-1) - \alpha_4(q-1)(q-t), \\ t(t-1)p' = -(3q^2-(2t+2)q+t)p^2 - \alpha_2(\alpha_1+\alpha_2) \\ \qquad -(1-2q+\alpha_0(2q-1)+\alpha_3(2q-t)+\alpha_4(2q-t-1))p, \end{cases}$$

$$\alpha = \frac{{\alpha_1}^2}{2}, \quad \beta = -\frac{{\alpha_4}^2}{2}, \quad \gamma = \frac{{\alpha_3}^2}{2}, \quad \delta = \frac{1}{2} - \frac{{\alpha_0}^2}{2},$$
$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$$

1) If $\alpha_4 = 0$, $q = 0$ satisfy the first equation. The second equation

$$t(t-1)p' = -tp^2 - (\alpha_3 t + \alpha_0 - 1) - \alpha_2(\alpha_1 + \alpha_2)$$

is the Riccati (solved by ${}_2F_1(\alpha_2+1, \alpha_2+\alpha_3+1, 1-\alpha_0; t)$).
p is not algebraic in general.
2) If $\alpha_4 = 0, \alpha_2 = 0$, $(q,p) = (0,0)$ is a rational solution.

We set

$$q = \frac{1}{q_4}, \;\; p = -q_4(\alpha_2 + q_4 p_4)$$

Then

$$\begin{aligned} t(t-1)H_4 &= t q_4^3 p_4^2 - q_4^2 p_4((t+1)p_4 - (2\alpha_2 + \alpha_4)t) \\ &\quad + q_4(p_4^2 - ((1-\alpha_0-\alpha_1)t - \alpha_1 - \alpha_3)p_4 + \alpha_2(\alpha_2+\alpha_4)t) - \alpha_1 p_4 \end{aligned}$$

If $\alpha_1 = 0, \alpha_2 = 0$, $(q_4, p_4) = (0, 0)$ is a rational solution.

This solution is OUTSIDE the standard chart $(q, p) \in \mathbb{C}^2$.

Remark. The initial value space of P6 is a union of six $\mathbb{C}^2$:

$$(q,p) = (p_1(\alpha_4 - q_1 p_1), 1/p_1), \quad (q,p) = (1 + p_2(\alpha_3 - q_2 p_2), 1/p_2),$$

$$(q,p) = (t + p_3(\alpha_0 - q_3 p_3), 1/p_3) \quad (q_4, p_4) = (p_5(\alpha_1 - q_5 p_5), 1/p_5)$$

The standard Hamiltonian is

$$H = f(t)F(q)p^2 + G(q)p + K(q)$$

$$F(q) = \begin{cases} 1/2, & \text{P1, P2} \\ q^2, & \text{P3, P4} \\ q(q-1)^2, & \text{P5} \\ q(q-1)(q-t), & \text{P6} \end{cases}$$

Therefore

$$q' = 2f(t)F(q)p + G(q).$$

If $F(q) \not\equiv 0$, p can be solved by q', q, t. And if q is rational (algebraic), p is also rational (algebraic).

The case $F(q) \equiv 0$ occurs for P4, P5, P6.

2.1 Remark

	singularity	alg. solution
P1, P2, P4	$t = \infty$	rational
P3, P5	$t = 0, \infty$	$f(t^{1/m})$, f: rational
P6	$t = 0, 1, \infty$	complicated

We may change the independent variable as $t = x^p$ for P3 and P5.

P3:

$$\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + p^2(\alpha y^2 + \beta)x^{p-2} + p^2\left(\gamma y^3 + \frac{\delta}{y}\right)x^{2p-2}$$

P5:

$$\frac{d^2y}{dx^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + p^2\alpha y(y-1)^2$$

$$+p^2\beta\frac{(y-1)^2}{y} + p^2\gamma x^{p-2}y + p^2\delta x^{2p-2}\frac{y(y+1)}{y-1}$$

2.2 the Riccati solution

The Riccati solution is called

a hypergeometric solution,

a transcendental classical solution...

Example The Painlevé II

$$y'' = 2y^3 + ty + \alpha$$

♣ If $\alpha = 0$, $y = 0$ is a rational solution (Gambier [1]).

♣ Riccati solution

$$\text{(H2)} \begin{cases} q' = -q^2 + p - \frac{t}{2}, \\ p' = 2pq + \alpha_1. \end{cases} \qquad (\alpha = \alpha_1 - 1/2)$$

$$\delta' = \delta + (-q^2 + p - \frac{t}{2})\frac{\partial}{\partial q} + (2pq + \alpha_1)\frac{\partial}{\partial p}$$

If $\alpha_1 = 0$, (p) is an invariant divisor: $\delta' p = 2qp$.

$q' = -q^2 - \frac{t}{2}$ is the Riccati equation (solved by Airy).

The transcendental classical solution:
1) $\forall K \supset C(t)$, $(K[q,p], \delta') \supset (K, \delta)$; extension of differential ring
2) If trans.deg $(K[q,p] : K) = 1$, it is a transcendental classical solution.

Condition (J):

$$\delta' = \delta + \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

There exist polynomials $F, G \in K[q,p]$ such that

$$\delta' F = GF$$

If condition (J) is satisfied, we call (F) an invariant divisor. On $F = 0$, the Painlevé equation reduced to the first order ODE (Riccati).

Condition (J) is known by Painlevé.
Umemura gave a strategy to classify invariant divisors.

3. Why do we study the special solutions?

A2.

1) P. Painlevé tried to find new transcendental functions. He classified six equations which are not reduced to known equations. But he claimed all solutions of P1 are transcendental, but he could not show a proof (Painlevé vs Roger Liouville, 1902).

2) To study special solutions, we can naturally understand the Bäcklund transformations are isomorphic to affine Weyl groups. Classical parameters are not natural.

3) τ-functions of rational solutions are represented by the Schur polynomials. This fact is related to similarity reduction from soliton equations.

4. The Painlevé equations [4]

$$\text{P1\_2)} \quad y'' = \alpha(2y^3 + ty) + \beta(6y^2 + t),$$

$$\text{P4\_34}') \quad y'' = \frac{y'^2}{2y} - \frac{\alpha}{2y} + \beta y(2y + t) + \gamma y(y + t)(3y + t),$$

$$\text{P3)} \quad y'' = \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y},$$

$$\text{P5)} \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1},$$

$$\text{P6)} \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right) y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right) y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right].$$

4.1 standard list of Painlevé equations

$$
\begin{aligned}
\text{P1)}\quad y'' &= 6y^2 + t,\\
\text{P2)}\quad y'' &= 2y^3 + ty + \alpha,\\
\text{P34}')\quad y'' &= \frac{y'^2}{2y} + 2y^2 + ty - \frac{\alpha}{2y},\\
\text{P4)}\quad y'' &= \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}.
\end{aligned}
$$

P1,2 By $y \to cy,\ t \to c^2 t$, $\mathrm{P1\_2}(\alpha,\beta) \to \mathrm{P1\_2}(c^6\alpha, c^5\beta)$.

(P1-A) $\alpha \neq 0$, P2
(P1-B) $\alpha = 0, \beta \neq 0$, P1
(P1-C) $\alpha = 0, \beta = 0$. quadrature

P4,34 By $y \to cy,\ t \to ct$, $\mathrm{P4\_34}'(\alpha,\beta,\gamma) \to \mathrm{P4\_34}'(\alpha,\ c^3\beta,\ c^4\gamma)$.

(P4-A) $\gamma \neq 0$, P4
(P4-B) $\beta \neq 0, \gamma = 0$, P34 = P2
(P4-C) $\beta = 0, \gamma = 0$. quadrature

P3 By $y \to cy$, $t \to dt$, $\mathrm{P3}(\alpha, \beta, \gamma, \delta) \to \mathrm{P3}(cd\alpha, \frac{d}{c}\beta, c^2d^2\gamma, \frac{d^2}{c^2}\delta)$.

(P3-A) $\gamma \neq 0, \delta \neq 0$ $D_6^{(1)}$

(P3-B) $\gamma \neq 0, \delta = 0$ or $\gamma = 0,\ \delta \neq 0$ $D_7^{(1)}$

(P3-C) $\gamma = 0,\ \delta = 0$ $D_8^{(1)}$

(P3-D) $\alpha = 0,\ \gamma = 0$ or $\beta = 0,\ \delta = 0.$ quadrature

P5 By $t \to ct$, $\mathrm{P5}(\alpha, \beta, \gamma, \delta) \to \mathrm{P5}(\alpha, \beta, c\gamma, c^2\delta)$.

(P5-A) $\delta \neq 0$ P5

(P5-B) $\gamma \neq 0, \delta = 0$ deg-P5 =P3($D_6^{(1)}$)

(P5-C) $\gamma = 0,\ \delta = 0.$ quadrature

Divided by scaling transformations, we get 10 different Painlevé.
(P2 and P34, P3($D_6^{(1)}$) and deg-P5 is equivalent, Gromak)

5. Algebraic solutions of Painlevé equations

Algebraic solution = **center type** + **Riccati type**

center type	parameter	Algebraic
P1	none	none
P2	(0)	$y = 0$
P34	$(1/4)$	$y = t/2$
P4	$(0, -2/9)$	$y = -2t/3$
$\mathrm{P3}'(D_8^{(1)})$	$(8h, -8h, 0, 0)$	$y = -\sqrt{t}$
$\mathrm{P3}'(D_7^{(1)})$	$(0, -2, 2, 0)$	$y = t^{1/3}$
$\mathrm{P3}'(D_6^{(1)})$	$(a, -a, 4, -4)$	$y = -\sqrt{t}$
deg-P5	$(h^2/2, -8, -2, 0)$	$y = 1 + 2\sqrt{t}/h$
P5	$(a, -a, 0, \delta)$	$y = -1$

	Riccati	Algebraic
P1	none	none
P2	$(-1/2)$: Airy	none
P34	(1):Airy	none
P4	$(1-s,-2s^2)$: Hermite-Weber	$(0,-2)$: $y=-2t$
$\mathrm{P3}(D_8^{(1)})$	none	none
$\mathrm{P3}(D_7^{(1)})$	none	none
$\mathrm{P3}(D_6^{(1)})$	$(4h,4(h+1),4,-4)$: Bessel	none
deg-P5	$(\alpha,0,\gamma,0)$: Bessel	none
P5	$\left(\frac{(\kappa_0+s)^2}{2},-\frac{\kappa_0^2}{2},-(s+1),-\frac{1}{2}\right)$: Kummer	$\left(\frac{(s+1)^2}{2},-\frac{1}{2},-(s+1),-\frac{1}{2}\right)$: $y=1+t/(s+1)$

Theorem All of classical solutions of P1 to P5 are equivalent to the above solutions up to the Bäcklund transformations.

5.1 Algebraic solutions of P1

[Theorem] [7]

Any solution of the first Painlevé equation is not algebraic.

Proof. 1) Assume y is algebraic and is not rational.
y has branch points at $t = c \in \mathbb{C}$.
Let $n > 1$ be the ramification index at $t = c$.

$$y = \sum_{k=s}^{\infty} a_k z^k, \quad z = (t-c)^{1/n}$$

Then

$$\sum_k (a_k k(k-n)/n^2) z^{k-2n} = 6\left(\sum a_k z^k\right)^2 + z^n + c.$$

$$\Longrightarrow s = -2n, \quad a_{-2n} = 1$$

($t = c$ may be a double pole...)

Let k_1 be the least k such that $a_k \neq 0$ and k is not divided by n. Comparing the coefficient of z^{k_1-2n}, we have

$$a_{k_1}k_1(k_1-n)/n^2 = 2a_{-2n}a_{k_1}(= 2a_{k_1}).$$

But this is a contradiction. Therefore y is rational.

2) Assume that y is rational. The Laurent expansion at ∞:

$$\sum_{k=0}^{\infty} b_k t^{m-k}.$$

We have

$$\sum_{k=0}^{\infty} b_k(m-k)(m-k-1)t^{m-k-2} = 6\left(\sum b_k t^{m-k}\right)^2 + t,$$

If $m > 0$, $6b_0^2t^{2m}$ is the leading term.
If $m \leq 0$, t is the leading term. This is a contradiction. $\square$

5.2 The irreducibility of P1

Consider the differential ring $(K[q,p],\delta')$

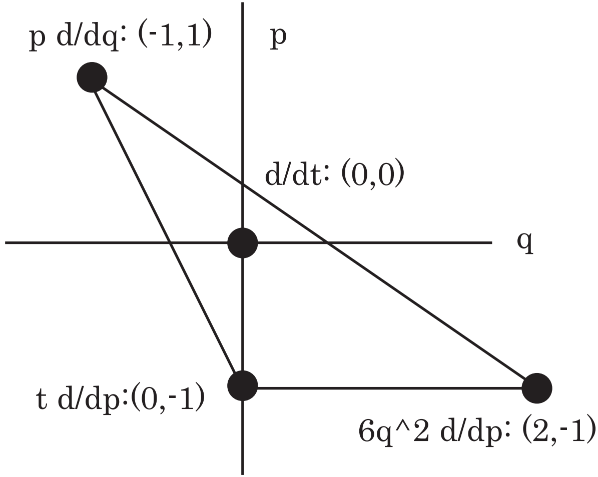
$$\delta' = \delta + p\frac{\partial}{\partial q} + (6q^2+t)\frac{\partial}{\partial p}$$

Condition (J): Find polynomials $F(q,p), G(q,p)$ such that

$$\delta' F = GF$$

$$\delta' = X_1 + X_0 + X_{-3}$$

$$\begin{aligned} X_1 &= p\frac{\partial}{\partial q} + 6q^2\frac{\partial}{\partial p} \\ X_0 &= \delta \\ X_{-3} &= t\frac{\partial}{\partial p} \end{aligned}$$



Set

$$\text{wt } q = 2, \quad \text{wt } p = 3$$

Then

$$\text{wt } p\frac{\partial}{\partial q} = \text{wt } 6q^2\frac{\partial}{\partial p} = 1$$

$$K[q,p] = \bigoplus_{j \in \mathbb{N}} R_j$$

[Lemma] $c \in K = R_0$ $A = p^2 - 4q^3$

1) $f \in R_j,\ X_1 f = 0 \Longrightarrow f = cA^n\ (j = 6n)$

2) $f \in R_{6j-1},\ X_1 f = cA^n \Longrightarrow f = 0$

3) $f \in R_{6j+2},\ X_1 f = cA^n p \Longrightarrow f = c(p^2 - 4q^3)^n q$

4) $f \in R_{6j+1},\ X_1 f = cA^n q \Longrightarrow f = 0$

Step.1

$(X_1 + X_0 + X_{-3})F = GF \Longrightarrow \text{wt } G \leq 1 \Longrightarrow G = g \in R_0$

Step.2

Set $F = \sum^m F_j$. Then

$$X_1 F_j = (-X_0 + g)F_{j+1} - X_{-3}F_{j+4}.$$

$$X_1 F_m = 0 \Longrightarrow F_m = cA^n \text{ by } (1).\ (m = 6n)$$

Step.3

$$X_1 F_{m-1} = (-X_0 + g)F_m = (-c' + gc)A^n$$
$$\Longrightarrow F_{m-1} = 0, c' + gc = 0 \text{ by } (2).$$
$$X_1 F_{m-2} = (-X_0 + g)F_{m-1} = 0 \Longrightarrow F_{m-2} = 0 \text{ by } (1).$$
$$X_1 F_{m-3} = (-X_0 + g)F_{m-2} = 0 \Longrightarrow F_{m-3} = 0 \text{ by } (1).$$

Step.4

$$X_1 F_{m-4} = (-X_0 + g)F_{m-3} - X_{-3}F_m = -2nctA^{n-1}p$$
$$\Longrightarrow F_{m-4} = -2nctA^{n-1}q \text{ by } (3).$$

Step.5

$$\begin{aligned} X_1 F_{m-5} = (-X_0 + g)F_{m-4} &= (2nc't + 2nc - 2nctg)A^{n-1}q \\ &= 2ncA^{n-1}q \end{aligned}$$

$$\Longrightarrow F_{m-5} = 0, 2nc = 0 \text{ by } (4).$$

Therefore $n = 0$, which means $F \in R_0$. Condition (J) fails. [7]

5.3 The second Painlevé equations

[Theorem]

The P2 has a unique rational solutions when α is an integer.

Proof. Expand at the infinity;

$$y = \sum_{n=0}^{\infty} \frac{c_n}{t^{3n+1}},$$

where

$$c_0 = -\alpha,\ c_1 = 2\alpha(\alpha^2 - 1),\ c_2 = -4\alpha(3\alpha^2 - 10)(\alpha^2 - 1), \cdots.$$

$$c_n = (3n-1)(3n-2)c_{n-1} + \sum_{i+j+k=n-1} c_i c_j c_k.$$

Therefore if there exists a rational solution, it is unique.

Assume that y has a pole at a finite point $t = c$.

$$y \sim \frac{\pm 1}{t-c} + \text{holomorphic}$$

By the residue theorem, α shod be an integer.

If $\alpha = 0$, $y = 0$ is a solution, which is evidently rational. Applying the Bäcklund transformation to this solution, we get a rational solution for any integer α. □

Gambier (1910): $y = 0$ is a solution of P2(0) [1]

Schubart (1956): P2 has a rational solution if and only if $\alpha = 0, \pm 1$. [5]

Yablonskii (1959): pointed out Schubart's mistake. [10]
Uniqueness, existence $|\alpha| \leq 5$

Vorob'ev (1965): found the Bäcklund transformation for rational solutions,
Existence [9]

Lukashevich (1971)*: found the Bäcklund transformation [2]
for any solution of P2

* I learned difference of the work by Vorob'ev and Lukashevich from Gromak after my talk.

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For classification of special solutions. see the book:

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