Local irreducibility of the first Painlevé equation

$$(P_1) \qquad \frac{d^2y}{dx^2} = 6y^2 + x$$

Its solutions are meromorphic on $\mathbb{C}$ and it cannot be *integrated* using solutions of *simpler* equations, *i.e.* it is *irreducible*.

<u>Lemma</u> (Painlevé)

The vector field

$$X_1 = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + (6y^2 + x)\frac{\partial}{\partial y'}$$

has no first integral in $\mathbb{C}(x, y, y')$.

Nishioka, Umemura (1987)

Irreducibility of P_1 for :

$$integrated = \text{find a particular solution}$$

$$simpler\ \text{equations} = \begin{cases} \text{- algebraic equations} \\ \text{- linear diff. equations} \\ \text{- order 1 diff. equations} \end{cases}$$

Lemma (Kolchin-Kovacic)

For any diff. extension $\mathbb{C}(x) \subset K$, there is no $P, L \in K[y, y']$ such that $X_1 P = LP$.

There is no ‘invariant divisor’ or ‘local Darboux polynomial’.

Replace

$$integrated = \text{find the } general \text{ solution}$$

First integrals and D.-V. reducibility of a vector field on $\mathbb{C}^3$.

L is a differential extension of $\mathbb{C}(x, y, y')$.
X is a vector field with coefficients in L such that $divX = 0$.

Let C_1 and C_2 be two differential indeterminates.
$L\{C_1, C_2\}$ is the differential ring generated by C_1 and C_2 and E is the differential ideal generated by

$$XC_1 = 0$$
$$XC_2 = 0$$

and the components of

$$dC_1 \wedge dC_2 = i_X dx \wedge dy \wedge dy'$$

Definition

X is Drach-Vessiot reducible over L if the ring

$$L\{C_1, C_2\}/E$$

has a non trivial differential ideal.

Theorem

For any diff. extension $\mathbb{C}(x) \subset K$, the vector field

$$X_1 = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + (6y^2 + x)\frac{\partial}{\partial y'}$$

is D.-V. irreducible over $K(y, y')$.

Examples of D.-V. reducible vector fields on $\mathbb{C}^3$.

Definition (Ritt, Kolchin)

Let $\widetilde{L}$ be a diff. extension of L diff. generated by $h_1, \ldots, h_k$ and L_n the field generated by the derivatives of the h of order less than n.
Then $transc.deg.L_n/L \sim an^b$. a and b are independent of the h.

b is the diff. dimension of $\widetilde{L}/L$

Lemma

If X has 2 first integrals in a diff. dim. 1 extension of L then it is D.-V. reducible.

Examples

- $\widetilde{L}$ is generated by a fondamental solution of $dH = H\Omega$, Ω a matrix of 1-forms with coefficients in L.

- $\widetilde{L}$ is diff. gen. by a non constant solution of $dh \wedge \omega = 0$, ω a 1-form with coefficients in L.

- $\widetilde{L}$ is diff. gen. by a fond.solution of $dH \wedge dt = H\Omega \wedge dt$, Ω a matrix of 1-forms with coefficients in L and $t \in L$.

Consequence of D.-V. reducibility on the flows of X.

The flows satisfy some differential equations:

$$\varphi^* X \wedge X = 0$$

and

$$\varphi^* i_X dx \wedge dy \wedge dy' = i_X dx \wedge dy \wedge dy'$$

Let $L^{(1)} \otimes_{\mathbb{C}} L^{(2)} \left[\varphi_i^\alpha \, | i = 1, 2, 3 \, ; \alpha \in \mathbb{N}^3, \frac{1}{\det \varphi_i^{\epsilon_j}} \right]$ with derivations

$$D_i = \partial_i^{(1)} + \varphi_j^{\epsilon_i} \partial_j^{(2)} + \ldots + \varphi_j^{\alpha+\epsilon_i} \frac{\partial}{\partial \varphi_j^\alpha} \cdots$$

be the ring of diff. equations with coefficients in L on germs of diffeomorphisms of $\mathbb{C}^3$

<u>Lemma</u>

Let I be a diff. ideal of $L\{C_1, C_2\}/E$ and J_I be the diff. ideal of $L \otimes L\{\varphi\}$ of equations satisfied by

$\{\varphi/(c_1, c_2)$ is solution of I iff $(c_1 \circ \varphi, c_2 \circ \varphi)$ is solution of $I\}$.

Then J_I is bigger that the ideal diff. generated by the known p.d.e.

$\sim$ Definition

The Galois groupoid of X over L is the groupoid over L defined by the ideal J_X in $L \otimes L\{\varphi\}$ of all the diff. equations satisfied by the flows of X.

The D.-V. irreducibility of X_1 is a consequence of

Theorem

For any diff. extension $\mathbb{C}(x) \subset K$, the ideal of the Galois groupoid of X_1 over $K(y, y')$ is diff. generated by the components of

$$\varphi^* X_1 \wedge X_1, \quad \varphi^* dx - dx$$

and

$$\varphi^*(i_{X_1} dx \wedge dy \wedge dy') - i_{X_1} dx \wedge dy \wedge dy'$$

Proposition ($\sim$ E. Cartan)

We are in one of these cases :

- there is an integrable 1-form vanishing on X_1 with algebraic coefficients over $K(y, y')$,
- there are two algebraic 1-foms $\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ vanishing on X_1 and a traceless matrix of 1-form Ω with algebraic coefficients over $K(y, y')$ such that
$$d\Theta = \Omega \wedge \Theta \text{ and } d\Omega = \Omega \wedge \Omega$$
- the theorem is true.

Corollary

If X is a reducible vector field over L with $divX = 0$ then

- $\exists \omega \in \Omega^1$ such that $dC_1 \wedge \omega = 0$ and $dC_1 \wedge dC_2 = i_X dx \wedge dy \wedge dy'$
- $\exists \Theta, \Omega$ a vector and a matrix with entries in Ω^1 such that $d\partial C = -\partial C \Omega$ and $d \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \partial C \Theta$.

Ingredients

- The change of variable $u = y'^2 + 4y^3$.

$$K[y, y'] = K[y, u, \sqrt{4y^3 - u}]$$

$$X_1 = \frac{\partial}{\partial x} + \sqrt{4y^3 - u}\frac{\partial}{\partial y} + x\frac{\partial}{\partial y'}$$

- The weight p : $p(y) = 2$ and $p(u) = 6$.

<u>The first case is impossible.</u>

- ω can be supposed polynomial and $i_{X_1} d\omega = 0$.

- $\omega = \sum_m^M \omega_h$, compute ω_M, ...

  The computation of ω_{M-5} leads to a contradiction.

<u>The second case is impossible.</u>

- Θ can be supposed to be $\begin{pmatrix} dy - y'dx \\ dy' - (6y^2 + x)dx \end{pmatrix}$

and

$$\Omega = A\theta_1 + B\theta_2 + \begin{pmatrix} 0 & 1 \\ 12y & 0 \end{pmatrix} dx$$

where A and B are matrices of polynomials s.t.

$$\begin{cases} X_1 A + 12yB - \begin{pmatrix} 0 & 0 \\ 12 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 12y & 0 \end{pmatrix}, A\right] \\ X_1 B + A = \left[\begin{pmatrix} 0 & 1 \\ 12y & 0 \end{pmatrix}, B\right] \\ \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y'} = [A, B] \end{cases}$$