Painlevé I asymptotics for orthogonal polynomials with respect to a varying quartic weight

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Overview

- Introduction of orthogonal polynomials on the real line wrt a weight parameterized by $t > 0$
- On formal grounds one expects a critical value at $t = -1/12$
- Introduction of orthogonal polynomials on contours in the complex plane wrt a weight parameterized by $t < 0$
- Painlevé I asymptotics for the polynomials at the critical value $t = -1/12$
- Overview of the proof

Orthogonal polynomials for $t > 0$

Define

$$V_t(x) = x^2/2 + tx^4/4.$$

For $t > 0$ let $p_{n,N}$ be the monic polynomial of degree n satisfying

$$\int_{\mathbb{R}} p_{n,N}(x) x^j \mathrm{e}^{-NV_t(x)}\, \mathrm{d}x = 0, \qquad j = 0, ..., n-1$$

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Properties:

- Three-term recurrence relation

$$p_{n+1,N}(x) = x p_{n,N}(x) - a_{n,N} p_{n-1,N}(x).$$

- Recurrence coefficients satisfy a nonlinear difference equation

$$a_{n,N} + t a_{n,N}(a_{n+1,N} + a_{n,N} + a_{n-1,N}) = \frac{n}{N}$$

Discrete Painlevé I equation. (Freud equation, string equation)

Asymptotics

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An important tool in the asymptotic analysis is the equilibrium measure μ_t that is defined as the unique minimizer of the energy functional

$$I(\nu) = -\iint \log|x-y| \, \mathrm{d}\nu(x)\mathrm{d}\nu(y) + \int V_t(x) \, \mathrm{d}\nu(x)$$

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among all Borel measure ν on $\mathbb{R}$ with unit mass.

It can be explicitly calculated:

$$\mathrm{d}\mu_t(x) = \frac{t}{2\pi}(x^2 - d_t^2)\sqrt{x^2 - c_t^2}\,\mathrm{d}x, \qquad \text{on } [-c_t, c_t]$$

where

$$c_t^2 = \frac{2}{3t}\left(\sqrt{1+12t} - 1\right) \qquad d_t^2 = -\frac{1}{3t}\left(\sqrt{1+12t} + 2\right)$$

Asymptotics

If we set $n = N$ and let $n \to \infty$ we find $\lim_{n\to\infty} a_{n,n}(t) = \frac{2}{1+\sqrt{1+12t}}$

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Note that

- the polynomials are not well-defined for $t < 0$
- the discrete Painlevé I equation is well-defined for $t < 0$
- The limiting value for $a_{n,n}(t)$ is well-defined for $-1/12 < t < 0$.
- The measure μ_t is well-defined for $-1/12 < t < 0$

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Formal calculation in physics literature show that the critical value $t = -1/12$ is described by a solution of the (continuous) Painlevé I equation

$$y''(x) = 6y^2(x) + x$$

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- definition of the orthogonal polynomials for $t < 0$. The orthogonality now takes place of contours in the complex plane.
- Formulation of a Riemann-Hilbert problem for orthogonal polynomials
- Lax pair formulation for the Riemann-Hilbert problem. The discrete Painlevé I equation is the compatibility equation of the Lax-pair.
- WKB analysis of the Riemann-Hilbert problem and the associated Lax-pair.

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Later Deift, Kriechenbauer, McLaughlin,Venakides and Zhou developed the Deift/Zhou steepest descent method for the Riemann-Hilbert problem associated to orthogonal polynomials on the real line. Powerful method to obtain asymptotics.

Goal

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Benefit:

- No use of the Lax pair
- No explicit use of the discrete Painlevé I equation
- Generic example for all cases where the "equilibrium measure" vanishes with a power 3/2 at the endpoints of the support.

$$\mathrm{d}\mu_{-1/12}(x) = \frac{1}{24\pi}(8 - x^2)^{3/2}\,\mathrm{d}x$$

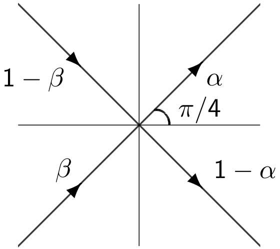
The bilinear form

Define

$$\Gamma_k = \mathrm{e}^{\mathrm{i}(\mathrm{k}\pi/2-\pi/4)}\mathbb{R}_+, \qquad k = 1, \dots, 4$$

Define the biliniear form $\langle\cdot,\cdot\rangle_{\alpha,\beta}$ on the space of polynomials by

$$\begin{aligned}\langle p,q\rangle_{\alpha,\beta} = \alpha \int_{\Gamma_1} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z + (1-\beta)\int_{\Gamma_2} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z \\ + \beta \int_{\Gamma_3} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z + (1-\alpha)\int_{\Gamma_4} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z.\end{aligned}$$



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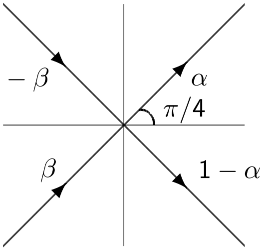
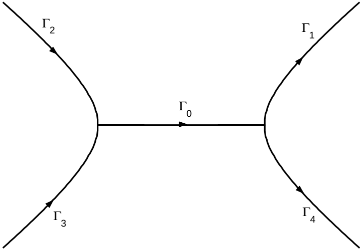
$$+\beta \int_{\Gamma_3} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z + (1-\alpha)\int_{\Gamma_4} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z.$$





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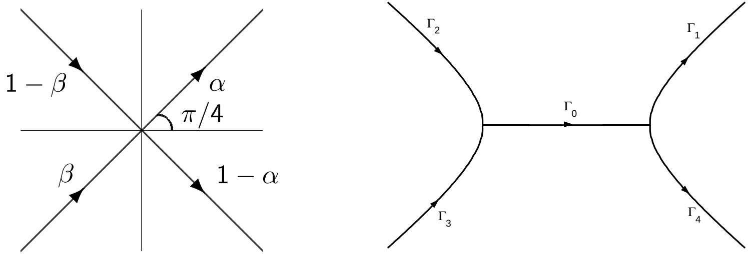

$\alpha = \beta$ corresponds to the situation considered by Fokas, Its, and Kitaev.

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Properties for n large enough

- $p_{n+1,n}, p_{n,n}$ and $p_{n-1,n}$ exist
- Recurrence

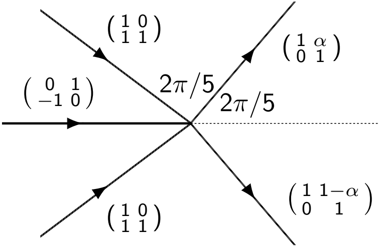
$$zp_{n,n}(z) = p_{n+1,n}(z) + b_{n,n}p_{n,n}(z) + a_{n,n}p_{n-1,n}(z)$$

- In general $a_{n,N}$ and $b_{n,N}$ satisfy a system of recurrence relations.
- If $\alpha = \beta$ then $b_{n,n} = 0$ and $a_{n,n}$ satisfies the discrete Painlevé I equation.

Ψ-function for the Painlevé I equation (Kapaev)

Let $\Psi : \mathbb{C} \to \mathbb{C}^{2\times 2}$ be the unique functions satisfying

- Ψ analytic except for the following rays, where it has the given jumps



- Ψ has asymptotics

$$\Psi(\zeta; x) = \frac{\zeta^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} \left(I + \mathcal{O}(\zeta^{-1/2}) \right) \mathrm{e}^{\theta(\zeta, x)\sigma_3},$$

for $\zeta \to \infty$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\theta(\zeta, x) = \frac{4}{5}\zeta^{5/2} + x\zeta^{1/2}$.

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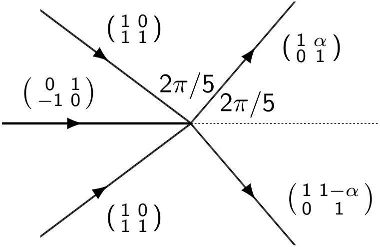

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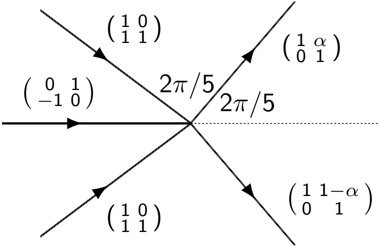

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Define

$$y_\alpha(x) = 2\mathrm{i}\,(\Psi_2(x))_{12}$$

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- All y_α have the common asymptotic series

$$y_\alpha(x) \sim \sqrt{-x/6}\left[1 + \sum_{k=1}^{\infty} a_k (-x)^{-5k/2}\right] \qquad \text{as } x \to -\infty$$

  for certain coefficients a_k.
- For y_1 this asymptotic series is even valid in the sector $\arg(x) = [3\pi/5, \pi]$.
- The other solutions differ from y_1 by only exponentially small terms

$$y_\alpha(x) = y_1(x) - \frac{\mathrm{i}(\alpha - 1)}{\sqrt{\pi}\, 2^{11/8} 3^{1/8} (-x)^{1/8}} \mathrm{e}^{-\frac{1}{5} 2^{11/4} 3^{1/4} (-x)^{5/4}} \left(1 + \mathcal{O}(x^{-3/8})\right)$$

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The y_α are uniquely characterized by this behavior.

Result

Theorem

Let $\alpha, \beta \in \mathbb{C}$, and let t depend on n such that

$$n^{4/5}(t + 1/12) = -c_1 x, \qquad c_1 = 2^{-9/5} 3^{-6/5},$$

remains fixed, where x is not a pole of y_α and y_β. Then

$$a_{n,n}(t) = 2 - c_2 \left(y_\alpha(x) + y_\beta(x)\right) n^{-2/5} + \mathcal{O}(n^{-3/5}), \qquad c_2 = 2^{3/5} 3^{2/5},$$

$$b_{n,n}(t) = c_3 \left(y_\beta(x) - y_\alpha(x)\right) n^{-2/5} + \mathcal{O}(n^{-3/5}), \qquad c_3 = 2^{1/10} 3^{2/5},$$

for $n \to \infty$.

Riemann-Hilbert problem (Fokas, Its and Kitaev)

Search for $Y : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$ with the properties

$$\begin{cases} Y(z) \text{is analytic in } \mathbb{C} \setminus \Gamma \\ \\ Y_+(z) = Y_-(z) \begin{pmatrix} 1 & \alpha(z)\mathrm{e}^{-NV_t(z)} \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma \\ \\ Y(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad z \to \infty, \end{cases}$$

where $\alpha(z) = 1$ on Γ_0, $\alpha(z) = \alpha$ on Γ_1, $\alpha(z) = 1 - \beta$ on Γ_2 and so on.

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Solution

$$Y(z) = \begin{pmatrix} p_{n,N}(z) & \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{p_{n,N}(s)\alpha(s)\mathrm{e}^{-NV_t(s)}}{s-z} \,\mathrm{d}s \\ \gamma_{n-1} p_{n-1,N}(s) & \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\gamma_{n-1} p_{n-1,N}(s)\alpha(s)\mathrm{e}^{-NV_t(s)}}{s-z} \,\mathrm{d}s \end{pmatrix},$$

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with $\gamma_{n-1} \neq 0$ and

$$a_{n,N} = (Y_1)_{12}\,(Y_1)_{21} \qquad\qquad b_{n,N} = \frac{(Y_2)_{12}}{(Y_1)_{12}} - (Y_1)_{22}\,.$$

Deift/Zhou steepest descent method

When $t > 0$, the orthogonal polynomials are characterized by the same RH-problem (but with $\Gamma = \mathbb{R}$).

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$$Y \mapsto T \mapsto S$$

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- simple and invertible
- based on the equilibrium measure μ_t

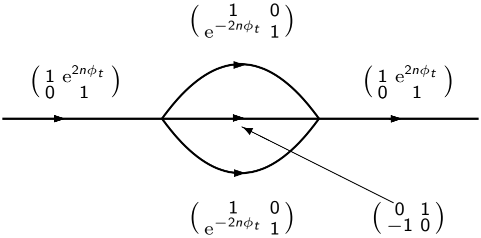
Since the equilibrium measure can be continued for $-1/12 < t < 0$, we can try the same transformations for these values of t.

Case $t > 0$

S is a solution of

$$\begin{cases} S(z) \text{ is analytic in } \mathbb{C} \setminus \Sigma_S \\ S_+(z) = S_-(z) v_S(z), & z \in \Sigma_S \\ S(z) = I + \mathcal{O}(1/z), & z \to \infty. \end{cases}$$

with jumps depending on a function ϕ_t



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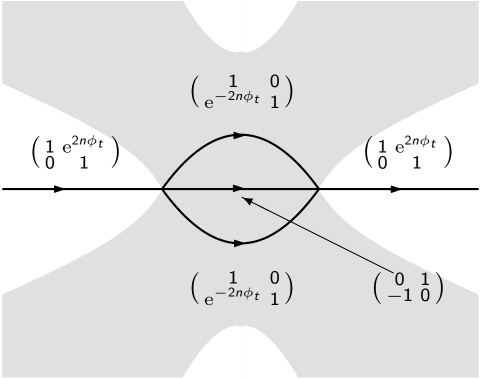
with jumps depending on a function ϕ_t (with $\mathrm{Re}\,\phi_t > 0$ in shaded area)

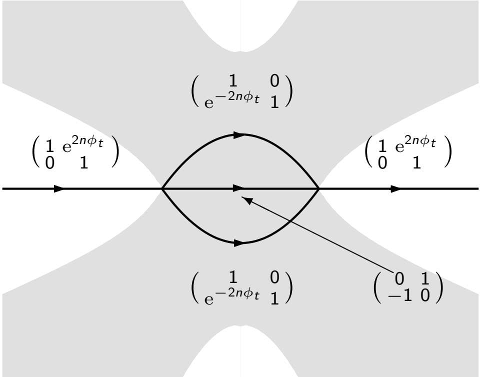


Case $t > 0$

Hope: S converges to M which is the solution of

$$\begin{cases} M(z) \text{ is analytic in } \mathbb{C} \setminus [-c_t, c_t] \\ M_+(z) = M_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in [-c_t, c_t] \\ M(z) = I + \mathcal{O}(1/z), & z \to \infty. \end{cases}$$

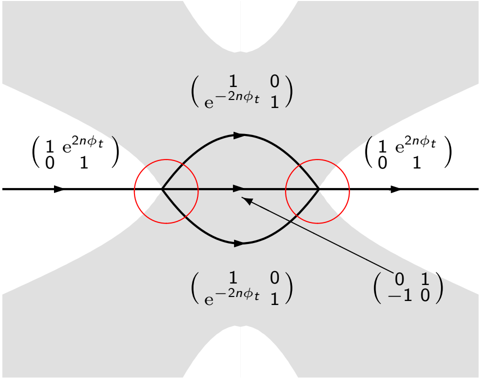
which can be explicitly calculated.



Case $t > 0$

<u>However:</u> This is not true near the endpoints! (near the endpoints the jumps do not converge uniformly to the identity)

Near the endpoint we need to make a special solution. In this case this is done by Airy functions

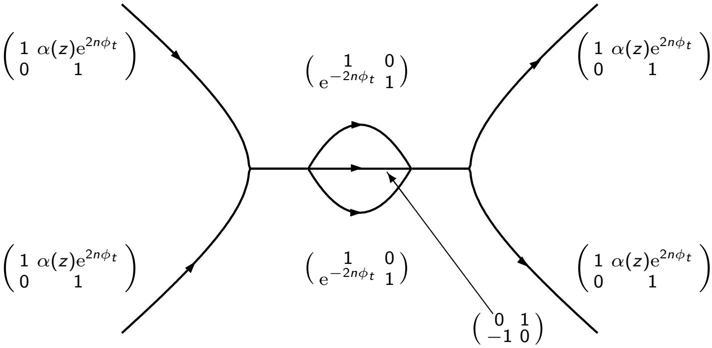


Case $-1/12 < t < 0$

Applying the same method for $-1/12 < t < 0$ based on the continuation of the equilibrium measure we find

$$\begin{cases} S(z) \text{ is analytic in } \mathbb{C} \setminus \Sigma_S \\ S_+(z) = S_-(z) v_S(z), & z \in \Sigma_S \\ S(z) = I + \mathcal{O}(1/z), & z \to \infty. \end{cases}$$

where

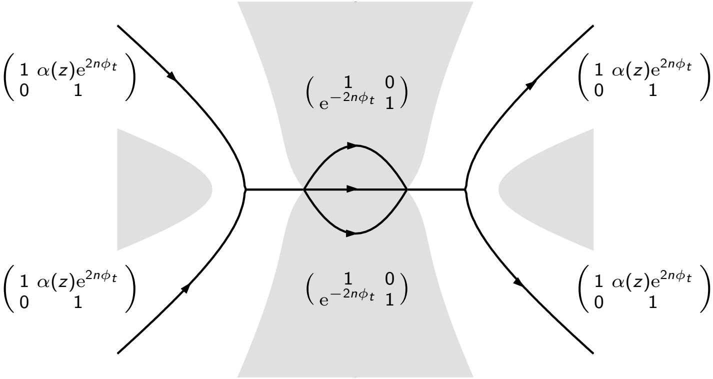


Case $-1/12 < t < 0$

Applying the same method for $-1/12 < t < 0$ based on the continuation of the equilibrium measure we find

$$\begin{cases} S(z) \text{ is analytic in } \mathbb{C} \setminus \Sigma_S \\ S_+(z) = S_-(z) v_S(z), \quad z \in \Sigma_S \\ S(z) = I + \mathcal{O}(1/z), \quad z \to \infty. \end{cases}$$

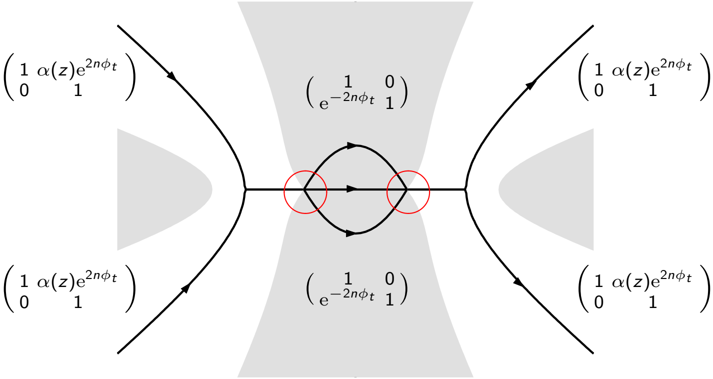
where (with $\operatorname{Re} \phi_t > 0$ in shaded area)



Case $-1/12 < t < 0$

As long as we deform the contour such that it stays away from the forbidden areas, we find the same behavior as in the case $t > 0$.

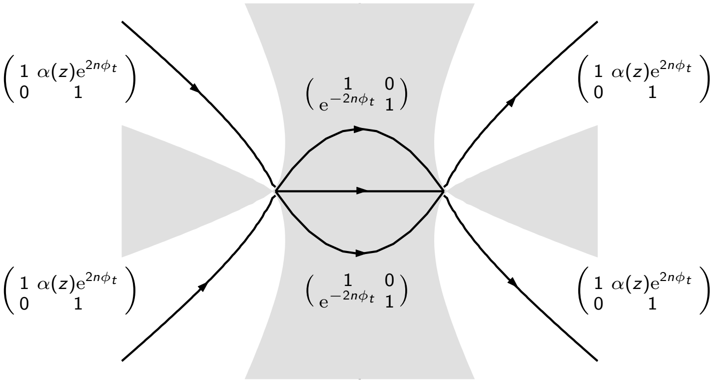
Near the endpoints the local solution is again described by Airy -functions.



Critical case $t = -1/12$

In the critical case the forbidden areas meet, and the contour has to leave into the complex plane already at the end-points.

At the end-points the local behavior is significantly different from the previous cases (equilibrium measure vanishes wit a power 3/2)



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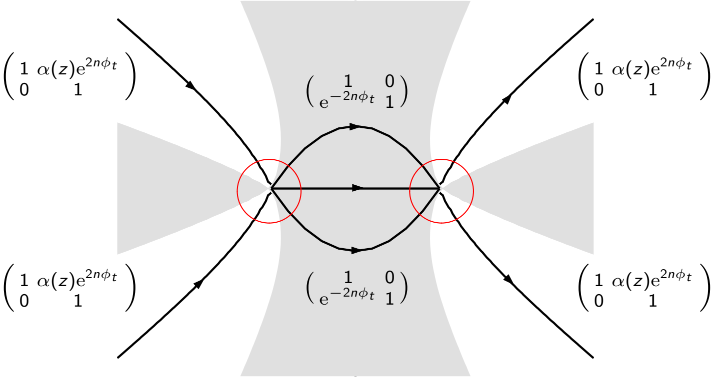
The local solution is now described by the Ψ-functions for the Painlevé I equation.



Modified equilibrium problem for the critical case

Define ν_t as the unique minimizer of the energy functional I defined by

$$I(\nu) = -\iint \log|x-y| \, \mathrm{d}\nu(x)\mathrm{d}\nu(y) + \int V_t(x) \, \mathrm{d}\nu(x)$$

among all signed measures ν with support $[-\sqrt{8}, \sqrt{8}]$.

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Claeys, Kuijlaars and Vanlessen and Bleher, Kuijlaars

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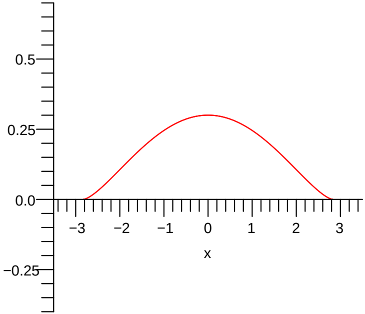

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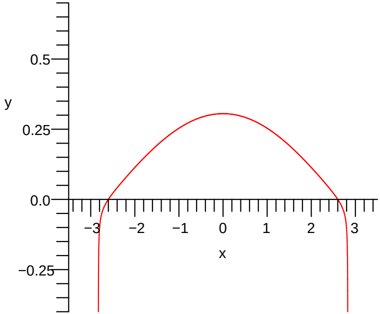

$t = -1/14$

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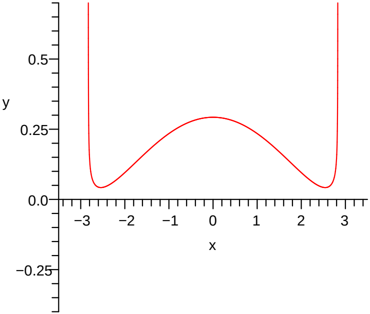

$t = -1/10$